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The Exponential-Generalized Truncated Geometric (EGTG) Distribution: A New Lifetime Distribution

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Abstract

This paper introduces a new two-parameter lifetime distribution, called the exponential-generalized truncated geometric (EGTG) distribution, by compounding the exponential with the generalized truncated geometric distributions. The new distribution involves two important known distributions, i.e., the exponential-geometric (Adamidis and Loukas, 1998) and the extended (complementary) exponential-geometric distributions (Adamidis et al., 2005; Louzada et al., 2011) in the minimum and maximum lifetime cases, respectively. General forms of the probability distribution, the survival and the failure rate functions as well as their properties are presented for some special cases. The application study is illustrated based on two real data sets.

Keywords: Lifetime distributions; exponential distribution; truncated geometric distribution; order statistics; failure rate; survival function.

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1. Introduction

Lifetime distributions are widely used in reliability theory, survival analysis and several areas of studies, such as finance, manufacture, biological sciences, physics and engineering. The exponential distribution is the most commonly used in reliability and lifetime testing, assuming the failure rate is constant (Balakrishnan and Basu, 1995; Barlow and Proschan, 1975). In recent years, some of research papers have been devoted to take into account data with increasing or decreasing failure rate functions. The motivation is to provide more convenient parametric fit for real data where the underlying hazard rates, arising on a latent complementary risk problem base, present monotone shapes (non-constant hazard rates). The genesis is stated on complementary risk scenarios with masked causes of failure (Basu and Klein, 1982; Cox and Oakes, 1984).

Several families of compound lifetime distributions are introduced as extensions of the exponential distribution, following Adamidis and Loukas (1998), using a mixture of discrete and continuous distributions. Adamidis and Loukas (1998) introduced the exponential-geometric (EG) lifetime distribution with decreasing failure rate (DFR) arising by mixing power-series distributions. In the same way, Adamidis et al. (2005) and Louzada et al. (2011) proposed the complementary (or extended) exponential-geometric (CEG) distribution with increasing failure rate (IFR). Other families of lifetime distributions have been investigated by several authors. For example, Kus (2007), Tahmasbi and Rezaei (2008), Chahkandi and Ganjali (2009), Barreto-Souza and Cribari-Neto (2009), Silva et al. (2010), Barreto-Souza et al. (2011), Cancho et al. (2011), Louzada-Neto et al. (2011), Morais and Barreto-Souza (2011), Hemmati et al. (2011), Alkarni and Orabi (2012), Nadarajah et al. (2013), Bakouch et al. (2014), and others.

The proposed distributions come from the idea of modelling the reliability of series and parallel systems based on the reliability of their components. They are carried out only for the first (minimum) or the last (maximum) order statistics.

The distributions of stochastic ordered statistics[1] are very interesting in many fields of statistics, particularly, the detection of outliers, quality control, auction theory, reliability and life testing models. In this paper, we generalize the distributions modelling the time to the first (or the last) failure, to a distribution more appropriate for modelling any order statistic (second, third, or any k^{th} lifetime). For example, one may be interested in the period of time patients may spend in hospital during a month before they leave. Let's suppose that number of patients fill beds in a hospital and leave during a month. We suppose that T_i is the period a patient spends before leaving and Z the random number of patients that leave the hospital. We may let $X_{(1)} < X_{(2)} < \dots < X_{(Z)}$ be the order statistics of z independent observations of time $\mathrm{T} = (\mathrm{T}_1, \mathrm{T}_2, \dots, \mathrm{T}_z)$. Maybe previous studies focused on the minimum, $X_{(1)} = \min\{T_i\}_{i=1}^{Z}$, or the maximum, $X_{(Z)} = \max\{T_i\}_{i=1}^{Z}$, duration of filling beds. We are interested here in the duration $X_{(k)}$ and then we determine the distribution for the observed k^{th} order statistic. Ordered random variables are already known for their ascending order. The concept of dual generalized ordered statistics, introduced by Burkschat et al. (2003), enables a common approach to the reverse (or descending) order statistics. Reverse order statistics have also wide range of applications in economics such as providing diverse distributive criteria in assessing welfare and inequalities of incomes and wealth (Weymark, 1981; Parrado-Gallardo et al., 2014).

We propose the new family of lifetime distributions by compounding the exponential with the generalized truncated geometric distributions, named the exponential-generalized truncated geometric (EGTG) distribution. We show that the minimum lifetime (Adamidis and Loukas, 1998) and the maximum lifetime (Adamidis et al., 2005; Louzada et al., 2011) are special cases of our EGTG distribution.

