The Upwind Finite Volume Element Method for Two-Dimensional Time Fractional Coupled Burgers’ Equation

Zechao Lv¹, Qing Yang¹

¹ School of Mathematics and Statistics, Shandong Normal University, 250014, Jinan, China

Abstract

The finite volume element method for approximating a two-dimensional time fractional coupled Burgers’ equation is presented. The linear finite volume element method is used for spatial discretization and the upwind technique is used for the nonlinear convective term to get the semi-discrete scheme. Further, the time-fractional derivative term is approximated by using L1 formula and the nonlinear convection term is treated by linearized upwind technique to get the fully discrete scheme. We prove that the semi-discrete scheme is convergent with one-order accuracy both in time and space in $L^2$-norm. Numerical experiments are presented finally to validate the theoretical analysis.

Keywords: two-dimensional time fractional coupled Burgers’ equation, finite volume element method, upwind technique, convergence

1. Introduction

In this paper, we consider the following two-dimensional time fractional coupled Burgers’ equation:

\[
\begin{align*}
C_0 \mathcal{D}_t^\alpha u + (u \cdot \nabla)u - \nu \Delta u &= f(x, t), (x, t) \in \Omega \times (0, T], \\
u(x, 0) &= u_0(x), x \in \Omega, \\
u(x, t)|_{\partial \Omega} &= \eta(x, t), (x, t) \in \Omega \times (0, T],
\end{align*}
\]

(1.1) \hspace{1cm} (1.2) \hspace{1cm} (1.3)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial \Omega$, $\nu > 0$ is the viscosity. Symbols $\Delta$, $\nabla$ denote the Laplacian and gradient operators, respectively. $u = u(x, t) = (u(x, t), v(x, t))$ is the unknown vector function which represents the velocity of the fluid. The Caputo fractional derivative $C_0 \mathcal{D}_t^\alpha$ is defined as

\[
C_0 \mathcal{D}_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1.
\]

Fractional differential operators, different from integer differential operators, are non-local and very suitable for describing materials with memory and heredity in the real world. Fractional differential equations are widely used in the fields of anomalous diffusion, viscoelastic mechanics, fluid mechanics, boundary layer effect of pipeline, signal recognition processing and system identification (Baleanu et al., 2012; DElia, 2020; Li, Changpin and Chen, An, 2018) etc.

Burgers’ equation is the simplest nonlinear time dependent partial differential equation which was first raised by Bateman in (Bateman, Harry, 1915) when he mentioned it as worthy of study and gave a special solution. It exists in plenty areas of applied mathematics, such as acoustic waves, heat conduction and modelling of dynamics (Caldwell, J et al. 1981; Cole, Julian D, 1951). Fractional Burgers’ equation can be used to describe the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the cumulative effect of the wall friction through the boundary layer (Sugimoto, Nobumasa, 1991; Inc, Mustafa, 2008). Therefore, fractional Burgers’ equations have been taken more and more seriously attention and numerical methods have been developed to provide numerical solutions for them. For example, Wang (Wang, Qi, 2006) discussed the adomian decomposition method for time and space fractional Burgers’ equations. Esen and Tasbozan (Esen et al., 2015) proposed quadratic B-spline Galerkin method for one-dimensional time fractional Burgers’ equation. Li et al. (Li, Dongfang et al. 2016) employed a linear implicit finite difference scheme for solving the generalized time fractional Burgers’ equation. Cao et al. (Cao, Wen et al., 2017) presented discontinuous Galerkin method to solve a two-dimensional time fractional Burgers’ equation with...
high and low Reynolds numbers on quasi-uniform triangular mesh. Wang et al. (Wang, Haifeng et al., 2021) proposed the weak Galerkin finite element method for a class of one-dimensional time fractional generalized Burgers’ equation. Hussein and Ahmed Jabbar (Hussein, Ahmed Jabbar, 2020) proposed a weak Galerkin finite element method for solving two-dimensional time fractional coupled Burgers’ equations on uniform triangular mesh. Qiao and Tang (Qiao, Leijie et al., 2022) introduce an accurate, robust, and efficient finite difference scheme with graded meshes for the time-fractional Burgers’ equation. Zhang and Feng (Zhang, Yadong et al., 2023) present a local projection stabilization virtual element method for the time-fractional Burgers’ equation with high Reynolds numbers. Kashif and Dwiwedi (Mohd, Kashif et al., 2022) use non-standard finite difference scheme and Fibonacci collocation method for fractional order Burgers equation. Wang et al. (Wang, Haifeng et al., 2021) proposed the weak Galerkin finite element method and cubic B-spline base function with quadratic weight function to approximate the numerical solution of the time-fractional Burgers equation.

The finite volume element method (FVEM) (Li, Ronghua et al., 2000; Cai, Zhiqiang et al., 1991; Cai, Zhiqiang, 1990; Ewing, Richard et al. 2000) is an important numerical method for solving partial differential equations. In some literature, FVEM is also called the control volume method, the covolume method, and the first-order generalized difference method. FVEM has been widely applied due to its various important features. For example, the grid is flexible and it inherits some physical conservation laws of the original problems locally which are very preferable in practical applications. And the computational effort is greater than in finite difference methods and less than finite element methods, while the accuracy is higher than finite difference methods and nearly the same as finite element methods. Liang (Liang, D, 1990; Liang, D, 1991) combined the upwind technique and the FVEM to solve the linear convection-dominated problems. Yang (Yang, Qing, 2013) used the upwind finite volume element method for two-dimensional integer-order Burgers’ equation. AA AL-saedi and J Rashidinia (AA, AL-saedi et al., 2023) present a numerical scheme based on the Galerkin finite element method and cubic B-spline base function with quadratic weight function to approximate the numerical solution of the semi-discrete scheme and the fully discrete scheme, respectively. In section 4, numerical experiments show that our method is effective for the approximation of the two-dimensional time coupled fractional Burgers’ equation (1.1). The upwind approximation is applied to handle the nonlinear convection term in view of its advantage to solve the convection-dominated problems. The semi-discrete and fully discrete schemes are defined. The linear finite volume element format is used for spatial discretization, and the upwind technique is used for the nonlinear convective term to get the semi-discrete scheme. Then L1 formula is used to approximate the time-fractional derivative term, and the nonlinear convection term is treated by linearized upwind technique to get the fully discrete scheme. We prove that the semi-discrete scheme is convergent with one-order accuracy in space in $L^2$-norm and the fully discrete scheme is convergent with one-order accuracy both in time and space in $L^2$-norm. Some numerical experiments show that our method is effective for the time fractional Burgers’ equation.

The outline of this paper is as follows. In Section 2, we build the upwind finite volume element semi-discrete scheme and fully discrete scheme for the coupled fractional Burgers’ equation, respectively. In section 3, we derive the $L^2$ norm error estimates for the semi-discrete scheme and the fully discrete scheme, respectively. In section 4, numerical experiments are presented to show the efficiency of proposed method and confirm our theoretical analysis.

