Boundary Null Controllability With Constrained Control for a Nonlinear Two Stroke System: Application to Boundary Sentinel and Identification of Parameters in a Nonlinear Population Dynamics Model

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Abstract

We first prove a new controllability result for a nonlinear two stroke system. The key to solve this controllability problem is an adapted Carleman inequality. Next, the obtained result is used to build a boundary sentinel to identify unknown parameters in a nonlinear population dynamics model with incomplete data.

Keywords: population dynamics, optimal control, controllability, sentinel, Carleman inequality

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$ an open and bounded subset, with $\partial\Omega = \Gamma$ of class C^∞. For the time $T > 0$ and $A > 0$,we consider $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$, $\Sigma = U \times \Gamma$, $\Sigma_1 = U \times \Gamma_1$, where $\Gamma_1 \subset \Gamma$ is open and nonempty. Let O and γ be some nonempty open subsets of $\Gamma\backslash\Gamma_1$ with $O \cap \gamma \neq \emptyset$. We consider the following nonlinear two stroke system:

$$\left\{\begin{array}{rcccl} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q & = & \beta q(t,0,x) G\left(\int_0^A \beta q da\right) & \text{in} & Q, \\ q & = & h_0\chi_O + (w_0 - v)\chi_\gamma & \text{on} & \Sigma, \\ q(T,a,x) & = & 0 & \text{in} & Q_A, \\ q(t,A,x) & = & 0 & \text{in} & Q_T, \end{array}\right. \tag{1}$$

where G is a given reel function, $h_0 \in L^2(U \times O)$, $(w_0 - v) \in L^2(U \times \gamma)$, χ_O and χ_γ are respectively the characteristics functions of O and γ. The functions $\beta(t, a, x) \geq 0$ and $\mu(t, a, x) \geq 0$ are specific for a given species.

Assume as in (Giovanna & Langlais,1982) that the following assumptions hold:

$$
(H1) \quad : \quad \begin{cases} \beta \in L^\infty(Q), \quad \beta(t,a,x) \geq 0 \text{ a.e. in } Q, \\ \sup_{(t,x)\in]0,T[\times\Omega} \int_{]0,A[} (|\beta(t,a,x)|^2 + |\nabla\beta(t,a,x)|^2 da), \\ \\ \exists\, \delta \in (0,\, A)\; s.t.\, \beta(a,.,) = 0 \text{ for a} \in (\delta,\, A), \end{cases}
$$

$$
(H2) \quad : \quad \mu \in C([0,\, T] \times [0,\, A] \times \bar{\Omega}),\ \mu(t,a,x) \geq 0 \text{ a.e} \quad \text{in} \quad Q,
$$

$$
(H3) \quad : \quad \begin{cases} \forall t,\ 0 < t < A, \quad \forall\, x \in \Omega,\ \lim_{a \longrightarrow A} \int_0^A \mu(\iota, a - t + \iota, x) d\iota = +\infty \quad , \\ \forall t,\ A < t < T, \quad \forall\, x \in \Omega,\ \lim_{a \longrightarrow A} \int_0^a \mu(t - a + \alpha, \alpha, x) d\alpha = +\infty \ , \\ \nabla\mu \in [L^\infty(Q)]^n, \\ \end{cases}
$$

$$
(H4) \quad : \quad G \in L^\infty(\mathbb{R}) \quad \text{and} \quad G \in C^1(\mathbb{R}).
$$

Since $h_0\chi_O + (w_0 - v)\chi_\gamma \in L^2(\Sigma)$, using the assumptions $(H1) - (H4)$ we have existence and uniqueness of a solution of problem (1) in $L^2(Q)$ (see Giovanna & Langlais,1982).

Assume that the following assumption hold : if ρ verifies

$$
\begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a} - \Delta\rho + \mu\rho & = & 0 & \text{in} & Q, \\ \rho & = & 0 & \text{in} & U \times \gamma, \\ \frac{\partial\rho}{\partial\nu} & = & 0 & \text{on} & U \times \gamma, \end{cases} \tag{2}
$$

then $\rho \equiv 0$ in Q.

For the sequel we need the following result:

Lemma 1 *Let $e_i \in L^2(\Sigma)$, $i \in \{1, \cdots, M\}$ such that $e_i\chi_{\Sigma_1}$, $i \in \{1, \cdots, M\}$ are linearly independent. Let $b \in L^\infty((0,T) \times \Omega)$ and θ a real positive function (θ will be defined later on by (13)). Let us consider the following system:*

$$
\begin{cases} \frac{\partial y_i}{\partial t} + \frac{\partial y_i}{\partial a} - \triangle y_i + \mu y_i & = & 0 & \text{in} & Q, \\ y_i(0,a,x) & = & 0 & \text{in} & Q_A, \\ y_i(t,0,x) & = & \int_0^A \beta y_i b(t,x) da & \text{in} & Q_T, \\ y_i & = & e_i\chi_{\Sigma_1} & \text{on} & \Sigma. \end{cases} \tag{3}
$$

Then the families of functions $\{\frac{\partial y_i}{\partial\nu}\chi_\gamma\}_{1\leq i\leq M}$ and $\{\frac{1}{\theta}\frac{\partial y_i}{\partial\nu}\chi_\gamma\}_{1\leq i\leq M}$ are linearly independent.

Proof Let $\alpha_i \in \mathbb{R}$, $1 \leq i \leq M$ be such that $\sum_{i=1}^M \alpha_i \frac{\partial y_i}{\partial\nu}\chi_\gamma = 0$. Set $k = \sum_{i=1}^M \alpha_i y_i$, using (3), k is solution of

$$
\begin{cases} \frac{\partial k}{\partial t} + \frac{\partial k}{\partial a} - \Delta k + \mu k & = & 0 & \text{in} & Q, \\ k(0,a,x) & = & 0 & \text{in} & Q_A, \\ k(t,0,x) & = & \int_0^A \beta k b(t,x) da & \text{in} & Q_T, \\ k & = & \sum_{i=1}^M \alpha_i e_i.\chi_{\Sigma_1} & \text{on} & \Sigma, \\ \frac{\partial k}{\partial\nu} & = & 0 & \text{on} & U \times \gamma. \end{cases} \tag{4}
$$

Assumption (2) allows us to say that k is identically zero. Therefore, we deduce that

$$
\sum_{i=1}^M \alpha_i e_i.\chi_{\Sigma_1} = 0 \quad \text{on} \ \ \Sigma. \tag{5}
$$

Since $e_i.\chi_{\Sigma_1}, 1 \leq i \leq M$ are linearly independent, we have $\alpha_i = 0$ for $1 \leq i \leq M$ and this implies that the family of functions $\{\frac{\partial y_i}{\partial \nu}\chi_\gamma\}_{1\leq i\leq M}$ is linearly independent, So is the family $\{\frac{1}{\theta}\frac{\partial y_i}{\partial \nu}\chi_\gamma\}_{1\leq i\leq M}$.