Finally, this paper is organized as follows: In section 2, we derive different statistical proprieties of our proposed distribution, including the probability density function (pdf), the moment generating function, the reliability and failure rate functions, the random number generation and the entropy function. The estimation of the parameters will be discussed in section 3. As an illustration of the estimation method, numerical computations will be performed in section 4. The application study is illustrated based on two real data sets in section 5. We conclude the paper in the last section.

2. The New Lifetime Distribution

2.1 The Distribution

The derivation of the new family of lifetime distributions is obtained by mixing the exponential with generalized truncated geometric distributions as follows:

Let $T = (\mathrm{T}_1, \mathrm{T}_2, \dots, \mathrm{T}_z)$ be iid exponential random variables with scale parameter $\theta > 0$ and pdf given by: $f(t) = \theta e^{-\theta t}$, for $t \geq 0$, where Z is a discrete random variable (the random number of unit in a system) following a truncated at $k-1$ geometric distribution with parameter $0 < \eta < 1$. The probability mass function (pmf), $P_{k-1}(Z = z)$, is:

$$P_{k-1}(Z = z) = (1 - \eta)\eta^{z-k} \;; 0 < \eta < 1 \;; \; k = 1, 2, \dots, z \tag{1}$$

where the pmf of the geometric distribution is given by:

$$P(Z = i) = (1 - \eta)\eta^{i-1} \,; 0 < \eta < 1 \; and \; i = 1, 2, \dots$$

Let $X_{(1)} = \min\{T_i\}_{i=1}^{Z}$ be the first order statistic (or the smallest order statistic) and $X_{(Z)} = \max\{T_i\}_{i=1}^{Z}$ the last order statistic (or the largest order statistic). The pdf of the k^{th} order statistic $X_{(k)}$ (the k^{th}-smallest value of the lifetime) is given by the equation (2) (David, 1981, p. 9; Balakrishnan and Cohen, 1991, p. 12):

$$f_k(x|z,\theta) = \frac{\theta\Gamma(z+1)}{\Gamma(k)\Gamma(z-k+1)} e^{-\theta(z-k+1)x}(1-e^{-\theta x})^{k-1} \quad ; \theta, x > 0 \tag{2}$$

The joint probability density function is obtained from equations (1) and (2) as follows:

$$g(x,z|\eta,\theta) = \begin{cases} \dfrac{\theta(1-\eta)\eta^{z-k}\Gamma(z+1)(1-e^{-\theta x})^{k-1}}{\Gamma(k)\Gamma(z-k+1)e^{\theta(z-k+1)x}} & (I) \;\; ; \;\; k = 1,2,\dots,z \\ \dfrac{\theta(1-\eta)\eta^{z-m-1}\Gamma(z+1)(1-e^{-\theta x})^{z-m-1}}{\Gamma(z-m)\Gamma(m+1)e^{\theta(m+1)x}} & (II) \; ; \; k = z, z-1,\dots,1 \;\; ; m = z-k \end{cases} \tag{3}$$

[1] Order Statistics play a key role in lifetime-distribution studies and they are widely used in statistical models and inference. One may refer to Balakrishnan and Rao (1998, 1998a) for theory, methods and some applications on the order statistics of random variables.

where, x is the lifetime of a system, z is the last order statistic and $m = 0,1, \ldots < z$. Equation (3) gives two definitions[2]: the first is for the ascending order statistics and the second is for the descending order. If we consider the ascending order $X_{(1)} < X_{(2)} < \ \ldots \ < X_{(Z)}$, the first part (I) of this joint probability density in equation (3) is reached by compounding a truncated at $l = k - 1$ geometric distribution and the pdf of the k^{th} order statistic ($k = 1,2, \ldots, z$). The truncated at $k - 1$ geometric distribution is motivated by mathematical interest because we are interested in the k^{th} order statistic. There is a left-truncation scheme, where only $(z - k + 1)$ individuals who survive a sufficient time are included. In comparison with the formulation of Adamidis and Loukas (1998), we consider the k^{th}-smallest value of lifetime instead of the minimum lifetime (the first order statistic), $X_{(1)} = \min\{T_i\}_{i=1}^{Z}$.

However, when we consider the reverse (descending) order $X_{(Z)} > X_{(Z-1)} > \ \ldots > X_{(1)}$, we obtain the second definition (II) in equation (3) by compounding the pdf of the $(z - m)^{th}$ order statistic and a truncated at m geometric distribution. We include in the sample only $(z - m)$ individuals who have experienced the event by the specified time $m = z - k$. In comparison with the formulation of Louzada et al. (2011), we consider the $(z - k)^{th}$-largest value of lifetime instead of the maximum lifetime (the last order statistics), $X_{(Z)} = \max\{T_i\}_{i=1}^{Z}$.