2. The Approximation Schemes

In this section, we introduce some important function spaces and give the semi-discrete scheme and the fully discrete scheme for problem (1.1)-(1.3).

Set $X = (H^1_0(\Omega))^2$, $Y = (L^2(\Omega))^2$. Let $T_h = \{K\}$ be a quasi-uniform triangulation of domain $\Omega$ so that $\bar{\Omega} = \bigcup_{K \in T_h} K$ and $N_h$ is the set of all nodal points of $T_h$, where $h = \max h_K$, $h_K$ is the diameter of element $K$. Associated with the triangulation $T_h$, we introduce the velocity approximation space:

$$X_h = \{ v_h \in X : v_h|_K \in (P_1(K))^2, K \in T_h \},$$

where $P_1(K)$ is the set of linear polynomials on element $K$.

In order to define the finite volume element method, we need a dual partition associated with the primal partition $T_h$. We construct the barycenter dual partition $T^*_h$ by connecting the barycenter to the midpoints of edges of each $K \in T_h$ by straight lines. Thus, for each nodal point $P$ in $T_h$, there exists a polygonal surrounding it, which is called the dual element or the control volume at point $P$ and denoted by $K^*_P$. So we have $T^*_h = \{ K^*_P, P \in N_h \}$. The test function space is defined as

$$V_h = \{ v_h \in (L^2(\Omega))^2 : v_h|_{K^*_P} \in P_0(K^*_P), \forall P \in N_h, v_h|_{K^*_P} = 0, \forall P \in \partial \Omega \},$$

where $P_0(K^*_P)$ is the constant set on element $K^*_P$. Clearly, the dimensions of $X_h$ and $V_h$ are the same.

Introduce the interpolation operators $R_h$: $(C(\bar{\Omega}))^2 \to X_h$ and $\gamma$: $X_h \to V_h$, respectively. Assuming that $u \in (H^2(\Omega))^2$, we
can easily get the following interpolation estimates:
\[
\|u - R_h u\|_{s} \leq h^{2-s} \|u\|_{2}, \quad s = 0, 1. \tag{2.1}
\]

**Lemma 1.** (Li, Ronghua et al., 2000) (i) The mapping γ is self-adjoint with respect to the \(L^2\)-inner product in \(X_h\):
\[
(u_h, \gamma v_h) = (\gamma u_h, v_h), \quad \forall u_h, v_h \in X_h. \tag{2.2}
\]
In particular, if \(u_h(\cdot, t) \in X_h, v_h \in X_h, t \in [0, T]\), it holds
\[
\left(\phi (\partial_t^2 u_h, \gamma v_h) = \left(\partial_t^2 u_h, v_h\right). \tag{2.3}
\]
(ii) The norm \(\|u_h\| = (u_h, \gamma u_h)^{1/2}\) is equivalent to the usual \(L^2\)-norm in \(X_h\), i.e.
\[
C_1 \|u_h\| \leq ||u_h|| \leq C_2 \|u_h\|, \quad \forall u_h \in X_h, \tag{2.4}
\]
where the positive constant \(C_1\) and \(C_2\) are independent of \(h\).

We start by testing the Eq.(1.1) by \(v \in X_h\),
\[
\left(\int_{\Omega} \partial_t^2 \mathbf{u} \cdot \gamma v \right) + \left(\mathbf{u} \cdot \nabla \mathbf{u}, \gamma v \right) - \nu \left(\Delta \mathbf{u}, \gamma v \right) = (f, \gamma v). \tag{2.5}
\]
Using integration by parts and the fact that \(\gamma v\) is a piecewise constant function on \(T_h^s\), we get
\[
-\nu \left(\Delta \mathbf{u}, \gamma v \right) = -\nu < \nabla \mathbf{u} \cdot \mathbf{n}, \gamma v >, \quad ((\mathbf{u} \cdot \nabla) \mathbf{u}, \gamma v) = < \mathbf{u} \cdot \mathbf{n}, \mathbf{u} \cdot \gamma v > - (\mathbf{u}, (\nabla \cdot \mathbf{u}) \gamma v),
\]
where
\[
< \nabla \mathbf{u} \cdot \mathbf{n}, \gamma v > = \sum_{P \in N_h} v(P) \cdot \int_{\partial K_t} \nabla \mathbf{u} \cdot \mathbf{n} ds, \quad < \mathbf{u} \cdot \mathbf{n}, \mathbf{u} \cdot \gamma v > = \sum_{P \in N_h} v(P) \cdot \int_{\partial K_t} (\mathbf{u} \cdot \mathbf{n}) \mathbf{u} ds, \quad (\mathbf{u}, (\nabla \cdot \mathbf{u}) \gamma v) = \sum_{P \in N_h} v(P) \cdot \int_{K_t} (\nabla \cdot \mathbf{u}) \mathbf{u} ds.
\]

Define
\[
A(\mathbf{u}, v) = -\nu < \nabla \mathbf{u} \cdot \mathbf{n}, \gamma v >, \quad b_1(\mathbf{u}, \mathbf{w}, v) = -(\mathbf{u}, (\nabla \cdot \mathbf{w}) \gamma v), \quad b_2(\mathbf{u}, \mathbf{w}, v) = < \mathbf{u} \cdot \mathbf{n}, \mathbf{w} \cdot \gamma v >.
\]
So (2.5) can be rewritten as
\[
\left(\int_{\Omega} \partial_t^2 \mathbf{u} \cdot \gamma v \right) + A(\mathbf{u}, v) + b_1(\mathbf{u}, \mathbf{u}, v) + b_2(\mathbf{u}, \mathbf{u}, v) = (f, \gamma v), \quad \forall v \in X_h. \tag{2.6}
\]

**Lemma 2.** (Li, Ronghua et al., 2000) For the bilinear form \(A(\cdot, \cdot)\), we have the following conclusions.
(i) For \(u_h, v_h \in X_h\), one has
\[
A(u_h, v_h) = A(v_h, u_h). \tag{2.7}
\]
(ii) There exists a positive constant \(C\) such that
\[
|A(\mathbf{u} - R_h \mathbf{u}, v_h)| \leq Ch \|\mathbf{u}\|_2 \|v_h\|_1, \quad \forall \mathbf{u} \in (H^2(\Omega))^2, v_h \in X_h. \tag{2.8}
\]
(iii) There exists a positive constant \(\mu\) such that
\[
|A(u_h, u_h)| \geq \mu \|u_h\|_2^2, \quad \forall u_h \in X_h. \tag{2.9}
\]