From now we set

$$K = \text{Span}\{\frac{\partial y_1}{\partial \nu}\chi_\gamma, ..., \frac{\partial y_M}{\partial \nu}\chi_\gamma\} \subset L^2(U \times \gamma), \tag{6}$$

and

$$K_\theta = \frac{1}{\theta}K \subset L^2(U \times \gamma).$$

Now we consider the following problem: *let $h_0 \in L^2(U \times O)$, $w_0 \in K_\theta$. Find $v \in L^2(U \times \gamma)$ such that*

$$v \in K^\perp, \tag{7}$$

and if $q = q(t, a, x; v)$ is a solution of

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \Delta q + \mu q &= \beta q(t, 0, x) G\left(\int_0^A \beta q da\right) & \text{in} \quad Q, \\ q &= h_0\chi_O + (w_0 - v)\chi_\gamma & \text{on} \quad \Sigma, \\ q(T, a, x) &= 0 & \text{in} \quad Q_A, \\ q(t, A, x) &= 0 & \text{in} \quad Q_T, \end{cases} \tag{8}$$

then q satisfies

$$q(0, a, x; v) = 0 \qquad \text{in} \quad Q_A. \tag{9}$$

The problem (7)-(9) is a null boundary controllability problem with constraint on the control.Controllability problems for two stroke system have been studied by several authors. For instance, Ainseba, B. and Langlais, M. proved that the set of profiles is approximatively reachable (Ainseba & Langlais,1996). It has been shown that in (Ainseba & Anita, 2001) if the initial distribution is small enough, we can find a control which leads to extinction of the population. The result was achieved by means of Carleman inequality for parabolic equation. Exact and approximate controllability results are obtained for a linear population dynamics problem structured in age and space by Ainseba, B. (Ainseba, 2002). Concerning the nonlinear population dynamics model, a null controllability result was established by Ainseba, B. and Iannelli, M. by means of Kakutani fixed point theorem (Ainseba & Innanelli, 2003). Using a derivation of Leray-Schauder fixed point theorem and Carleman inequality for the adjoint system,Traoré, O. showed that for all given initial density, there exists an internal control acting on a small open set of the domain and leading the population to extension in (Traoré, 2006). Sawadogo, S. and Mophou, G. (Sawadogo & Mophou, 2012) gave a null controllability result for population dynamics model with constraints on the state when the age of the population belongs to (γ, A) for any $\gamma > 0$. Following this work, Mercan, M. and Mophou, G. (Mercan & Mophou, 2014) proved a null controllability problem with constraints on the state for an adjoint system of population dynamics model. The result was achieved by means of Carleman inequality adapted to the constraints. Simporé, Y. and Traoré, O. solve the internal null controllability without constraint on the control for a nonlinear dissipative system (Simporé & Traoré, 2016). They use their result to detect incomplete parameter in a nonlinear population dynamics model. Soma, M. and Sawadogo, S. build a boundary sentinel with given sensitivity in order to identify unknown parameters in the linear population dynamics model with incomplete data (Soma & Sawadogo, 2019).

In this paper we solve the nonlinear null boundary controllability problem with constraint on the control (7)-(9) which seems new to us and which we will apply in the construction of a boundary sentinel in the nonlinear case. More precisely, we have the following result:

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ an open subset with $\partial\Omega = \Gamma$ of class C^∞. Let $\Gamma_1 \subset \Gamma$ a nonempty open subset. Let also O and γ be two nonempty open subsets of $\Gamma \setminus \Gamma_1$ such that $O \cap \gamma \neq \emptyset$ and G a real function verifying (H_4). Assume that the assumption (2) hold. For any function $h_0 \in L^2(U \times O)$ with $\theta h_0 \in L^2(U \times O)$ (θ will be given later on by (13)), there exists a unique control $\hat{v} \in \mathrm{K}^\perp$ such that the pair $(\hat{v}, \hat{q})$ with $\hat{q} = q(\hat{v})$ is a solution of the problem (7)-(9).*

2. Resolution of the Problem (7)-(9)

2.1 An Adapted Carleman Inequality

There exists a function $\psi \in C^2(\overline{\Omega})$ such that:

$$\begin{array}{rcl} \psi(x) & > & 0, \forall x \in \Omega, \\ \nabla\psi & > & \alpha, \forall x \in \overline{\Omega}, \\ \psi(x) & = & 0, \forall x \in \Gamma \setminus \gamma, \\ \dfrac{\partial\psi}{\partial\nu} & < & 0, \forall x \in \Gamma \setminus \gamma, \end{array} \tag{10}$$

(see Fursikov & Imanuvilov, 1996). Consider the functions :

$$\varphi(t,a,x) = \frac{e^{\lambda(m|\psi|_\infty + \psi(x))}}{at(A-a)(T-t)}, \tag{11}$$

$$\eta(t,a,x) = \frac{e^{2\lambda m|\psi|_\infty} - e^{\lambda(m|\psi|_\infty + \psi(x))}}{at(A-a)(T-t)}, \tag{12}$$

with $m \geq 1$ and $\lambda > 0$. Since $\varphi \neq 0$ in Q, we set

$$\frac{1}{\theta^2} = \min\left[e^{-2s\eta}\left(\varphi^{-1}, \varphi, \varphi^3, \varphi\left|\frac{\partial\psi}{\partial\nu}\right|\right)\right], \tag{13}$$

with $s > 0$ and we consider the notations below :

$$\left\{\begin{array}{rcl} L & = & \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - \Delta + \mu I, \\ L^* & = & -\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial a} - \Delta + \mu I, \\ \mathcal{V} & = & \left\{\rho \in C^\infty\left(\overline{Q}\right), \rho = 0 \text{ on } \Sigma\right\}. \end{array}\right. \tag{14}$$

Using the notations given by (14) and the definition given by (11)-(12), we have the following boundary Carleman inequality:

Proposition 1 *Let ψ, φ and η be respectively defined by (10)-(12). Then, there exists numbers $\lambda_0 = \lambda_0(\gamma,\mu) > 1, s_0 = s_0(\gamma,\mu,T) > 1, C_0 = C_0(\gamma,\mu) > 0$ and $C_1 = C_1(\gamma,\mu) > 0$ such that for any $\lambda \geq \lambda_0$, for any $s \geq s_0$, for any $\rho \in \mathcal{V}$, the following inequality holds :*

$$\begin{aligned} &\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(|\rho_t + \rho_a|^2 + |\Delta\rho|^2\right) dadtdx + \int_Q e^{-2s\eta}\left(s\lambda^2\varphi|\nabla\rho|^2 + s^3\lambda^4\varphi^3|\rho|^2\right) dtdadx \\ &+C_0 \int_0^T \int_0^A \int_{\Gamma\setminus\gamma} se^{-2s\eta}\varphi\left(-\frac{\partial\psi}{\partial\nu}\right)|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma \\ &\leq C_1\left[\int_Q e^{-2\eta}|L\rho|^2 dtdadx + \int_0^T \int_0^A \int_\gamma se^{-2\eta}\varphi|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma\right]. \end{aligned} \tag{15}$$

Proof. See (Nakoulima & Sawadogo, 2007)

As ψ belong to $C^2(\overline{\Omega})$ and $\varphi e^{-2s\eta}$ is bounded, then $\frac{1}{\theta}$ is also bounded in Q. Hence, from Proposition 1, we have this other inequality :

Proposition 2 *Let θ be defined by (13). Then, there exists numbers $\lambda_0 = \lambda_0(\Omega, \gamma, \mu) > 1$, $s_0 = s_0(\Omega, \gamma, \mu, T) > 1$, $C_0 = C_0(\Omega, \gamma, \mu) > 0$, and $C_1 = C_1(\Omega, \gamma, \mu) > 0$ such that, for any $\lambda \geq \lambda_0$, for any $s \geq s_0$, and for any $\rho \in \nu$,*

$$\begin{aligned} &\int_U \int_\Omega \tfrac{1}{\theta^2}\left(|\frac{\partial\rho}{\partial\nu}|^2 + |\Delta\rho|^2 + |\nabla\rho|^2 + |\rho|^2\right) dtdad\Gamma + C_0 \int_U \int_\Gamma \tfrac{1}{\theta^2}|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma \\ &\leq C_1\left[\int_Q |L\rho|^2 dtdadx + \int_U \int_\gamma |\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma\right]. \end{aligned} \tag{16}$$

To prove the adapted Carleman inequality, we need the following Lemma

Lemma 2 *Assume that the assumptions of Lemma 1 hold. Let* K *be the real vector subspace of* $L^2(U \times \gamma)$ *of finite dimensional defined in (6). Then any function* ρ *such that*

$$\begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial a} - \triangle\rho + \mu\rho &= 0 \quad \text{in} \quad Q \\ \rho(0,.,.) &= 0 \quad \text{in} \quad Q_A \\ \rho &= 0 \quad \text{on} \quad \Sigma \setminus \Sigma_1 \\ \frac{\partial\rho}{\partial\nu}|_\gamma \in \mathrm{K} \end{cases} \tag{17}$$

is identically zero.