So, our proposed new family of lifetime distributions (EGTG) is the marginal density function of x given by:

$$g(x|\eta,\theta) = \begin{cases} \dfrac{\theta k(1-\eta)e^{-\theta x}(1-e^{-\theta x})^{k-1}}{[1-\eta e^{-\theta x}]^{k+1}} & k = 1,2,\ldots,z \\ \dfrac{(m+1)\theta(1-\eta)e^{-(m+1)\theta x}}{[1-\eta(1-e^{-\theta x})]^{m+2}} & k = z, z-1, \ldots, 1 \ \ and \ m = z-k \end{cases} \quad (4)$$

where $0 < \eta < 1$ is the shape parameter and $\theta > 0$ is the scale parameter.

Our distribution is more appropriate for modelling any order statistic (2^{nd}, 3^{rd} or any k^{th} lifetime). We show later that the minimum lifetime (Adamidis and Loukas, 1998) is a special case of the first part of the EGTG distribution, and that the maximum lifetime (Adamidis et al., 2005; Louzada et al., 2011) is a special case of the second part of the EGTG distribution. It should be noted that we have two separate definitions. The first case is a generalization of the minimum lifetime distribution using the ascending order of lifetime. The second case is a generalization of the maximum lifetime distribution using the descending order. Also, the cumulative distribution function (cdf) of x corresponding to the pdf in equation (4) is given by:

$$G(x|\eta,\theta) = \begin{cases} \left[\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right]^{k} & k = 1,2,\ldots,z \\ 1-\left[\dfrac{e^{-\theta x}}{1-\eta(1-e^{-\theta x})}\right]^{m+1} & k = z, z-1, \ldots, 1 \ ; \ m = z-k \end{cases} \quad (5)$$

In table (1), we present the pdf in equation (4) for some special cases at the first, second and third order statistics and at the last, last-1 and last-2 order statistics. Table (1) shows that the particular case of our EGTG density function, for $k = 1$, is the EG distribution modelling the time to the first failure (Adamidis and Loukas, 1998). Figure (1) illustrates some possible shapes of the pdf for some selected cases of the order statistics (k=1,2; m=0,1) and some selected values of (η, θ).

The pdf decreases strictly in x and tends to zero as $x \to \infty$. Note that, for $k = 1$ the EGTG distribution is strictly decreasing with modal value $\theta(1 - \eta)^{-1}$ at $x = 0$. The median is calculated to be $\theta^{-1}\ln(2 - \eta)$ for $k = 1$. As $\eta \to 0$ and $k = 1$, the EGTG distribution tends to an exponential distribution with parameter θ. The graphs of the density resemble those of the exponential and Pareto II distributions.

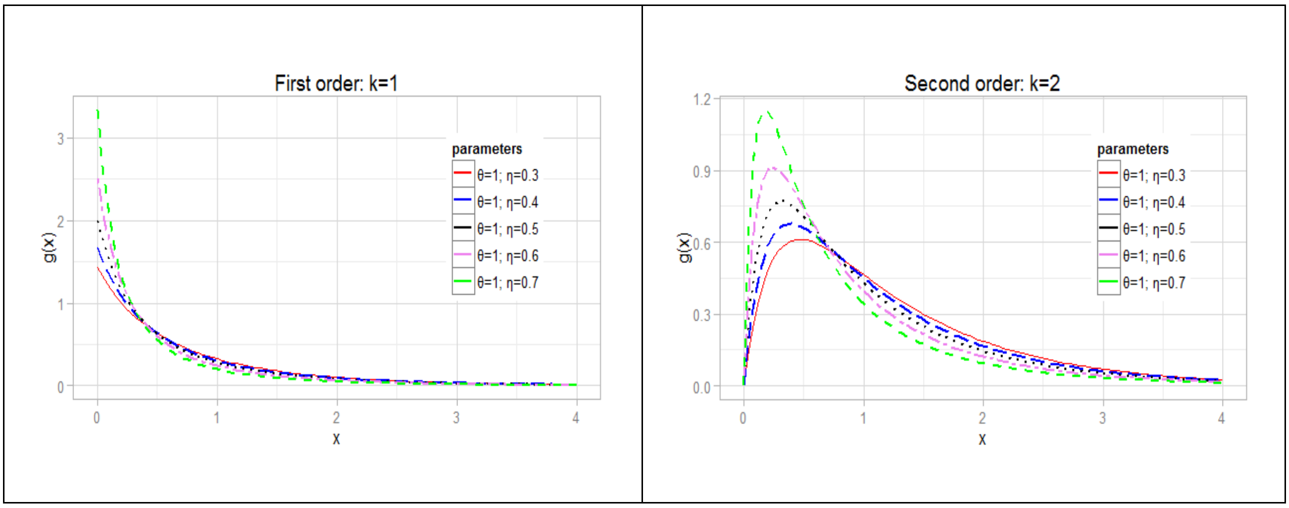
The particular case of the pdf for the last order statistic ($m = 0$) is the maximum lifetime distribution (Adamidis et al., 2005; Louzada et al., 2011). As $\eta \to 0$ and $m = 0$, the EGTG distribution converges to the exponential distribution with parameter θ. The EGTG density function is decreasing and its mode is $\theta(1 - \eta)$ at $x = 0$ for $\eta \leq \frac{1}{2}$. It is increasing and uni-modal at $-\frac{1}{\theta} log \frac{1-\eta}{\eta}$ for $\eta > \frac{1}{2}$. As noted by Louzada et al. (2011), the parameters η and θ may be interpreted