**Lemma 3.** Let \(u_h(\cdot, t) \in X_h\), then
\[
\left(\int_{\Omega} \partial_t^2 u_h \cdot \gamma u_h \right) = \frac{1}{2} \int_{\Omega} \|u_h\|^2. \tag{2.10}
\]
Proof. It is known by the definition of the fractional derivative that
\[
\begin{align*}
\left( \frac{c}{0}D_t^\alpha \mathbf{u}_h, \gamma \mathbf{u}_h \right) & = \int_\Omega \frac{c}{0}D_t^\alpha \mathbf{u}_h \cdot \gamma \mathbf{u}_h \, dx = \int_\Omega \frac{1}{\Gamma(1-\alpha)} \frac{1}{(t-\tau)^\alpha} d\tau \cdot \gamma \mathbf{u}_h \, dx \\
& = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_\Omega \mathbf{u}_{h,\tau} \cdot \gamma \mathbf{u}_h \, dx \, d\tau \\
& = \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_\Omega \mathbf{u}_h \cdot \gamma \mathbf{u}_h \, dx \, d\tau \\
& = \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_\Omega (\mathbf{u}_h, \gamma \mathbf{u}_h) \, d\tau \\
& = \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \| \mathbf{u}_h \|^2 \, d\tau \\
& = \frac{1}{2} \frac{c}{0}D_t^\alpha \| \mathbf{u}_h \|^2.
\end{align*}
\]

We will approximate $b_2(\mathbf{u}, \mathbf{v}, \mathbf{w})$ by using the upwind technique. Let $\Lambda_j = \{ j : P_j$ is adjoint with $P_i \}$. Assuming that $j \in \Lambda_i$, let $\Gamma_{ij} = \partial K_{P_j} \cap \partial K_{P_i}$ and $\gamma_{ij}$ is the length of $\Gamma_{ij}$. Denote by $\mathbf{n}_{ij}$ the unit outward normal vector of $\Gamma_{ij}$ when $\Gamma_{ij}$ is regarded as the boundary of $K_{P_j}$. Define
\[
\beta_{ij}(\mathbf{u}) = \int_{\Gamma_{ij}} \mathbf{n}_{ij} \cdot \mathbf{u} \, ds.
\]

Let
\[
\beta_{ij}^*(\mathbf{u}) = \max \left\{ \beta_{ij}(\mathbf{u}), 0 \right\}, \quad \beta_{ij}^*(\mathbf{u}) = \max \left\{ -\beta_{ij}(\mathbf{u}), 0 \right\},
\]

\[
\int_{\partial K_{P_i}} (\mathbf{u} \cdot \mathbf{n}) \mathbf{v} \, ds = \sum_{j \in \Lambda_i} \left\{ \beta_{ij}^*(\mathbf{u}) \mathbf{v}(P_i) - \beta_{ij}^*(\mathbf{u}) \mathbf{v}(P_j) \right\}.
\]

The upwind approximation form of the nonlinear term $b_2(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is defined by the form
\[
b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{P_i \in N_h} \sum_{j \in \Lambda_i} \left( \beta_{ij}^*(\mathbf{u}) \mathbf{v}(P_i) - \beta_{ij}^*(\mathbf{u}) \mathbf{v}(P_j) \right) \cdot \mathbf{w}(P_i).
\]

Using the Heaviside function
\[
H(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0, \end{cases}
\]

we can write $b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ as
\[
b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{P_i \in N_h} \sum_{j \in \Lambda_i} \beta_{ij}(\mathbf{u}) \left( H(\beta_{ij}(\mathbf{u})) \mathbf{v}(P_i) + (1 - H(\beta_{ij}(\mathbf{u}))) \mathbf{v}(P_j) \right) \cdot \mathbf{w}(P_i). \tag{2.11}
\]

Lemma 4. (Yang, Qing, 2016) For $\mathbf{u} \in \left( H^1(\Omega) \right)_2$, $\mathbf{v} \in \left( H^1(\Omega) \right)_2$, $\mathbf{u}_h \in X_h$ and $\mathbf{w}_h \in X_h$, one has
\[
|b_2(\mathbf{u}, \mathbf{v}, \mathbf{w}_h) - b_{2h}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| \\
\leq \left| \mathbf{w}_h \right|_1 \left| h \right| \| \mathbf{u} \|_1 + \| \mathbf{v} \|_1 \left( \| \mathbf{u} - \mathbf{u}_h \| + h \| \mathbf{u} - \mathbf{u}_h \|_1 \right). \tag{2.12}
\]

Then, the semi-discrete finite volume element scheme of problem (1.1)-(1.3) is as follows: seek $u_h(\cdot, t) \in X_h$ for $t \in [0, T]$ such that
\[
\left( \frac{c}{0}D_t^\alpha \mathbf{u}_h, \gamma \mathbf{v}_h \right) + A(\mathbf{u}_h, \mathbf{v}_h) + b_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b_{2h}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (f, \gamma \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h, \tag{2.13}
\]
\[
u_h(\mathbf{x}, 0) = R_h u_0(\mathbf{x}). \tag{2.14}
\]

Let $T_n = [t_n]_{n=0} = [nt_n = n\tau, 0 \leq n \leq N]$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$. Based on a piecewise linear interpolation, the L1-approximation to the Caputo fractional derivative is given by
\[
\frac{c}{0}D_t^\alpha \mathbf{u}_{h,T_n} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^n \frac{u(x, t_j) - u(x, t_{j-1})}{\tau} \int_{t_{j-1}}^{t_j} \frac{1}{(t - s)^\alpha} ds + Q^n \\
= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j} (u(x, t_j) - u(x, t_{j-1})) + Q^n.
\]
where
\[
a_i = (i + 1)^{1-\alpha} - i^{1-\alpha}.
\]

If \(u \in C^2([0, T]; L^2(\Omega))\), the truncation error \(Q^n\) satisfies (Sayevand, 2016)
\[
\|Q^n\| = O(\tau^{2-\alpha}).
\]

For a sequence \(\{\omega^n\}_{n=0}^N\), we define
\[
D^\omega_n := \frac{\tau^{-\alpha}}{1 - \alpha} \sum_{j=1}^{n} a_{n-j} \omega^j = \frac{\tau^{-\alpha}}{1 - \alpha} \sum_{j=0}^{n} b_{n-j} \omega^j, \quad n = 1, \ldots, N,
\]  
(3.15)

where \(\delta^\omega_n = \omega^n - \omega^{n-1}\) and
\[
b_0 = a_0, \quad b_n = -a_{n-1}, \quad b_{n-j} = a_{n-j} - a_{n-j-1}, \quad j = 1, \ldots, n - 1.
\]

With the above notations, the fully discrete finite volume element scheme for (1.1)-(1.3) seeks \(u_h^n \in X_h\) such that
\[
(D^\omega_n u_h^n, \gamma v_h) + A(u_h^n, v_h) + b_1(u_h^{n-1}, u_h^n, v_h) + b_2(b(u_h^{n-1}, u_h^n, v_h) = (f, \gamma v_h), \quad n \geq 1, \gamma v_h \in X_h,
\]  
(3.16)

\[
\sum\nabla^2 u_h^0 = R_h u_0.
\]  
(3.17)

3. Error Analysis

In this section, we present the error estimates for the semi-discrete scheme (2.13)-(2.14) and the fully discrete scheme (2.16)-(2.17) in \(L^2\)-norm.