Proof. For any ρ verifying (17), there exists $\alpha_i \in \mathbb{R}$, $1 \leq i \leq M$, such that $\frac{\partial\rho}{\partial\nu} = \sum_{i=1}^{M} \alpha_i \frac{\partial y_i}{\partial\nu}$. We set $z = \rho - \sum_{i=1}^{M} \alpha_i y_i$. Using (3), we have

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - \Delta z + \mu z &= 0 \quad \text{in} \quad Q, \\ z(0,.,.) &= 0 \quad \text{in} \quad Q_A, \\ z &= 0 \quad \text{on} \quad \Sigma \setminus \Sigma_1, \\ \frac{\partial z}{\partial\nu} &= 0 \quad \text{on} \quad U \times \gamma. \end{cases} \tag{18}$$

Since $\gamma \subset \Gamma \setminus \Gamma_1$, we have $z = 0$ and $\frac{\partial z}{\partial\nu} = 0$ in $U \times \gamma$. Then it follows from assumption (2) that $z = 0$ in Q. Consequently, $\rho = \sum_{i=1}^{M} \alpha_i y_i$ and $\sum_{i=1}^{M} \alpha_i e_i = 0$ on Σ_1. Hence, it follows from the assumptions of Lemma 1 that $\alpha_i = 0$ for $1 \leq i \leq M$. Thus, $\rho = 0$ in Q.

Proposition 3 [Adapted Carleman inequality] *Under the Assumption of Lemma 1 . Let* K *defined in (6) and* P *be the orthogonal projection operator from* $L^2(U \times \gamma)$ *into* K*. Then, there exists some numbers* $\lambda_0 = \lambda_0(\Omega, \gamma, \mu) > 1$, $s_0 = s_0(\Omega, \gamma, \mu, T) > 1$, $C_0 = C_0(\Omega, \gamma, \mu) > 0$ *and* $C_1 = C_1(\Omega, \gamma, \mu) > 0$ *such that, for any* $\lambda \geq \lambda_0$*, for any* $s \geq s_0$*, and for any* $\rho \in \mathcal{V}$,

$$\int_U \int_\Gamma \frac{1}{\theta^2} \left|\frac{\partial\rho}{\partial\nu}\right|^2 dtdad\Gamma \leq C_1 \left[\int_Q \left|L\rho\right|^2 dtdadx + \int_U \int_\gamma \left|P\frac{\partial\rho}{\partial\nu} - \frac{\partial\rho}{\partial\nu}\right|^2 dtdad\Gamma\right]. \tag{19}$$

Proof. As in (Nakoulima & Sawadogo, 2007) and (Mophou & Nakoulima,2008), we use a well known compactness-uniqueness argument and the inequality (15). Indeed, suppose that (19) does not hold. Then for any $j \in \mathbb{N}$, there exists $\rho_j \in \nu$ such that

$$\int_U \int_\Omega \left|L\rho_j\right|^2 dtdadx \leq \frac{1}{j}, \tag{20}$$

$$\int_U \int_\gamma \left|P\frac{\partial\rho_j}{\partial\nu} - \frac{\partial\rho_j}{\partial\nu}\chi_\gamma\right|^2 dtdad\Gamma \leq \frac{1}{j}, \tag{21}$$

$$\int_U \int_\Gamma \frac{1}{\theta^2} \left|\frac{\partial\rho_j}{\partial\nu}\right|^2 dtdad\Gamma = 1. \tag{22}$$

In what follows, we prove in three steps that (20)-(22) yields contradiction.

Step 1. We have

$$\begin{aligned} \int_U \int_\gamma \frac{1}{\theta^2} \left|P\frac{\partial\rho_j}{\partial\nu}\right|^2 dtdad\Gamma &\leq 2 \int_U \int_\gamma \frac{1}{\theta^2} \left|P\frac{\partial\rho_j}{\partial\nu}\right|^2 dtdad\Gamma \\ &+ 2 \int_U \int_\gamma \frac{1}{\theta^2} \left|P\frac{\partial\rho_j}{\partial\nu} - \frac{\partial\rho_j}{\partial\nu}\chi_\gamma\right|^2 dtdad\Gamma. \end{aligned} \tag{23}$$

Since $\frac{1}{\theta^2}$ is bounded, using (20) and (21), it follows that there exists a positive constant C such that

$$\forall j \in \mathbb{N}, \quad \int_U \int_\gamma \left| P \frac{\partial \rho_j}{\partial \nu} \right|^2 dt da d\Gamma \leq C. \tag{24}$$

As $\frac{\partial \rho_j}{\partial \nu}\chi_\gamma = P\frac{\partial \rho_j}{\partial \nu}\chi_\gamma + (\frac{\partial \rho_j}{\partial \nu}\chi_\gamma - P\frac{\partial \rho_j}{\partial \nu}\chi_\gamma)$, using (22) and (24),we obtain

$$\left\| \frac{\partial \rho_j}{\partial \nu} \right\|^2_{L^2(U\times\gamma)} \leq C. \tag{25}$$

Step 2. Let $L^2\left(\frac{1}{\theta}, U\times\gamma\right) = \left\{\rho \in L^2(U\times\Omega); \int_U \int_\Gamma \frac{1}{\theta^2} \left|\frac{\partial \rho}{\partial \nu}\right|^2 dt da d\Gamma < \infty\right\}.$

Then in view of (22) and (25), we deduce from (16) that , $\left(\frac{\partial \rho_j}{\partial t} + \frac{\partial \rho_j}{\partial a}\right)$, $(\frac{\partial \rho_j}{\partial \nu})$, $(\nabla \rho_j)$, (ρ_j) and $(\Delta \rho_j)$ are bounded in $L^2\left(\frac{1}{\theta}, U\times\gamma\right)$. Let us take a subsequence still denoted by (ρ_j) such that

$$\rho_j \rightharpoonup \rho \quad \textit{weakly in } L^2\left(\frac{1}{\theta}, U\times\gamma\right), \tag{26}$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \textit{weakly in } L^2\left(\frac{1}{\theta}, U\times\gamma\right). \tag{27}$$

Then it follows from (10)-(12) and the definition of $\frac{1}{\theta}$ given by (13) that (ρ_j) and $(\Delta\rho_j)$ are bounded in $L^2(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Omega)$ for any $\beta > 0$ and any $\alpha > 0$. In particular, for all $\beta > 0$ and any $\alpha > 0$, we have

$$\rho_j \rightharpoonup \rho \quad \textit{weakly in } L^2\left(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Omega\right)$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \textit{weakly in } L^2\left(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Sigma\right),$$

which implies that

$$\rho_j \rightharpoonup \rho \quad \textit{weakly in } D'(Q)$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \textit{weakly in } D'(\Sigma).$$

Therefore, we get from (26)-(27) that

$$L\rho_j \longrightarrow L\rho = 0 \quad \text{strongly in } L^2(U\times\Omega), \tag{28}$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \text{strongly in } L^2(U\times\gamma). \tag{29}$$

And, since P is a compact operator, we deduce from (29)that

$$P\frac{\partial \rho_j}{\partial \nu} \longrightarrow P\frac{\partial \rho}{\partial \nu} \quad \text{strongly in } L^2(U\times\gamma). \tag{30}$$