[2] The proofs of all steps and equations are available upon request.

directly in terms of complementary risk problems in presence of latent risks (see also, Louzada-Neto, 1999). In fact, $(1-\eta)$ and θ represent the mean of the number of complementary risks and the lifetime failure rate, respectively. A detailed presentation of the competing risks theory can be found in David and Moeschberger (1978) and Crowder (2001).

Table 1. The probability density function for some special cases

Order statistics	k, m	Probability density function
First	k=1	$\frac{\theta(1-\eta)e^{-\theta x}}{[1-\eta e^{-\theta x}]^2}$
Second	k=2	$\frac{2\theta(1-\eta)e^{-\theta x}(1-e^{-\theta x})}{[1-\eta e^{-\theta x}]^3}$
Third	k=3	$\frac{3\theta(1-\eta)e^{-\theta x}(1-e^{-\theta x})^2}{[1-\eta e^{-\theta x}]^4}$
Last	m=0	$\frac{\theta(1-\eta)e^{-\theta x}}{[1-\eta(1-e^{-\theta x})]^2}$
Last -1	m=1	$\frac{2\theta(1-\eta)e^{-2\theta x}}{[1-\eta(1-e^{-\theta x})]^3}$
Last-2	m=2	$\frac{3\theta(1-\eta)e^{-3\theta x}}{[1-\eta(1-e^{-\theta x})]^4}$



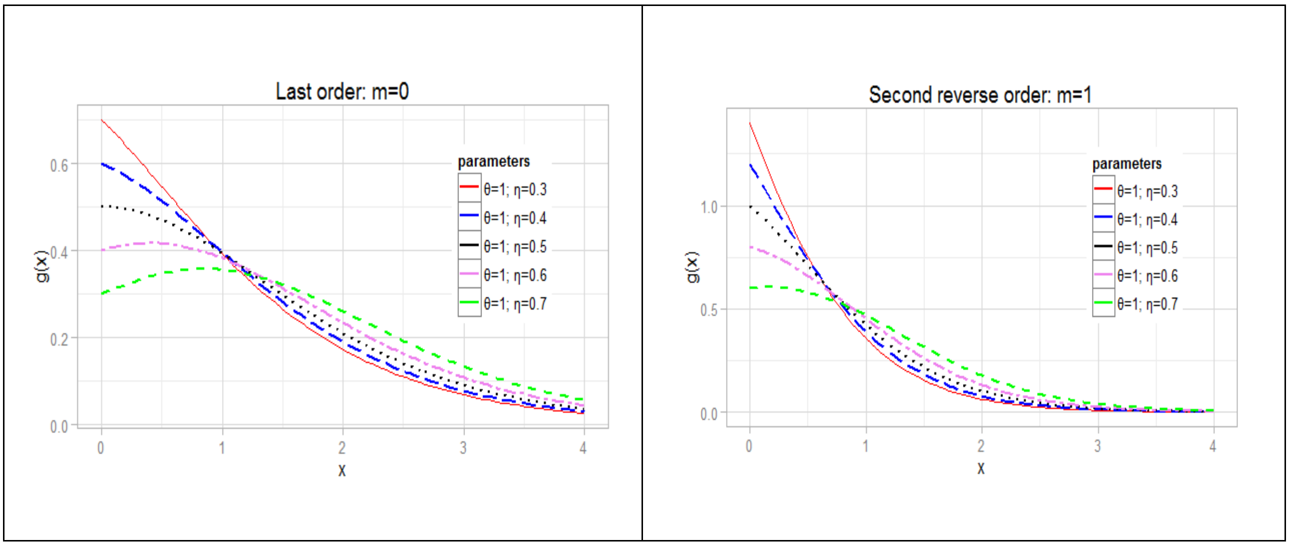


Figure 1. Density of the EGTG distribution for k=(1,2) and m=(0,1)

2.2. Moment Generating Function and r^{th} Moment

The moment generating function (mgf) is an alternative specification of the probability distribution of a random variable. It allows for the study of the characteristics and the features of a distribution through its moments, such as the mean and variance.