3.1 Error Estimates for the Semi-Discrete Scheme

**Lemma 5.** (Zhu, Tao, 2018) Let \(0 < T < \infty, \beta \in (0, 1), a(t), l(t)\) and \(u(t)\) are continuous, nonnegative functions on \([0, T]\) with
\[
u(t) \leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} l(s) u(s) ds.
\]  
(3.1)

Then
\[
u(t) \leq \left( A(t) + \int_0^t L(s) A(s) \exp \left( \int_s^t L(\tau) d\tau \right) ds \right)^{\alpha}, \quad t \in [0, T).
\]  
(3.2)

If \(a(t)\) is nondecreasing on \([0, T]\), then the above inequality is reduced to
\[
u(t) \leq \left( A(t) \exp \left( \int_0^t L(s) ds \right) \right)^{\alpha},
\]  
(3.3)

where \(A(t) = 2^{\frac{1}{\alpha}-1} a^\frac{1}{\alpha}(t), L(t) = \frac{2^{\frac{1}{\alpha}-1}}{1 + \beta} \left( \Gamma \left( \frac{\alpha}{\alpha - \beta} \right) \Gamma \left( \frac{\alpha - \beta}{\alpha} \right) \right) \right) \frac{1}{\tau^{\frac{\alpha-\beta}{\alpha}}} \int_0^t l^\beta(t)\) and \(0 < \alpha < \beta < 1\). If \(a(t) \equiv 0\) on \([0, T]\), then
\[
u(t) \equiv 0.
\]  
(3.4)

**Theorem 6.** Let \(u\) and \(u_h\) be the solutions of (1.1)-(1.3) and (2.13)-(2.14), respectively. Also assume that \(u\) satisfies the necessary regularities. Then, for \(t \in [0, T]\) and sufficiently small \(h > 0\),
\[
\|u(t) - u_h(t)\| \leq Ch,
\]  
(3.5)

where \(C\) is a constant independent of \(h\) and dependent on principally \(\|u_0\|_2, \|u\|_{L^\infty((H^1(\Omega))^2)}, \max_{0 \leq t \leq T} C^{\|D_t u\|_2}\) and \(\|u\|_{L^\infty((W^{1,\infty}(\Omega))^2)}\).

**Proof.** Subtract (2.13) from (2.6) to obtain that
\[
(D^\omega_n u - D^\omega_n u_h, \gamma v_h) + A(u - u_h, v_h) + b_1(u, u, v_h) - b_1(u_h, u_h, v_h) + b_2(u, u, v_h) - b_2(b(u_h^{n-1}, u_h^n, v_h) = 0.
\]  
(3.6)

Letting \(u - u_h = (u - R_h u) + (R_h u - u_h) = E + e\), we rewrite (3.6) as
\[
(D^\omega_n e, \gamma v_h) + A(e, v_h) + b_1(u, u, v_h) - b_1(u_h, u_h, v_h) + b_2(u, u, v_h) - b_2(b(u_h^{n-1}, u_h^n, v_h) = - (D^\omega_n E, \gamma v_h) - A(E, v_h).
\]  
(3.7)
We choose \( v_h = e \) in (3.7) to get

\[
\left( \frac{C}{0} \frac{\partial}{\partial t} e, ye \right) + A(e, e) = - \left( \frac{C}{0} \frac{\partial}{\partial t} E, ye \right) - A(E, e) \\
- [b_1 (u, u, e) - b_1 (u_h, u_h, e)] - [b_2 (u, u, e) - b_2 (u_h, u_h, e)].
\]

(3.8)

Using Lemmas 2 and 3 for the left two terms in (3.8), we have

\[
\frac{1}{2} \frac{C}{0} \frac{\partial}{\partial t} \| e \|^2 + \mu \| e \|^2 \leq - \left( \frac{C}{0} \frac{\partial}{\partial t} E, ye \right) - A(E, e) \\
- [b_1 (u, u, e) - b_1 (u_h, u_h, e)] - [b_2 (u, u, e) - b_2 (u_h, u_h, e)].
\]

(3.9)

For the first term on the right-hand side of (3.9), using Young’s inequality, we get

\[
\left( \frac{C}{0} \frac{\partial}{\partial t} E, ye \right) \leq C(\| \frac{C}{0} \frac{\partial}{\partial t} E \|^2 + \| e \|^2).
\]

(3.10)

For the second term, applying Lemma 2, we obtain

\[
\| A(E, e) \| \leq Ch\| u \|_2 \| e \|_1 \leq Ch^2 \| u \|_2^2 + \varepsilon \| e \|_1^2.
\]

(3.11)

For \( \| \frac{C}{0} \frac{\partial}{\partial t} E \| \) in (3.10), we have

\[
\left\| \frac{C}{0} \frac{\partial}{\partial t} E \right\| = \left\| \frac{1}{\Gamma(1-\alpha)} \int_0^\gamma \frac{(u - R_h u)_{\tau}}{(t - \tau)^\alpha} d\tau \right\| \leq \frac{1}{\Gamma(1-\alpha)} \int_0^\gamma \\| u - R_h u \|_{\tau} (t - \tau)^\alpha d\tau \leq \frac{h^2}{\Gamma(1-\alpha)} \int_0^\gamma \| u \|_1 \| (u - R_h u) \| \| t - \tau \|^\alpha d\tau \leq Ch^2 \frac{C}{0} \frac{\partial}{\partial t} \| u \|_2.
\]

(3.12)

Now we bound the last two terms on the right-hand side of (3.9). We need the following induction hypothesis:

\[
\left( \log \frac{1}{h} \right)^{1/2} \| e(s) \| \rightarrow 0, \ h \rightarrow 0, \ 0 \leq s \leq t, \ 0 < t < T.
\]

(3.13)

Also we have the following norm inequalities

\[
\| w \|_\infty \leq C \left( \log \frac{1}{h} \right)^{1/2} \| w \|_1, \quad \forall w \in X_h.
\]

(3.14)

\[
\| w \|_1 \leq Ch^{-1} \| w \|, \quad \forall w \in X_h.
\]

(3.15)

Using hypothesis (3.13), (3.14) and Young’s inequality, we have

\[
\| b_1 (u, u, e) - b_1 (u_h, u_h, e) \| \\
= \| (u, (V \cdot u) ye) - (u_h, (V \cdot u_h) ye) \| \\
= \| (u, (V \cdot u) ye) - (u_h, (V \cdot u_h) ye) \| + \| (u_h, (V \cdot u) ye) - (u_h, (V \cdot u_h) ye) \| \\
\leq C (\| V \cdot u \|_\infty \| u - u_h \| + \| V \cdot u \| \| u - u_h \|_\infty + \| R_h u \|_\infty) \| e \| \\
\leq C (\| E \|_1 + \| e \|_1 + (\| E \|_1 + \| e \|_1) (\log \frac{1}{h})^{1/2} \| e \|_1 + \| u \|_1) \| e \| \\
\leq C (\| e \|_1 + \| E \|_1 + (\log \frac{1}{h})^{1/2} \| e \|_1 + \| E \|_1) \| e \| \\
\leq C \| E \|_1^2 + \| e \|_1^2 + C (\log \frac{1}{h})^{1/2} \| e \| \| e \|_1^2 \\
\leq Ch^2 \| u \|_2^2 + 2\varepsilon \| e \|_1^2 + C \| e \|_1^2.
\]