In view of (21), we also have

$$\frac{\partial \rho_j}{\partial \nu} - P\frac{\partial \rho_j}{\partial \nu} \longrightarrow 0 \quad \text{strongly in } L^2(U\times\gamma). \tag{31}$$

Thus combining (30) and (31), we get

$$P\frac{\partial \rho_j}{\partial \nu} \longrightarrow \frac{\partial \rho_j}{\partial \nu} \quad \text{strongly in } L^2(U\times\gamma). \tag{32}$$

The uniqueness of the limit in $L^2(U\times\gamma)$, the relations (30)-(31) and (32) imply that $P\frac{\partial \rho}{\partial \nu} = \frac{\partial \rho}{\partial \nu}\chi_\gamma$. This means that $\frac{\partial \rho}{\partial \nu}\chi_\gamma \in Y$. We thus have proved that ρ verifies (17). Hence thanks to Lemma 2, ρ is identically zero. Therefore, (30) becomes

$$\frac{\partial \rho_j}{\partial \nu} \longrightarrow 0 \quad \text{strongly in } L^2(U\times\gamma). \tag{33}$$

Step 3. Since $\rho_j \in \mathcal{V}$, it follows from the observability inequality (16) that

$$\int_U \int_\Gamma \frac{1}{\theta^2} \left|\frac{\partial \rho_j}{\partial \nu}\right|^2 dtdad\Gamma \leq C_1 \left[\int_Q \left|L\rho_j\right|^2 dtdadx + \int_U \int_\gamma \left|\frac{\partial \rho_j}{\partial \nu}\right|^2 dtdad\Gamma\right].$$

Therefore passing this latter inequality to the limit while using (28) and (33), we obtain

$$\lim_{j\longrightarrow\infty} \int_U \int_\Gamma \left|\frac{\partial \rho_j}{\partial \nu}\right|^2 dtdadx = 0.$$

The contradiction occurs with (22).

2.2 Resolution of an Intermediary Problem

In this subsection, we consider the following problem : *given $h_0 \in L^2(U \times O)$, $w_0 \in K_\theta$, find $v \in L^2(U \times \gamma)$ such that*

$$v \in K^\perp, \tag{34}$$

and if $q = q(t, a, x; v)$ is a solution of

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q = \beta q(t,0,x)b(t,x) & \text{in } Q, \\ q = h_0\chi_O + (w_0 - v)\chi_\gamma & \text{on } \Sigma, \\ q(T,a,x) = 0 & \text{in } Q_A, \\ q(t,A,x) = 0 & \text{in } Q_T, \end{cases} \tag{35}$$

then q satisfies

$$q(0, a, x; v) = 0 \qquad \text{in } Q_A. \tag{36}$$

The following result hold:

Theorem 4 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary Γ of class C^∞. Let Γ_1 be a nonempty open subset of Γ. Let also O and γ be two nonempty open subsets of $\Gamma \setminus \Gamma_1$ such that $O \cap \gamma \neq \emptyset$. Let $b \in L^\infty((0,T) \times \Omega)$. Then there exists a positive real weight function θ a precise definition of θ given by (13) such that, for any function $h_0 \in L^2(U \times O)$ with $\theta h_0 \in L^2(U \times O)$, there exists a unique control $\hat{v} \in K^\perp$ such that the pair $(\hat{v}, \hat{q})$ with $\hat{q} = q(\hat{v})$ is solution of the null boundary controllability problem with constraint on the control (34)-(36). Moreover, the control $\hat{v}$ is given by*

$$\hat{v} = (I - P)(w_0\chi_\gamma - \frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma),$$

where P is the orthogonal projection operator from $L^2(U \times \gamma)$ into K and $\hat{\rho}$ satisfies

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial t} + \frac{\partial \hat{\rho}}{\partial a} - \Delta \hat{\rho} + \mu \hat{\rho} = 0 & \text{in } Q, \\ \hat{\rho}(t,0,x) = b(t,x)\int_0^A \beta \hat{\rho} da & \text{in } Q_T, \\ \hat{\rho} = 0 & \text{on } \Sigma. \end{cases}$$

Proof. We use a penalization argument which will be divided in three steps.

Step 1. Let $w_0 \in K_\theta$. If $v \in K^\perp$ and if q is a solution of (35) then $q(0, ., .) \in L^2(Q_A)$ and we define the functional

$$J_\epsilon(v) = \frac{1}{2}\|w_0 - v\|^2_{L^2(U\times\gamma)} + \frac{1}{2\epsilon}\|q(0, ., .)\|^2_{L^2(Q_A)}. \tag{37}$$

We consider the optimal control problem: Find $v_\epsilon \in K^\perp$ such that

$$J_\epsilon(v_\epsilon) = \min_{v\in K^\perp} J_\epsilon(v) \ . \tag{38}$$

Since $K^\perp$ is a closed and convex subset of $L^2(U \times \gamma)$, it is classical to prove existence and uniqueness of the solution of (38). If we write q_ϵ the solution of (35) corresponding to v_ϵ using an adjoint state ρ_ϵ , we have that the triplet $(q_\epsilon, \rho_\epsilon, v_\epsilon)$

is solution of the first order optimality system:

$$\begin{cases} L^* q_\epsilon &= \beta q_\epsilon(t,0,x) b(t,x) & \text{in} \quad Q, \\ q_\epsilon(T,a,x) &= 0 & \text{in} \quad Q_A, \\ q_\epsilon(t,A,x) &= 0 & \text{in} \quad Q_T, \\ q_\epsilon &= h_0\chi_O + (w_0 - v_\epsilon)\chi_\gamma & \text{on} \quad \Sigma, \end{cases} \tag{39}$$

$$\begin{cases} L\rho_\epsilon &= 0 & \text{in} \quad Q, \\ \rho_\epsilon(0,a,x) &= \frac{1}{\epsilon} q_\epsilon(0,a,x) & \text{in} \quad Q_A, \\ \rho_\epsilon(t,0,x) &= b(x,t)\displaystyle\int_0^A \beta\rho_\epsilon da & \text{in} \quad Q_T, \\ \rho_\epsilon &= 0 & \text{on} \quad \Sigma, \end{cases} \tag{40}$$

$$v_\epsilon = \left(w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\right) - P\left(w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\right) \in K^\perp. \tag{41}$$

Step 2. Multiplying the state equation (39) by ρ_ϵ and integrating by parts over Q, we get

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} = \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma + \int_U\int_\gamma (w_0 - v_\epsilon)\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma,$$

which in view of (41) and the fact that $v_\epsilon \in K^\perp$ give

$$\begin{aligned} \frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &\quad + \int_U\int_\gamma (w_0 - v_\epsilon)\left(w_0 - v_\epsilon - P\left(w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\right)\right)dtd\Gamma. \\ &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &\quad - \|w_0 - v_\epsilon\|_{L^2(U\times\gamma)} + \|Pw_0\chi_\gamma\|_{L^2(U\times\gamma)} + \int_U\int_\gamma w_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma. \end{aligned}$$

Since on $U\times\gamma$

$$w_0 - v_\epsilon = Pw_0\chi_\gamma + (I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma,$$

we have that

$$\|w_0 - v_\epsilon\|_{L^2(U\times\gamma)} = \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} + \|Pw_0\chi_\gamma\|^2_{L^2(U\times\gamma)},$$

so that

$$\begin{aligned} \frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &\quad + \int_U\int_\gamma w_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} &\le \left(\int_U\int_O(\theta h_0)^2 dtd\Gamma\right)^{\frac{1}{2}}\left(\int_U\int_\gamma\frac{1}{\theta^2}\frac{\partial\rho_\epsilon}{\partial\nu}^2 dtd\Gamma\right)^{\frac{1}{2}} \\ &\quad + \left(\int_U\int_O(\theta w_0)^2 dtd\Gamma\right)^{\frac{1}{2}}\left(\int_U\int_\gamma\frac{1}{\theta^2}\frac{\partial\rho_\epsilon}{\partial\nu}^2 dtd\Gamma\right)^{\frac{1}{2}}. \end{aligned} \tag{42}$$