If x has the pdf in equation (4), then the mgf is calculated to be:

$$M_x(t)=E(e^{tx})=\begin{cases} k(1-\eta)\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}C_i^{k+i}C_j^{k-1}(-1)^j\eta^i\left[i+j+1-\dfrac{t}{\theta}\right]^{-1} \quad ; \quad k=1,2,\dots,z \\ (m+1)(1-\eta)\sum_{i=0}^{\infty}\sum_{j=0}^{i}C_i^{m+i+1}C_j^{i}(-1)^j\eta^i\left[m+j+1-\dfrac{t}{\theta}\right]^{-1} \\ \qquad k=z,z-1,\dots,1;\ m=z-k \end{cases} \tag{6}$$

The mgf in equation (6) generates the moments of X by differentiation. In other words, the rth moment can be obtained by evaluating the rth order derivative of $M_x(t)$ at $t=0$ as follows:

$$E(x^r)=M_X^{(r)}(0)=\left.\frac{d^rM_X(t)}{dt^r}\right|_{t=0}$$

The mean is the first order derivative, $\mu=E(X)=M_X^{(1)}(0)$, and the second order derivative is $E(X^2)=M_X^{(2)}(0)$. Hence, the variance is $\sigma^2=E(X^2)-\mu^2$.

From equation (6), the r^{th} moment is given by:

$$E(x^r)=\begin{cases} \dfrac{\Gamma(r+1)k(1-\eta)}{\theta^r}\sum_{i=0}^{\infty}\sum_{j=0}^{k-1}C_i^{k+i}C_j^{k-1}(-1)^j\eta^i\left[i+j+1\right]^{-(r+1)} & k=1,2,\dots,z \\ \dfrac{\Gamma(r+1)(m+1)(1-\eta)}{\theta^r}\sum_{i=0}^{\infty}\sum_{j=0}^{i}C_i^{m+i+1}C_j^{i}(-1)^j\eta^i\left[m+j+1\right]^{-(r+1)} & k=z,z-1,\dots,1\,;m=z-k \end{cases} \tag{7}$$

Table (2) provides the explicit expressions in terms of the polylogarithm function for the first and second order moments of the random variable X. $Li_s(\eta)$ is the generalization of Euler's dilogarithm function of η. It is known as the polylogarithm function defined by the power series (Erdelyi et al., 1953, p. 31):

$$Li_s(\eta) = \sum_{t=1}^{\infty} \frac{\eta^t}{t^s} = \eta + \frac{\eta^2}{t^s} + \frac{\eta^3}{t^s} + \cdots$$

where, $Li_0(\eta) = \frac{\eta}{1-\eta}$ and $Li_1(\eta) = -\ln(1-\eta)$ (Jodra, 2008; Adamidis and Loukas, 1998).