(3.16)

Next, we write

\[
\| b_2 (u, u, e) - b_2 (u_h, u_h, e) \| \\
\leq \| b_2 (u, u, e) - b_2 (u_h, R_h u, e) \| + \| b_2 (u_h, e, e) \| \\
= D_1 + D_2.
\]

(3.17)
where $D_1 = |b_2(u, u, e) - b_2(h, R_0 u, e)|$ and $D_2 = |b_2(h, e, e)|$. By choosing $v = u$, $v_h = u_h$ and $w_h = e$ in Lemma 4, using (3.15) and Young’s inequality, we can obtain

$$
D_1 \leq \|e\|_{L^\infty} \|u\|_{L^\infty} |u_h| + |u|_{L^\infty} (\|u - u_h\| + h |u - u_h|) \\
\leq C \left( h^2 |u|^2 + \|e\|^2 + \|e\|^2 \right) + \epsilon |e|^2. \tag{3.18}
$$

By the definition of $b_2(\cdot, \cdot, \cdot)$ and then by (3.13)-(3.15), we have

$$
D_2 = \left| \sum_{P \in N_h} e(P_i) \cdot \sum_{i \in N_h} \int_{T_i} (u_h \cdot n) ds \times \left[ H(\beta_{ij}) e(P_i) + \left( 1 - H(\beta_{ij}) \right) e(P_j) \right] \right| \\
\leq \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i \in \mathcal{N}_K} \left| \epsilon(P_i) - \epsilon(P_j) \right| \times \int_{T_i \cap K} \left| H(\beta_{ij}) e(P_i) + \left( 1 - H(\beta_{ij}) \right) e(P_j) \right| \\
\times \int_{T_i \cap K} \left| \epsilon \cdot n \right| ds \\
+ \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i \in \mathcal{N}_K} \left| \epsilon(P_i) - \epsilon(P_j) \right| \\
\times \left| H(\beta_{ij}) e(P_i) + \left( 1 - H(\beta_{ij}) \right) e(P_j) \right| \times \int_{T_i \cap K} |R_h u \cdot n| ds \\
\leq C \left( \|e\|_{L^\infty} \|u\|_{L^\infty} + h |u|_{L^\infty} \right) \|e\|_{L^\infty} + C \|e\|^2 + \epsilon |e|^2. \tag{3.19}
$$

Now, substituting (3.10)-(3.19) into (3.9), using Lemma 3, we find

$$
\frac{1}{2} \mathcal{C} D_3^\alpha \|e\|^2 + \mu \|e\|^2 \leq C h^2 \left( h^2 \mathcal{C} D_3^\alpha \|u\|^2 + \|u\|^2 \right) + C \|e\|^2 + 6\epsilon \|e\|^2. \tag{3.20}
$$

Taking $\epsilon = \frac{\mu}{6}$, we have

$$
\mathcal{C} D_3^\alpha \|e\|^2 \leq C h^2 \left( h^2 \mathcal{C} D_3^\alpha \|u\|^2 + \|u\|^2 \right) + C \|e\|^2. \tag{3.21}
$$

Therefore, making a fractional integral of order $\alpha$ on both sides of (4.21), noting that $e(0) = 0$, so

$$
\|e(t)\|^2 \leq \frac{Ch^4}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \left( \mathcal{C} D_3^\alpha \|u(\tau)\|^2 \right) d\tau + \frac{Ch^2}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \|u(\tau)\|^2 d\tau \\
+ \frac{C}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \|e(\tau)\|^2 d\tau \leq C h^2 + \frac{C}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \|e(\tau)\|^2 d\tau. \tag{3.22}
$$

Noting Lemma 1, we have

$$
\|e(t)\|^2 \leq C h^2 + \frac{C}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} \|e(\tau)\|^2 d\tau. \tag{3.23}
$$

Using Lemma 5 in (3.23), we get

$$
\|e(t)\| \leq Ch. \tag{3.24}
$$

By Lemma 2, we know

$$
\|E(t)\| \leq C h^2 \|u(t)\|_2 \leq C h^2 (\|u(0)\|_2 + \int_0^t \|u(\tau)\|_2 d\tau). \tag{3.25}
$$

Combining (3.24) with (3.25) and using triangle inequality, we obtain

$$
\|u(t) - u_h(t)\| \leq Ch, \tag{3.26}
$$

where $C$ depends on $\|u(0)\|_2$, $\|u\|_{L^\infty((H^1(\Omega))^2)}$, max $0 \leq t \leq T \|u\|_{L^\infty((H^1(\Omega))^2)}$ and $\|u\|_{L^\infty((W^1,\infty(\Omega))^2)}$. 

37
We now prove the hypothesis (3.13). Noting that \(|e(0)| = 0\), so it holds when \(s = 0\) clearly. Suppose it holds when \(0 \leq s < t\), we know from (3.24) that
\[
(\log \frac{1}{h})^{1/2}\|e(t)\| \leq C(\log \frac{1}{h})^{1/2}h \rightarrow 0, h \rightarrow 0,
\]
i.e. (3.13) is true for \(s = t\). Thus, the hypothesis (3.13) holds for any \(t \in [0, T]\). □

3.2 Error Estimates for the Fully Discrete Scheme

**Lemma 7.** (Li, Dongfang, 2018) Suppose that the nonnegative sequence \(\{\omega^n\}_{n=0}^\infty\) and \(\{g^n\}_{n=0}^\infty\) satisfy
\[
D^n_\tau \omega^n = \lambda_1 \omega^n + \lambda_2 \omega^{n-1} + g^n, \quad n \geq 1,
\]
where \(\lambda_1\) and \(\lambda_2\) are both positive constants independent of the time step \(\tau\). Then, there exists a positive constant \(\tau^*\) such that, when \(\tau \leq \tau^*\),
\[
\omega^n \leq 2 \left( \omega^0 + \frac{\tau^n}{\Gamma(1 + \alpha)} \max_{0 \leq s \leq t} g^s \right) E_n(2\lambda^n), \quad 1 \leq n \leq N,
\]
here, \(E_n(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\alpha)}\) is the Mittag-Leffler function and \(\lambda = \lambda_1 + \frac{\lambda_2}{2^{1-\alpha}}\).

**Theorem 8.** Let \( u \) and \( \{u^n_h\}_{h=0}^\infty \) be the solutions of (1.1)-(1.3) and (2.16)-(2.17), respectively. Also assume that \( u \) satisfies the necessary regularities and the discretization parameters obey the relation \( \tau = O(h) \). Then
\[
\max_{0 \leq n \leq N} \|u^n - u^n_h\| \leq C(h + \tau),
\]
where \( C \) depends on \(\|u_0\|_{2r}, \|u\|_{L^\infty((i\Omega)^r)}, \|u\|_{L^\infty((i\Omega)^r)}\), and \(\|u\|_{L^\infty((i\Omega)^r)}\).