Applying the adapted Carleman inequality (19) to ρ_ϵ we obtain

$$\int_U\int_\Gamma\frac{1}{\theta^2}|\frac{\partial\rho_\epsilon}{\partial\nu}|^2 dtd\Gamma \le C\int_U\int_\gamma|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma|^2 dtda\Gamma, \tag{43}$$

where $C > 0$ is independent of ϵ. From (42), the choice of $w_0 \in Y_\theta$ and the hypothesis on h_0, we deduce that

$$\begin{aligned} &\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \frac{1}{2}\|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} \\ &\qquad \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}, \end{aligned} \tag{44}$$

and then

$$\|v_\epsilon\|^2_{L^2(U\times\gamma)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}; \tag{45}$$

$$\|q_\epsilon\chi_\omega\|^2_{L^2(U\times\gamma)} \leq C\sqrt{\epsilon}\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}. \tag{46}$$

Then, the properties of the equation (39) allow us to conclude that

$$\|q_\epsilon\|^2_{L^2(Q)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}. \tag{47}$$

In view of (43) and (44), we get

$$\|\frac{1}{\theta}\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(\Sigma)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}, \tag{48}$$

and using (44) and the fact that $\frac{1}{\theta}$ is bounded, we have

$$\|\frac{1}{\theta}P\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}$$

Therefore, K being a finite dimensional vector subspace of $L^2(U\times\gamma)$, we deduce that

$$\|P\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}. \tag{49}$$

from which we deduce, using (44) that

$$\|\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}. \tag{50}$$

Using Proposition 2 , we have that

$$\begin{aligned} &\int_U\int_\Omega \frac{1}{\theta^2}\left(|\frac{\partial\rho_\epsilon}{\partial\nu}|^2 + |\Delta\rho_\epsilon|^2 + |\nabla\rho_\epsilon|^2 + |\rho_\epsilon|^2\right) dtdad\Gamma \\ &\leq C\left(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma\right)^{\frac{1}{2}}. \end{aligned} \tag{51}$$

Step 3. We prove the convergence of $(v_\epsilon, q_\epsilon)_\epsilon$ and ρ_ϵ towards $\hat{v}$, $\hat{q}$ and $\hat{\rho}$ as $\epsilon \longrightarrow 0$. According to (48), (50) and (54) we can extract subsequences of $(v_\epsilon, q_\epsilon)_\epsilon$ (still called $(v_\epsilon, q_\epsilon)_\epsilon$) such that

$$v_\epsilon \rightharpoonup \widetilde{v} \quad \text{weakly in } L^2(U\times\gamma), \tag{52}$$

$$q_\epsilon \rightharpoonup \widetilde{q} \quad \text{weakly in } L^2(U; H^1(\Omega)), \tag{53}$$

$$\frac{1}{\theta}\rho_\epsilon \rightharpoonup \widetilde{\rho} \quad \text{weakly in } L^2(\frac{1}{\theta}, Q). \tag{54}$$

As v_ϵ belong to $K^\perp$ which is closed vector subspace of $L^2(U\times\gamma)$, we have

$$\widetilde{v} \in K^\perp. \tag{55}$$

So, using (52) and (53) while passing (39) to the limit as $\epsilon \longrightarrow 0$, we can prove that $\widetilde{q}$ is solution of

$$\left\{\begin{array}{rclll} L^{*}\tilde{q} & = & \beta\tilde{q}(t,0,x)b(t,x) & \text{in} & Q, \\ \tilde{q}(T,a,x) & = & 0 & \text{in} & Q_A, \\ \tilde{q}(t,A,x) & = & 0 & \text{in} & Q_T, \\ \tilde{q} & = & h_0\chi_O + (w_0 - v_0)\chi_\gamma & \text{on} & \Sigma, \end{array}\right. \tag{56}$$

and it follows from (44) that

$$q_\epsilon(0,.,.) \rightharpoonup \widetilde{q}(0,.,.) = 0 \text{ weakly in } L^2(Q_A). \tag{57}$$

In view of (55), (56) and (57), $(\widetilde{v}, \widetilde{q})$ verifies the null controllability (34)-(36) and there exists a solution to the boundary null controllability problem. Moreover, it is clear from (40) that $\widetilde{\rho}$ satisfies

$$\left\{\begin{array}{rclll} L\widetilde{\rho} & = & 0 & \text{in} & Q \\ \tilde{\rho}(t,0,x) & = & \displaystyle\int_0^A \beta\tilde{\rho}b(t,x)da & \text{in} & Q_T, \\ \widetilde{\rho} & = & 0 & \text{on} & \Sigma. \end{array}\right.$$

From (48)

$$\frac{\partial\rho_\epsilon}{\partial\nu} \rightharpoonup \frac{\partial\widetilde{\rho}}{\partial\nu} \qquad \text{weakly in } L^2(U\times\gamma). \tag{58}$$

We have on the one hand that $(\widetilde{v}, \widetilde{q})$ is solution to null controllability (34)-(36) , and on the other other hand that, there exists a unique $\hat{v} \in \varepsilon$ such that $(w_0 - \hat{v})$ is of minimal norm in $L^2(U\times\gamma)$. If we denote by $\hat{q}$ the corresponding solution to (35), we have $\hat{q}(0,.,.) = 0$ and, as $\widetilde{v} \in \varepsilon$,

$$\frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)} \leq J_\epsilon(v_\epsilon) \leq J_\epsilon(\hat{v}) = \frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)}$$

and

$$\frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)} \leq \frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)}.$$

Using (52)

$$\liminf_{\epsilon\longrightarrow 0}\frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)} \geq \frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)}$$

Hence,

$$\widetilde{v} = \widehat{v}$$

and

$$v_\epsilon \rightharpoonup \widetilde{v} \quad \textit{strongly in } L^2(U\times\gamma).$$

Writing $\widetilde{\rho} = \widehat{\rho}$, we obtain

$$\widehat{v} = (I-P)\left(w_0\chi_\gamma - \frac{\partial\widehat{\rho}}{\partial\nu}\chi_\gamma\right).$$

2.3 Controllability Result in the Nonlinear Case

Let $\Lambda : L^2((0,T)\times\Omega) \to L^2((0,T)\times\Omega)$ defined by

$$\Lambda(\eta) = \int_0^A \beta(t,a,x)q_\epsilon(\eta)da, \tag{59}$$

where $q_\epsilon(\eta)$ verifies:

$$\left\{\begin{array}{rclll} L^{*}q_\epsilon & = & \beta q_\epsilon(t,0,x)G(\eta) & \text{in} & Q, \\ q_\epsilon(T,a,x) & = & 0 & \text{in} & Q_A, \\ q_\epsilon(t,A,x) & = & 0 & \text{in} & Q_T, \\ q_\epsilon & = & h_0\chi_O + (w_0 - v_\epsilon)\chi_\gamma & \text{on} & \Sigma. \end{array}\right. \tag{60}$$

We set

$$b(t,x) = G(\eta).$$

The following proposition hold.

Proposition 5 *The operator Λ is continuous, bounded and compact. Then Λ admits a fixed point.*

Proof. The proof will be done in two steps.