Table 2. The first and second order moments for some special cases

Order statistics	**k, m**	$\boldsymbol{E}(\boldsymbol{x})$	$\boldsymbol{E}(\boldsymbol{x}^2)$
First	k=1	$\frac{1-\eta}{\eta\theta} Li_1(\eta)$	$\frac{2(1-\eta)}{\eta\theta^2} Li_2(\eta)$
Second	k=2	$\frac{1-\eta^2}{\eta^2\theta} Li_1(\eta) - \frac{1-\eta}{\eta\theta}$	$\frac{2(1-\eta^2)}{\eta^2\theta^2} Li_2(\eta) - \frac{2(1-\eta)^2}{\eta^2\theta^2} Li_1(\eta)$
Third	k=3	$\frac{1-\eta^3}{\eta^3\theta} Li_1(\eta) - \frac{3}{2}\frac{(1-\eta)}{\eta\theta} - 2\frac{1-\eta}{\eta^2\theta}$	$\frac{2(1-\eta^3)}{\eta^3\theta^2} Li_2(\eta) - \frac{3(1-\eta)^2(1+\eta)}{\eta^3\theta^2} Li_1(\eta) + \frac{(1-\eta)^2}{\eta^2\theta^2}$
Last	m=0	$\frac{1}{\eta\theta} Li_1(\eta)$	$\frac{2}{\eta\theta^2}\left\{Li_2(1) - Li_2(1-\eta) + \frac{1}{2}[Li_1(\eta)]^2 + \ln(\eta) Li_1(\eta)\right\}$
Last -1	m=1	$\frac{1}{\eta^2\theta} Li_1(\eta) - \frac{1}{\eta\theta}$	$\frac{2}{\eta^2\theta^2}\left\{Li_2(1) - Li_2(1-\eta) + \frac{1}{2}[Li_1(\eta)]^2 + [\ln(\eta) - 1] Li_1(\eta)\right\}$
Last-2	m=2	$\frac{1}{\eta^3\theta} Li_1(\eta) - \frac{1}{2}.\frac{1}{\eta\theta} - \frac{1}{\eta^2\theta}$	$\frac{2}{\eta^3\theta^2}\left\{Li_2(1) - Li_2(1-\eta) + \frac{1}{2}[Li_1(\eta)]^2 + \left[\ln(\eta) - \frac{3}{2}\right] Li_1(\eta) + \frac{\eta}{2}\right\}$

2.3. Reliability and Failure Rate Functions

The reliability (or the survival) function is the probability of being alive just before a duration x, given by $S(x) = Pr\{X > x\} = 1 - G(x) = \int_x^\infty f(t)dt$ which is the probability that the event under study has not occurred by duration x. So the reliability function corresponding to the pdf in equation (4) is given by:

$$S(x|\eta,\theta,m) = \begin{cases} 1 - \left[\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right]^k & k = 1,2,\ldots,z \\ \left[\dfrac{e^{-\theta x}}{1-\eta(1-e^{-\theta x})}\right]^{m+1} & k = z, z-1,\ldots,1 \;\; and \; m = z-k \end{cases} \qquad (8)$$

In other words, the reliability, $S(x)$, is the probability that a system operates properly in the interval from time 0 to time x, where X is a random variable denoting the time-to-failure or failure time. One may refer to the literature on the theory and applications of reliability (see Barlow and Proschan, 1975, 1981; Basu and Klein, 1982).

The hazard rate $h(x)$, known as failure rate function, is the instantaneous rate of occurrence of the event of interest at duration x (i.e. the rate of event occurrence per unit of time). Mathematically, it equals the probability density $g(x)$ of events at x, divided by the probability, $S(x)=1-G(x)$, of surviving to that duration without experiencing the event. Thus, we define a failure rate function as in Barlow and Proschan (1965) by $h(x)=g(x)/S(x)$. The failure rate function corresponding to the pdf in equation (4) is given by:

$$h(x|\eta,\theta)=\begin{cases}\dfrac{\theta k(1-\eta)e^{-\theta x}(1-e^{-\theta x})^{k-1}}{[1-\eta e^{-\theta x}]^{k+1}}\left[1-\left(\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right)^{k}\right]^{-1} & k=1,2,\ldots,z\\ \dfrac{\theta(m+1)(1-\eta)}{1-\eta(1-e^{-\theta x})} & k=z,z-1,\ldots,1 \ \ and\ m=z-k\end{cases} \qquad (9)$$

The hazard rate function is analytically related to the time-to-failure probability distribution. It leads to the examination of the IFR or the DFR properties of life-length distributions. G is an IFR distribution, if $h(x)$ increases for all X such that $G(X)<1$. The motivation of the EGTG lifetime distribution is the realistic features of the hazard rate in many real-life physical and non-physical systems, which is not a monotonically increasing, decreasing or constant failure rate. Note that for the first part of equation (9), if $k=1$, the hazard rate function is decreasing (Adamidis and Loukas, 1998). In fact, if $x\to 0$ then $h_k(x\backslash\eta,\theta)=\theta(1-\eta)^{-1}$ and if $x\to\infty$ then $h_k(x\backslash\eta,\theta)\to\theta$.

If $k>1$, there is an IFR distribution. Indeed, if $x\to 0$ then $h_k(x\backslash\eta,\theta)\to 0$. If $x\to\infty$ then $h_k(x\backslash\eta,\theta)=\theta$.