**Proof.** Subtract (2.16) from (2.6) to obtain that
\[
\left( \sum_{i,j=1}^d D^n_{i,j} u^n - D^n_{i,j} u^n_h, v_h \right) + \sum_{i,j=1}^d \left( -b_1(u^n, u^n, v_h) - b_2(u^n, u^n, v_h) \right)
\]
Taking \( v_h = e^n \), we have
\[
(D^n_\tau e^n, ye^n) + A(e^n, e^n) = - \left( \sum_{i,j=1}^d D^n_{i,j} u^n - D^n_{i,j} u^n_h, ye^n \right) - (D^n_\tau E^n, ye^n) - A(E^n, e^n)
\]
\[
- \left[ b_1(u^n, u^n, e^n) - b_2(u^n, u^n, e^n) \right] - \left[ b_2(u^n, u^n, e^n) - b_2(u^{n-1}, u^n, e^n) \right].
\]
For the left two terms of (3.32), from Lemma 3, we have
\[
(D^n_\tau e^n, ye^n) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( a_0 e^n - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) e^j - a_{n-1} e^0, ye^n \right)
\]
\[
\geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left( a_0 \|e^n\|^2 - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) \|e^j\|^2 + \|e^0\|^2 + \|e^n\|^2 \right)
\]
\[
= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \left( a_0 \|e^n\|^2 - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) \|e^j\|^2 - a_{n-1} \|e^0\|^2 \right)
\]
\[
= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{j=0}^{n-1} b_{n-j} \|e^j\|^2
\]
and
\[
\|A(e^n, e^n)\| \geq \mu \|e^n\|^2.
\]
The five terms on the right-hand side of (3.32) are represented by $T_1$, $T_2$, ..., $T_5$, respectively. So, (3.32) can be rewritten as

$$\frac{1}{2}D^n_T \| e^n \|_2^2 + \mu \| e^n \|_1^2 \leq T_1 + T_2 + \cdots + T_5.$$  

(3.35)

Now, let’s estimate the terms $T_1, T_2, ..., T_5$ one by one. Using the property of L1-approximation we have

$$|T_1| \leq C \left\| D^n_T (\mathbf{u}^n - D^n_T \mathbf{u}^n) \right\|_2 \| e^n \| \leq C (\| Q_{\mathbf{u}} \| \left\| e^n \right\| \leq C (\| Q_{\mathbf{u}} \| ^2 + \| e^n \| ^2).$$

(3.36)

For $T_2$, we first have

$$|T_2| \leq \left\| D^n_T \mathbf{e}^n \right\| \| e^n \| \leq C \left( \frac{(\pi r)^{1-a}}{2} \max_{t \leq \tau} (R_h - I) \mathbf{u}_t \right) \| e^n \| \leq C (\pi h \max_{t \leq \tau} \| (R_h - I) \mathbf{u}_t \|).$$

(3.37)

Then using Young’s inequality we get

$$|T_2| \leq \left\| D^n_T \mathbf{e}^n \right\| \| e^n \| \leq C \left( \frac{(\pi r)^{1-a}}{2} \max_{t \leq \tau} (R_h - I) \mathbf{u}_t \right) \| e^n \| \leq C (\pi h \max_{t \leq \tau} \| (R_h - I) \mathbf{u}_t \|).$$

(3.38)

For the next term, by applying Lemma 2, we have

$$|T_3| \leq A (\mathbf{u}^n - R_h \mathbf{u}^n, e^n) \leq Ch \| \mathbf{u}^n \|_2 \| e^n \| \leq C h^2 \| \mathbf{u}^n \|_2^2 + \varepsilon \| e^n \|_1^2.$$  

(3.39)

Next, we make the following induction hypothesis:

$$\| e^n \| (\log \frac{1}{h})^{1/2} \rightarrow 0, \quad h \rightarrow 0, \quad 0 \leq s \leq n - 1, \quad n \leq N.$$  

(3.40)

For $T_4$, using the similar argument as (3.15) and noting (3.40), we deduce that

$$|T_4| \leq C \left( \| \nabla \cdot \mathbf{u}^n \|_\infty \| \mathbf{u}^n - \mathbf{u}^{n-1} \| \| e^n \| + \| \nabla \cdot \mathbf{u}^n - \mathbf{u}_h^n \| \| e^n \| + \| R_h \mathbf{u}^{n-1} \| \| e^n \| + \| \nabla \cdot \mathbf{u}^{n-1} \| \| e^n \| \right)$$

$$\leq C \left( \left\| \frac{\partial \mathbf{u}^n}{\partial t} \right\|_2 \left( \frac{1}{h} \right)^{1/2} + \| \mathbf{E}^{n-1} \| + \| e^{n-1} \| \right) \| \nabla \cdot \mathbf{u}^n \|_\infty \| e^n \|$$

$$+ C \left( \| \mathbf{E}^n \|_1 + \| e^n \| \right) \left( \log \frac{1}{h} \right)^{1/2} \| e^n \| \| \mathbf{u}^{n-1} \| \| e^n \|$$

$$\leq C \left( \frac{r^2 \max_{t \leq \tau} \| \mathbf{u}^n \|_2^2 + \| \mathbf{E}^{n-1} \|_1^2 + \| \mathbf{E}^{n-1} \|_1^2 + \| e^n \|_1^2 + 2\varepsilon \| e^n \|_1^2 \right).$$

(3.41)

Next, we write

$$|T_5| \leq \left| b_2 (\mathbf{u}^n, \mathbf{e}^n) - b_2 (\mathbf{u}^{n-1}, R_h \mathbf{u}^n, e^n) \right| + \left| b_{2h} (\mathbf{u}^n, \mathbf{e}^n) \right| = E_1 + E_2,$$  

(3.42)

where $E_1 = \left| b_2 (\mathbf{u}^n, \mathbf{e}^n) - b_2 (\mathbf{u}^{n-1}, R_h \mathbf{u}^n, e^n) \right|$ and $E_2 = \left| b_{2h} (\mathbf{u}^n, \mathbf{e}^n) \right|$. $E_1$ and $E_2$ can be handled as $D_1$ and $D_2$ in Theorem 6. Thus, we have

$$E_1 \leq C \| e^n \| \| u^n \|_\infty \left( h \| u^n \|_1 + \left( \| u^n - u^{n-1} \| + h \| u^n - u^{n-1} \| \right) \right)$$

$$\leq C \| e^n \| \| u^n \|_\infty \left( \| u^n - u^{n-1} \| + \| e^{n-1} \| + \left( \frac{r}{h} \right)^{1/2} \right)$$

$$\leq C \left( \| e^{n-1} \| + \| e^{n-1} \|_1 + \left( \frac{r}{h} \right)^{1/2} \right)$$