Step 1.(boundedness and compactness of Λ)

Let

$$Y(t,a,x) = \int_0^A \beta(t,a,x) q_\epsilon da, \tag{61}$$

we have Y is the solution of the following system:

$$\begin{cases} \frac{\partial Y(\eta)}{\partial t} - \Delta Y(\eta) + \displaystyle\int_0^A \mu\beta(t,a,x) q_\epsilon da & = & Z_1(\eta) & \text{in} & Q_T; \\ Y(\eta)(t,x) & = & Z_2(\eta) & \text{on} & (0,T)\times\partial\Omega; \\ Y(\eta)(0,x) & = & 0 & \text{on} & \Omega; \end{cases} \tag{62}$$

with

$$Z_1(\eta) = \int_0^A \left(\frac{\partial\beta}{\partial t} q_\epsilon + \frac{\partial\beta}{\partial a} q_\epsilon + \beta^2 q_\epsilon(t,0,x) G'(\eta) + q_\epsilon \Delta\beta + 2\nabla\beta\nabla q_\epsilon \right) da$$

and

$$Z_2(\eta) = \int_0^A \beta \left(h_0 \chi_O + (w_0 - \upsilon_\epsilon)\chi_\gamma \right) da.$$

Under assumption $(H_1) - (H_4)$ and the result and owing to the estimation on q_ϵ the functions $Z_1(\eta)$ and $Z_2(\eta)$ satisfies

$$\|Z_1(\eta)\|_{L^2((0,T)\times\Omega)} \leq C\left(\|\beta_t\|_\infty, \|\beta_a\|_\infty, \|\beta\|_\infty, \|\Delta\beta\|_\infty, \|\nabla\beta\|_\infty \right), \tag{63}$$

$$\|Z_2(\eta)\|_{L^2((0,T)\times\Omega)} \leq C'(\|h_0\|_{L^2((0,T)\times O)} + \|w_0 - \upsilon_\epsilon\|_{L^2((0,T)\times\gamma)}). \tag{64}$$

From system (62) and using the Lions-Aubin lemma we conclude that Λ is bounded and compact in $L^2(Q_T)$.

Step 2.(Continuity of Λ)

Let $\eta_k \to \eta$ strongly in $L^2(Q)$.
In view of (45)-(46) we get for all $\eta_k \in \mathbb{R}$, $q_\epsilon(\eta_k)$ and v_ϵ are bounded independently to η_k. Therefore $Z_1(\eta_k)$ and $Z_2(\eta_k)$ are bounded respectively in $L^2(Q_T)$ and $L^2((0,T)\times O)$. Then we can extract a subsequence $X(\eta_{k_j})$ such that

$$Y_{\eta_{k_j}} \longrightarrow Y(\eta)$$

$$Z_{1_{\eta_{k_j}}} \longrightarrow Z_1(\eta)$$

$$\int_0^A \mu\beta q_\epsilon(\eta_{k_j}) da \longrightarrow \int_0^A \mu\beta q_\epsilon(\eta) da$$

weakly in $L^2(Q)$.
Then $Y(\eta)$ is a solution of (62). We deduce that the sequence $(Y(\eta_k))$ converge to $Y(\eta)$ so that Λ is continuous.
Since the operator Λ is continuous, bounded, and compact on $L^2(Q_T)$ onto $L^2(Q_T)$, Schauder's fixed-point theorem implies that Λ admits a fixed point.

There exists $\eta \in L^2(Q_T)$ such that

$$\Lambda(\eta) = \eta = \int_0^A \beta q_\epsilon da$$

Then q_ϵ is solution of

$$\begin{cases} -\frac{\partial q_\epsilon}{\partial t} - \frac{\partial q_\epsilon}{\partial a} - \triangle q_\epsilon + \mu q_\epsilon & = & \beta q_\epsilon(t,0,x) G\left(\displaystyle\int_0^A \beta q_\epsilon da\right) & \text{in} & Q, \\ q_\epsilon & = & h_0\chi_O + (w_0 - v_\epsilon)\chi_\gamma & \text{on} & \Sigma \\ q_\epsilon(T,a,x) & = & 0 & \text{in} & Q_A, \\ q_\epsilon(t,A,x) & = & 0 & \text{in} & Q_T. \end{cases} \tag{65}$$

From (47) we have that q_ϵ is bounded in $L^2(Q)$.Then we can extract a subsequence of (q_ϵ) still denoted by (q_ϵ) such that

$$\begin{aligned} q_\epsilon &\rightharpoonup q \text{ in } Ł^2(Q); \\ \int_0^A \beta q_\epsilon da &\rightharpoonup \int_0^A \beta q da \text{ in } Ł^2(Q_T). \end{aligned} \tag{66}$$

Then there exists a subsequence still denoted by $\int_0^A \beta q_\epsilon da$ such that

$$\int_0^A \beta q_\epsilon da \longrightarrow \int_0^A \beta q da \text{ in } Ł^2(Q_T). \tag{67}$$

Now since G is continuous, then

$$G\left(\int_0^A \beta q_\epsilon da\right) \longrightarrow G\left(\int_0^A \beta q da\right) \text{ in } Ł^2(Q_T). \tag{68}$$

From (45) by the same idea

$$v_\epsilon \chi_\gamma \rightharpoonup v\chi_\gamma \text{ in } L^2(U \times \gamma). \tag{69}$$

Therefore, one derives that q solves the following system:

$$\left\{ \begin{array}{rclll} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q &=& \beta q(t,0,x) G\left(\int_0^A \beta q da\right) & \text{in} & Q, \\ q &=& h_0\chi_O + (w_0 - v)\chi_\gamma & \text{on} & \Sigma \\ q(T,a,x) &=& 0 & \text{in} & Q_A, \\ q(t,A,x) &=& 0 & \text{in} & Q_T, \end{array} \right. \tag{70}$$

and we have also, for $\epsilon \longmapsto 0$, $v \in K^\perp$ and $q(0,a,x;v) = 0$ in Q_A.

3. Application to Build the Boundary Sentinel and Detection of Unknown Parameters in a Nonlinear Population Dynamics Model With Incomplete Data

3.1 Nonlinear Population Dynamics Model With Incomplete Data

For a given positive real function F, we consider the following nonlinear population dynamics model:

$$\left\{ \begin{array}{rclll} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y &=& 0 & \text{in} & Q, \\ y(0,a,x) &=& y^0 + \tau \hat{y}^0 & \text{in} & Q_A, \\ y(t,0,x) &=& F\left(\int_0^A \beta(t,a,x) y(t,a,x) da\right) & \text{in} & Q_T, \\ y &=& \left\{ \begin{array}{ll} \xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i, & \text{on} \quad \Sigma_1, \\ 0 & \text{on} \quad \Sigma \setminus \Sigma_1, \end{array} \right. && \end{array} \right. \tag{71}$$

where :

- $y(t,a,x)$ is the distribution of a-year old individuals at time t at the point $x \in \Omega$.
- $\beta(t,a,x) \geq 0$ and $\mu(t,a,x) \geq 0$ are respectively the natural fertility and the natural death rate of age a at time t and position $x \in \Omega$.
- Thus, the formula $\int_0^A \beta(t,a,x)y(t,a,x)da$ denotes the distribution of newborn individuals at time t and location x. In an oviparous species it denotes the total eggs at time t and position x. Therefore, the quantity $F\left(\int_0^A \beta(t,a,x)y(t,a,x)da\right)$ is the distribution of eggs that hatch at time t and position x.

- The boundary condition is unknown on a part Σ_1 of the boundary and represents a pollution with a structure of the form $\xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i$. In this structure, the functions ξ and $\hat{\xi}_i$, $i = 1, \dots M$ are known where as the real λ_i, $i = 1, \dots M$ are unknown.

- The initial distribution of individuals is unknown and its structure is of the form is $y^0 + \tau\hat{y}^0$ is where the function y^0 is known and the term $\tau\hat{y}^0$ is unknown.