In the second part of equation (9), the EGTG is an IFR distribution. The initial rate value is finite and given by $h_m(0\backslash\eta,\theta)=\theta(m+1)(1-\eta)$. The long term hazard value is finite and given by $\theta(m+1)$. Indeed, we have $\lim_{x\to 0}h_m(x|\eta,\theta)\le\lim_{x\to\infty}h_m(x|\eta,\theta)$. Figure (2) illustrates the shapes of the hazard rate function for some cases (k=1,2; m=0,1) and some selected values of (η,θ).

$$\lim_{x\to 0}h_k(x|\eta,\theta)=\begin{cases}\theta(1-\eta)^{-1} & k=1\\ 0 & k=2,3,\ldots,z\end{cases}$$

$$\lim_{x\to\infty}h_k(x|\eta,\theta)=\theta \qquad k=1,2,3,\ldots,z$$

$$\lim_{x\to 0}h_m(x|\eta,\theta)=\theta(m+1)(1-\eta) \qquad k=z,z-1,\ldots,1\ ;\ m=z-k$$

$$\lim_{x\to\infty}h_m(x|\eta,\theta)=\theta(m+1) \qquad k=z,z-1,\ldots,1\ ;\ m=z-k$$

In table (3), we present the reliability and failure rate functions in equations (8) and (9) for some special cases at the first, second and third order statistics, and at the last, last-1 and last-2 order statistics.

Table 3. The reliability and failure functions for some special cases

Order statistics	k, m	Survival function	Hazard rate
First	k=1	$1-\left[\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right]$	$\dfrac{\theta}{1-\eta e^{-\theta x}}$
Second	k=2	$1-\left[\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right]^2$	$\dfrac{2\theta(1-\eta)e^{-\theta x}(1-e^{-\theta x})}{[1-\eta e^{-\theta x}]^3}\left[1-\left(\dfrac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right)^2\right]^{-1}$

Third	k=3	$1-\left[\frac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right]^3$	$\frac{3\theta(1-\eta)e^{-\theta x}(1-e^{-\theta x})^2}{[1-\eta e^{-\theta x}]^4}\left[1-\left(\frac{1-e^{-\theta x}}{1-\eta e^{-\theta x}}\right)^3\right]^{-1}$
Last	m=0	$\frac{e^{-\theta x}}{1-\eta(1-e^{-\theta x})}$	$\frac{\theta(1-\eta)}{1-\eta(1-e^{-\theta x})}$
Last -1	m=1	$\left[\frac{e^{-\theta x}}{1-\eta(1-e^{-\theta x})}\right]^2$	$\frac{2\theta(1-\eta)}{1-\eta(1-e^{-\theta x})}$
Last-2	m=2	$\left[\frac{e^{-\theta x}}{1-\eta(1-e^{-\theta x})}\right]^3$	$\frac{3\theta(1-\eta)}{1-\eta(1-e^{-\theta x})}$