(3.43)
Then we have the source term functions

\[ E_2 \leq \frac{1}{2} \sum_{k \in \mathcal{T}_h} \sum_{i,j \in \mathcal{N}_k} \left| e^k(P_i) - e^k(P_j) \right| \times \left| H \left( \beta_{ij}(u_{h1}^{-1}) \right) e^k(P_i) + \left( 1 - H \left( \beta_{ij}(u_{h1}^{-1}) \right) \right) e^k(P_j) \right| \]

\[ \times \int_{\Gamma_{1, i,j}} \left| e^{n-1} \cdot n_i \right| ds + \frac{1}{2} \sum_{k \in \mathcal{T}_h} \sum_{i,j \in \mathcal{N}_k} \left| e^k(P_i) - e^k(P_j) \right| \times \left| H \left( \beta_{ij}(u_{h1}^{-1}) \right) e^k(P_i) + \left( 1 - H \left( \beta_{ij}(u_{h1}^{-1}) \right) \right) e^k(P_j) \right| \int_{\Gamma_{1, i,j}} \left| R_h u^{n-1} \cdot n_i \right| ds \] (3.44)

Substituting the previously mentioned estimates (3.36)-(3.44) into (3.35), we get

\[ \left\| e^n \right\|^2 \leq C \left( \left\| e^{n-1} \right\|^2 + \left\| e^n \right\|^2 + \frac{\tau^2 \max_{0<s<\tau} \left\| u_s \right\|^2 + h^2 \left\| u^n \right\|^2 + h^2 \max_{0<s<\tau} \left\| u_s \right\|^2 \right. \]

\[ + \left. ||E_{n-1}||^2 + \left\| E^n \right\|^2 + \left\| Q^n \right\|^2 \right) + 6\varepsilon \left\| e^n \right\|^2. \]

Taking \( \varepsilon = \mu/6 \), we have

\[ D_\tau \left\| e^n \right\|^2 \leq C \left( \left\| e^n \right\|^2 + \left\| e^{n-1} \right\|^2 \right) + O(\tau^2 + h^2). \]

4. Numerical Examples

In this section, we will show the numerical performances of proposed finite volume element method on the convergence and efficiency by several examples.

Example 1. In this example, the exact solutions of coupled Burgers’ equation are given by

\[ u(x, y, t) = r^{n+1} \sin(\pi x) \sin(\pi y), \quad v(x, y, t) = r^{n+1} \sin(\pi x) \cos(\pi y). \]

Then we have the source term functions

\[ f_1(x, y, t) = \Gamma (1 + \alpha) r^n \sin(\pi x) \sin(\pi y) + \pi r^{2n+2} \sin(\pi x) \cos(\pi y) \sin(\pi y) \]

\[ + \pi r^{2n+2} (\sin(\pi y))^2 \sin(\pi y) + 2\pi r^{2n+1} \sin(\pi y) \sin(\pi y), \]

and

\[ f_2(x, y, t) = \Gamma (1 + \alpha) r^n \sin(\pi x) \cos(\pi y) + \pi r^{2n+2} \sin(\pi x) \cos(\pi y) \sin(\pi y) \cos(\pi y) \]

\[ - \pi r^{2n+2} (\sin(\pi y))^2 \sin(\pi y) \cos(\pi y) + 2\pi r^{2n+1} \sin(\pi y) \cos(\pi y) \]

by computing accordingly. We take the viscosity \( \nu = 1 \), the spatial interval \( \Omega = (0, 1) \times (0, 1) \) and the time interval \( [0, T] = [0, 1] \). In all runs, we use the uniform mesh \( h = \tau \) and choose the time \( t = 1 \). Table 1 shows the numerical results of \( \|u - u_h\| \) and \( \|v - v_h\| \) for \( \alpha = 0.3 \) and \( \alpha = 0.7 \). As seen in the table, the errors decrease by a factor of about two as
$h$ decreases by the factor of two. This indicates that all $L^2$-norm error estimates are of first-order convergence, which is consistent with our theoretical analysis. Surface plots of the exact solution $u$ and the numerical solution $u_h$ for $\alpha = 0.3$ are given in Figure 1 and Figure 2. Figure 3 shows the error between the exact and numerical solutions. As can be seen, there is a good agreement between the exact solution and numerical solution.

Table 1. Numerical results for $\alpha = 0.3$ and $\alpha = 0.7$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u - u_h|$</th>
<th>order</th>
<th>$|v - v_h|$</th>
<th>order</th>
<th>$|u - u_h|$</th>
<th>order</th>
<th>$|v - v_h|$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{16}$</td>
<td>7.1904e-03</td>
<td>-</td>
<td>2.7828e-03</td>
<td>-</td>
<td>6.0983e-03</td>
<td>-</td>
<td>2.9354e-03</td>
<td>-</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>3.1573e-03</td>
<td>1.18</td>
<td>1.1943e-03</td>
<td>1.22</td>
<td>2.6444e-03</td>
<td>1.21</td>
<td>1.2921e-03</td>
<td>1.18</td>
</tr>
<tr>
<td>$\frac{1}{64}$</td>
<td>1.4709e-03</td>
<td>1.10</td>
<td>5.5057e-04</td>
<td>1.11</td>
<td>1.2289e-03</td>
<td>1.11</td>
<td>6.0639e-04</td>
<td>1.09</td>
</tr>
<tr>
<td>$\frac{1}{128}$</td>
<td>7.0889e-04</td>
<td>1.05</td>
<td>2.6416e-04</td>
<td>1.06</td>
<td>5.9325e-04</td>
<td>1.05</td>
<td>2.9432e-04</td>
<td>1.04</td>
</tr>
</tbody>
</table>

Example 2. We consider the following time fractional Burgers’ equation:

$$
\begin{align*}
\mathcal{C}_0^\alpha &\partial_t^\alpha u + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \nu \Delta u = 0, \quad (x, y) \in \Omega, \ t \in (0, T] \\
\mathcal{C}_0^\alpha &\partial_t^\alpha v + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \nu \Delta v = 0, \quad (x, y) \in \Omega \times (0, T], \\
u(x, 0) = \sin(\pi x)\sin(\pi y), & \quad \nu(x, 0) = xy(x - 1)(y - 1), \\
u = 0, & \quad v = 0, \quad (x, t) \in \partial \Omega \times (0, T].
\end{align*}
$$

Figure 1. The exact solution $u$ for $t = 1$, $\nu = 1$ and $\alpha = 0.3$

Figure 2. The numerical solution $u_h$ for $t = 1$, $\nu = 1$, $h = 1/32$ and $\alpha = 0.3$

Figure 3. The error $u - u_h$ for $t = 1$, $\nu = 1$, $h = 1/64$ and $\alpha = 0.3$
where $\Omega = (0, 1) \times (0, 1)$, $T = 1$ and $\nu = 0.01$. In this example, the exact solution is not given beforehand, so we take the numerical solution with $h = \frac{1}{150}$ as the reference solution to calculate the error and convergence rate. In Table 2, we present the numerical results for $\alpha = 0.2$ and $\alpha = 0.8$. It can be seen in the table that the format is convergent with one-order accuracy. Surface plots of the reference solution $u$ and numerical solution $u_h$ for $\alpha = 0.8$ are given in Figure 4 and Figure 5. As can be seen from the table and figure, it is consistent with the results of our theoretical analysis.