System (71) is a system with incomplete data because the information on the boundary condition as well as on the initial condition are partially or completely unknown. Here, the pollution is isolated on the boundary $\Gamma \setminus \Gamma_1$ and we do not know with certainly the number of individuals living on the other part of the boundary Γ_1 . The missing term in the initial condition expresses the fact that we do not know exactly the initial density.
In what follows, We also assume that:

- y^0 and $\hat{y}^0$ belong to $L^2(Q_A)$, ξ and $\hat{\xi}_i$ belong to $L^2(\Sigma)$.

- the reels τ, λ_i $1 \leq i \leq M$ are sufficiently small and $\|\hat{y}^0\|_{L^2(Q_A)} \leq 1$, and we set $\lambda = (\lambda_1, \dots, \lambda_M)$.

- $F \in L^\infty(\mathbb{R})$ and $F \in C^1(\mathbb{R})$

Under the above assumptions on the data, one can prove as in (Ouédraogo & O. Traoré, 2003) that the system (71) has a unique solution $y = y(\lambda, \tau)$ in $L^2(Q)$. Moreover, if we denote by $I \subset \mathbb{R}$ a neighbourhood of 0, the applications $\tau \longmapsto y(\lambda, \tau)$ and $\lambda_i \longmapsto y(\lambda, \tau)$, $(1 \leq i \leq M)$ are in $C^1\left(I; L^2(Q)\right)$.

For more literature on the model describing the dynamics of population with age dependence and spatial structure as well as for some existence results on such problem, we refer for instance to (Langlais, 1985), (Ainseba & Langlais, 2000), (Giovanna & Langlais, 1982); (Ouédraogo & Traoré, 2003) and the reference there in.

For the model (71), we are interested in identifying the unknown parameters λ_i without any attempt at computing $\tau\hat{y}^0$. To identify these parameters, we use the theory of sentinel in a general framework. More precisely, let O be a nonempty open subset of $\Gamma \backslash \Gamma_1$ and let $y = y(t, a, x; \lambda, \tau) = y(\lambda, \tau)$ be the solution of (71).

3.2 Sentinel

For any nonempty open subset γ of $\Gamma \backslash \Gamma_1$ such that $O \cap \gamma \neq \emptyset$, we look for a function $S(\lambda, \tau)$ solution to the following problem : Given $h_0 \in L^2(U \times O)$, find $w \in L^2(U \times \gamma)$ such that

i) the function S defined by

$$S(\lambda, \tau) = \int_U \int_O h_0 \frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma + \int_U \int_\gamma w \frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma, \tag{72}$$

satisfies :

- S is stationary to the first order with respect to missing term $\tau\hat{y}^0$

$$\frac{\partial S}{\partial \tau}(0,0) = 0 \qquad \forall\, \hat{y}^0. \tag{73}$$

- S is sensitive to the first order with respect to pollution terms $\lambda_i \hat{\xi}_i$:

$$\frac{\partial S}{\partial \lambda_i}(0,0) = c_i \qquad 1 \leq i \leq M, \tag{74}$$

where $c_i,\ 1 \leq i \leq M$, are given constants not all identically zero.

ii) The control w is of minimal norm in $L^2(U \times \gamma)$ among " the admissible controls", i.e.

$$\|w\|^2_{L^2(U\times\gamma)} = \min_{\tilde{w}\in E} \|\tilde{w}\|^2_{L^2(U\times\gamma)}. \tag{75}$$

Where

$$E = \left\{\tilde{w} \in L^2(U \times \gamma), \text{such that } (\tilde{w}, S(\tilde{w})) \text{ satisfies } (72) - (75)\right\}. \tag{76}$$

Remark 1 J.L.Lions (Lions, 1992) refers to the function S as a sentinel with given sensitivity c_i. The c_i are chosen according to the importance which is conferred to the component ξ_i of the pollution.

Remark 2 Notice that for the J.L.Lions's sentinels defined by (72)-(75), the observatory $O \subset (\Gamma \setminus \Gamma_1)$ is also the support of the control function w.

For more information on the theory of sentinel, we refer to (Mophou &Nakoulima, 2008), (Nakoulima &Sawadogo, 2007) and the reference therein. We set $y_0 = y(0,\ 0) \in L^2(Q)$, the solution of (1) when $\lambda = 0$ and $\tau = 0$ we denote respectively by y_τ and y_{λ_i}, the derivatives of y at (0, 0) with respect to τ and λ_i, i.e. :

$$y_\tau = \lim_{\tau \to 0} \frac{y(0,\tau) - y(0,0)}{\tau}.$$

and

$$y_{\lambda_i} = \lim_{\lambda_i \to 0} \frac{y(\lambda_i, 0) - y(0,0)}{\lambda}.$$

Then y_τ and y_{λ_i} are respectively solutions of

$$\begin{cases} \frac{\partial y_\tau}{\partial t} + \frac{\partial y_\tau}{\partial a} - \triangle y_\tau + \mu y_\tau & = & 0 & \text{in} \quad Q, \\ y_\tau(0,a,x) & = & \hat{y}^0 & \text{in} \quad Q_A, \\ y_\tau(t,0,x) & = & \displaystyle\int_0^A \left(\beta F'\left(\int_0^A \beta y(0,0) da\right) y_\tau\right) da & \text{in} \quad Q_T, \\ y_\tau & = & 0 & \text{on} \quad \Sigma, \end{cases} \tag{77}$$

and

$$\begin{cases} \frac{\partial y_{\lambda_i}}{\partial t} + \frac{\partial y_{\lambda_i}}{\partial a} - \triangle y_{\lambda_i} + \mu y_{\lambda_i} & = & 0 & \text{in} \quad Q, \\ y_{\lambda_i}(0,a,x) & = & 0 & \text{in} \quad Q_A, \\ y_{\lambda_i}(t,0,x) & = & \displaystyle\int_0^A \left(\beta F'\left(\int_0^A \beta y(0,0) da\right) y_{\lambda_i}\right) da & \text{in} \quad Q_T, \\ y_{\lambda_i} & = & \hat{\xi}_i \chi_{\Sigma_1} & \text{on} \quad \Sigma, \end{cases} \tag{78}$$

where χ_X denote now and in the sequel, the characteristic function of the set X. Under the above assumptions on the data, we have that (77)and (78) has respectively a unique solution y_τ and y_{λ_i} in $L^2(Q)$) (see Giovanna & Langlais, 1982, Ouédraogo & Traoré, 2003).

From now on, we assume that the functions

$$\hat{\xi}_{i\chi_{\Sigma_1}}, 1 \leq i \leq M \text{ are linearly independent}, \tag{79}$$

and we set

$$Y = \text{Span}\{\frac{\partial y_{\lambda_1}}{\partial \nu}\chi_\gamma, ..., \frac{\partial y_{\lambda_M}}{\partial \nu}\chi_\gamma\}, \tag{80}$$

the vector subspace of $L^2(U \times \gamma)$ generated by M functions $\{y_{\lambda_i}\chi_\gamma\}_{i=1}^M$, and

$$Y_\theta = \frac{1}{\theta} Y,$$

the vector subspace of $L^2(U \times \gamma)$ generated by M functions $\{\frac{1}{\theta}\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$, where θ is the positive function defined by (13)

Remark 3 Lemma 2 allows us to say that the functions $\{\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$ and $\{\frac{1}{\theta}\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$ are linearly independent.