**Table 2. Numerical results for $\alpha = 0.2$ and $\alpha = 0.8$**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$h$</th>
<th>$|u - u_h|$</th>
<th>order</th>
<th>$|v - v_h|$</th>
<th>order</th>
<th>$|u - u_h|$</th>
<th>order</th>
<th>$|v - v_h|$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.2$</td>
<td>$\frac{1}{150}$</td>
<td>$9.1067 \times 10^{-3}$</td>
<td>-</td>
<td>$5.9229 \times 10^{-4}$</td>
<td>-</td>
<td>$8.7381 \times 10^{-4}$</td>
<td>-</td>
<td>$4.9997 \times 10^{-5}$</td>
<td>-</td>
</tr>
<tr>
<td>$\alpha = 0.8$</td>
<td>$\frac{1}{150}$</td>
<td>$7.5809 \times 10^{-3}$</td>
<td>1.01</td>
<td>$4.9387 \times 10^{-4}$</td>
<td>0.97</td>
<td>$7.0824 \times 10^{-4}$</td>
<td>1.15</td>
<td>$4.0657 \times 10^{-5}$</td>
<td>1.13</td>
</tr>
<tr>
<td>$\alpha = 0.2$</td>
<td>$\frac{1}{250}$</td>
<td>$4.1389 \times 10^{-3}$</td>
<td>1.18</td>
<td>$2.7062 \times 10^{-4}$</td>
<td>1.18</td>
<td>$3.6608 \times 10^{-4}$</td>
<td>1.29</td>
<td>$2.1167 \times 10^{-5}$</td>
<td>1.28</td>
</tr>
<tr>
<td>$\alpha = 0.8$</td>
<td>$\frac{1}{250}$</td>
<td>$2.1722 \times 10^{-3}$</td>
<td>1.59</td>
<td>$1.4230 \times 10^{-4}$</td>
<td>1.58</td>
<td>$1.8710 \times 10^{-4}$</td>
<td>1.66</td>
<td>$1.0862 \times 10^{-5}$</td>
<td>1.65</td>
</tr>
</tbody>
</table>

**Figure 4.** The reference solution $u$ for $t = 1$, $\nu = 0.01$, $h = 1/150$ and $\alpha = 0.8$  

**Figure 5.** The numerical solution $u_h$ for $t = 1$, $\nu = 0.01$, $h = 1/250$ and $\alpha = 0.8$

**Example 3.** We consider the following time fractional Burgers’ equation:

\[
\begin{align*}
\frac{\partial}{\partial t}u + \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} - \nu \Delta u &= 0, & (x, y) &\in \Omega, t \in (0, T], \\
\frac{\partial}{\partial t}v + \frac{\partial u}{\partial y} + \nu \frac{\partial v}{\partial x} - \nu \Delta v &= 0, & (x, y) &\in \Omega \times (0, T], \\
u(x, 0) &= 0, & v(x, 0) &= 0,
\end{align*}
\]

where $\Omega = (0, 1) \times (0, 1)$, $T = 1$ and $\nu = 0.001$, and the boundary value functions $t \in (0, T]$

\[
u(x, y, t) = \begin{cases} 
0, & y = 1, 0 < x < 0.3 \\
1, & y = 1, 0.3 \leq x < 1 \\
0, & x = 0 \text{ or } x = 1 \text{ or } y = 0
\end{cases}
\]

\[
u(x, y, t) = \begin{cases} 
0, & x = 0 \text{ or } y < 0.3 \\
1, & x = 0, 0.3 \leq y < 1 \\
0, & x = 1 \text{ or } y = 0 \text{ or } y = 1
\end{cases}
\]

In order to show that our method keeps stable when $\nu$ is smaller, we give the the numerical solution by FVEM with and without upwinding technique for $\nu = 0.001$ in Figure 6 and Figure 7. We can see that our method can perform well but the approximation without upwinding technique produces nonphysical oscillations. These are seen much more clearly in Figure 8 and Figure 9, which shows that the oscillations will increase further when $\nu$ is smaller and smaller. In contrast, our method still keeps stable.
Figure 6. The numerical solution $u_h$ with upwinding for $t = 1$, $\nu = 0.001$, $h = 1/32$ and $\alpha = 0.3$

Figure 7. The numerical solution $\tilde{u}_h$ without upwinding for $t = 1$, $\nu = 0.001$, $h = 1/32$ and $\alpha = 0.3$

Figure 8. The numerical solution $u_h$ with upwinding for $t = 1$, $\nu = 0.0002$, $h = 1/32$ and $\alpha = 0.3$

Figure 9. The numerical solution $\tilde{u}_h$ without upwinding for $t = 1$, $\nu = 0.0002$, $h = 1/32$ and $\alpha = 0.3$

5. Conclusions

In this article, a upwind finite volume element scheme has been derived for the two-dimensional time fractional coupled Burgers’ equation. We discrete the equation using L1 formula in the time direction. We then discrete the resulting equations in space domain using upwind finite volume element method. The convergence analyses of semi-discrete scheme and fully discrete scheme are proved. Then, the numerical experiments are given to verify the effectiveness of the proposed scheme. And our method keeps stable when $\nu$ is smaller by contrast with finite volume element method without upwinding technique. Due to the influence of the linear scheme and the upwinding technique, the convergence order of the scheme is $O(\tau + h)$. In future works, we will use this method to solve more nonlinear fractional partial differential equations, such as nonlinear convection-diffusion equation and Navier-Stokes equation. And for the Caputo fractional derivative with $\alpha \in (0, 1)$, we will try to use other approximation methods (such as L1-2, L2-1, $\sigma$ formulas) and high-precision upwinding technique to improve the convergence order.

References


**Acknowledgements**

The authors would like to thank the editor and all the anonymous referees for their valuable comments which greatly improved the presentation of this article. We greatly appreciate the support of the Shandong normal University as well as the National Science Foundation of China (12172202).

**Authors contributions**

Mr. Lv was responsible for creating the mathematical models and drafting the manuscript. Dr. Yang was responsible for reviewing the research questions and revising the manuscript. Both authors approved the final manuscript.

**Funding**

This work is supported by National Science Foundation of China (12172202).

**Competing interests**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Informed Consent**

Obtained.

**Ethics approval**

The Publication Ethics Committee of the Canadian Center of Science and Education.

The journals policies adhere to the Core Practices established by the Committee on Publication Ethics (COPE).

**Provenance and peer review**

Not commissioned; externally double-blind peer reviewed.

**Data availability statement**

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

**Data sharing statement**

No additional data are available.

**Open access**

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

**Copyrights**

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.