3.3 Equivalence Between Sentinel Problem and Controllability

In view of (72), the stationary condition (73) and respectively the sensitivity condition (74) hold if and only if

$$\int_U \int_O h_0 \frac{\partial y_\tau}{\partial \nu} dtdad\Gamma + \int_U \int_\gamma w \frac{\partial y_\tau}{\partial \nu} dtdad\Gamma = 0. \qquad \forall\ \hat{y}^0,\ \|\hat{y}^0\|_{L^2(Q_A)} \leq 1; \tag{81}$$

and

$$\int_U \int_O h_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma + \int_U \int_\gamma w \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma = c_i. \quad 1 \leq i \leq M. \tag{82}$$

Therefore, in order to transform equation (81) we introduce the classical adjoint state. More precisely, we consider the following problem :

$$\left\{ \begin{array}{cclcc} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q & = & \beta q(t,0,x) F'\left(\int_0^A \beta y(0,0) da \right) & \text{in} & Q, \\ q & = & h_0 \chi_O + w \chi_\gamma & \text{on} & \Sigma \\ q(T,a,x) & = & 0 & \text{in} & Q_A, \\ q(t,A,x) & = & 0 & \text{in} & Q_T. \end{array} \right. \tag{83}$$

Since $h_0\chi_O + w\chi_\gamma \in L^2(\Sigma)$ and under the assumptions $(H1)-(H3)$, we can prove that (83) has a unique solution $q \in L^2(Q)$. Now multiplying both sides of the differential equation in (83) by y_τ solution of (77) and integrating by parts in Q, we get

$$\int_U \int_O h_0 \frac{\partial y_\tau}{\partial \nu} dtdad\Gamma + \int_U \int_\gamma w \frac{\partial y_\tau}{\partial \nu} dtdad\Gamma = \int_0^A \int_\gamma q(0,a,x) \hat{y}^0 dadx \ \forall\ \hat{y}^0 \in L^2(Q_A). \tag{84}$$

Thus, the condition (73) or (81) holds if and only if

$$q(0,a,x;v) = 0 \qquad \text{in } Q_A, \tag{85}$$

then, multiplying both sides of the differential equation in (83) by y_{λ_i} solution of (78) and integrating by parts in Q, we have

$$\int_U \int_O h_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma + \int_U \int_\gamma w \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma = \int_{\Sigma_1} \frac{\partial q}{\partial \nu} \hat{\xi}_i . \chi_{\Gamma_1} dtda, \quad 1 \leq i \leq M.$$

Thus, the condition the condition (74) or (82) is equivalent to

$$\int_{\Sigma_1} \frac{\partial q}{\partial \nu} \hat{\xi}_i . \chi_{\Gamma_1} dtda = c_i, \quad 1 \leq i \leq M. \tag{86}$$

Now, consider the matrix

$$\left(\int_0^T \int_0^A \int_\gamma \frac{1}{\theta} \frac{\partial y_{\lambda_j}}{\partial \nu} \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma \right)_{1 \leq i,j \leq M}.$$

This matrix is symmetric positive definite therefore, there exists a unique $w_0 \in Y_\theta$ such that

$$c_i - \int_U \int_O h_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma = \int_U \int_\gamma w_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma. \quad 1 \leq i \leq M. \tag{87}$$

Consequently, combining(82) with (87), we observe that condition (74) holds if and only if

$$w - w_0 = -v \in Y^\perp,$$

where Y is given by (80). Replacing w by $w_0 - v$ in the second expression of (83), we have just proved that the sentinel problem (72)-(75) hold if and only if the following null boundary controllability problem with constraint on the control has a solution: Given $h_0 \in L^2(U \times O)$, $w_0 \in Y_\theta$. find $v \in L^2(U \times \gamma)$ such that

$$v \in Y^\perp, \tag{88}$$

and if q = q(t, a, x; v) is solution of

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q &= \beta q(t,0,x)F'\left(\int_0^A \beta y(0,0)da\right) & \text{in} \quad Q, \\ q &= h_0\chi_O + (w_0 - v)\chi_\gamma & \text{on} \quad \Sigma \\ q(T,a,x) &= 0 & \text{in} \quad Q_A, \\ q(t,A,x) &= 0 & \text{in} \quad Q_T, \end{cases} \tag{89}$$

$$q(0,a,x;v) = 0 \qquad \text{in } \; Q_A. \tag{90}$$

Remark 4 Let us notice that if v exists, the set

$$\mathcal{E} = \{\overline{v} \in Y^\perp \text{such that } (\overline{v}, \overline{q} = q(t,a,x;\overline{v})) \text{ satisfies } (88) - (90)\} \tag{91}$$

is a nonempty closed, and convex set in $L^2(U \times \gamma)$. Therefore there exists $v \in \mathcal{E}$ of minimal norm.

Proposition 6 *Assume that the hypotheses of Theorem 1 are satisfied. Then there exists a positive real weight function θ given by (13) such that, for any function $h_0 \in L^2(U \times O)$ with $\theta h_0 \in L^2(U \times O)$ there exists a unique control $\hat{v} \in L^2(U \times \gamma)$ such that $(\hat{v},\ \hat{q})$ with $\hat{q} = q(\hat{v})$ is solution of null boundary controllability problem with constraint on the control (88)-(90) and provides a control $\hat{w} = w_0\chi_\gamma - \hat{v}$ of the sentinel problem satisfying (75). Moreover, the control $\hat{w}$ is given by:*

$$\hat{w} = P(w_0) + (I - P)(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma), \tag{92}$$

Proof. Replacing $b(t,x)$ in (35) by $F'\left(\int_0^A \beta y(0,0)da\right)$

3.4 Identification of Unknown Parameters λ_i

Using proposition 6, if we replace in (72) w with

$$\hat{w} = P(w_0) + (I - P)\left(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma\right),$$

then the function S is defined by

$$S(\lambda,\tau) = \int_U \int_O h_0 \frac{\partial y}{\partial \nu}(\lambda,\tau)dtdad\Gamma + \int_U \int_\gamma \left(P(w_0) + (I-P)\left(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma\right)\right)\frac{\partial y}{\partial \nu}(\lambda,\tau)dtdad\Gamma,$$

and the pair $(\hat{w}, S(\hat{w}))$ verify (72)-(75). To estimate the parameters λ_i , one proceeds as follows: assume that the solution of (71) when $\lambda = 0$ and $\tau = 0$ is known. Then, one has the following information

$$S(\lambda,\tau) - S(0,0) \approx \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \nu}(0,\ 0).$$

Therefore, fixing $i \in \{1, \ldots M\}$ and choosing

$$\frac{\partial S}{\partial \lambda_j}(0,\ 0) = 0 \; for \; j \neq i \; and \; \frac{\partial S}{\partial \lambda_i}(0,\ 0) = c_i,$$

one obtains the following estimate of the parameter λ_i:

$$\lambda_i \approx \frac{1}{c_i}\left(S(\lambda,\tau) - S(0,0)\right).$$

We deduce that

$$\begin{aligned} \lambda_i \approx & \frac{1}{c_i}\left\{\int_U \int_O h_0 \left(m_0 - \frac{\partial y_0}{\partial \nu} dtdad\Gamma\right)\right\} \\ & + \frac{1}{c_i}\left\{\int_U \int_\gamma \left(P(w_0) + (I-P)\left(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma\right)\right)\left(m_0 - \frac{\partial y_0}{\partial \nu}\right) dtdad\Gamma\right\}, \end{aligned}$$

where m_0 is a measure of the flux of the concentration of the pollutant taken on the observatory $O \cup \gamma$ and y_0 is solution of (71) when $\lambda = 0$ and $\tau = 0$.

4. Conclusion

In this paper, using an adapted Carleman inequality and Schauder's fixed-point theorem, we solve a boundary null controllability problem with constraints on the control for a nonlinear two stroke system. The obtained results are used to build a new mathematical tool of analysis which is the boundary sentinel with given sensitivity. The obtained sentinel is also used to identify unknown parameters in a nonlinear population dynamics model with incomplete data. The sentinel method is the best one to use in the inverses problems.

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