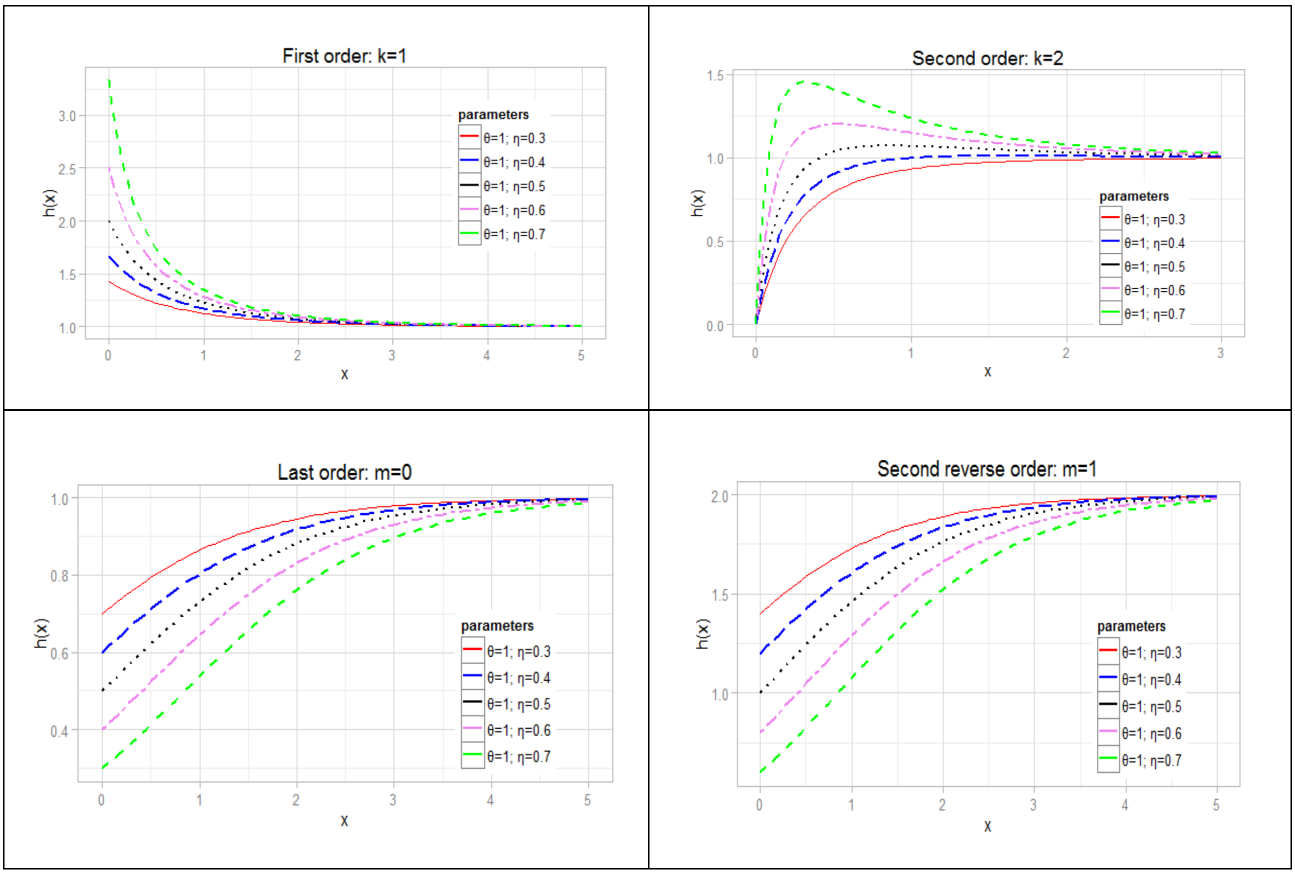


Figure 2. Hazard rate of the EGTG distribution for k=(1,2) and m=(0,1)

2.4. Random Number Generation

We can generate a random variable from the cdf of x in equation (5) using the following steps:

- Generate a random variable U from the standard uniform distribution.
- Calculate the values of X such as:

$$X = \begin{cases} \dfrac{1}{\theta}\ln\left[\dfrac{1-\eta U^{\frac{1}{k}}}{1-U^{\frac{1}{k}}}\right] & k=1,2,\dots,z \\ \dfrac{1}{\theta}\ln\left[\dfrac{1-\eta(1-U)^{\frac{1}{m+1}}}{(1-\eta)(1-U)^{\frac{1}{m+1}}}\right] & k=z,z-1,\dots,1 \;\; and\; m=z-k \end{cases} \tag{10}$$

We can determine the quantiles by dividing the set of observations into equal sized groups. The median is computed by letting $U = 0.5$. The equation (10) can be used for a simulation study of the EGTG distribution.

2.5. Entropy

Entropy is a measure of the uncertainty that was introduced by Shannon (1948) as a basic concept in information and communication theory, measuring the average missing information on a random source (Lesne, 2014). Shannon defined the entropy as:

$$H_b(X) = E\left[log_b\left(\frac{1}{g(x)}\right)\right] = \int_0^\infty g(x)\, log_b\left(\frac{1}{g(x)}\right)dx \;= -\int_0^\infty g(x)\,[log_b g(x)] \tag{11}$$

$log_b\left(\frac{1}{g(x)}\right)$ is also called an uncertainty. The probability distribution already describes the probability characteristics of a random variable. However, if we have two or more probability distributions we do not know exactly which variable is more random than the other. In this case, a comparison between probability distributions is possible thanks to entropy whereas a probability distribution describes the randomness of one random variable. The entropy appears as the average information required to specify the outcome X when the distribution $g(x)$ is known. It equivalently measures the amount of uncertainty represented by a probability distribution (Jaynes, 1957). When the entropy is large this means that the uncertainty associated to the random variable is large, and vice versa. Let the entropy of a random variable X_1 be 0.8 and that of another variable X_2 be 0.1. It is observed that the first random variable is more uncertain than the second.

From equation (11), the entropy for the new family distribution is given by:

$$H(x) = \begin{cases} -\log[\theta k(1-\eta)] + k(k-1)(1-\eta)\displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\eta^{j}}{i}\frac{(k+j)!}{j!k!}B(i+j+1,k) \\ -k(k+1)(1-\eta)\displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\eta^{i+j}}{i}\frac{(k+j)!}{j!k!}B(i+j+1,k) \;;k=1,2,\dots,z \\ \\ -\log[\theta(m+1)(1-\eta)] + (m+1)\theta\mu - (m+1)(m+2)(1-\eta) \\ \displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\eta^{i+j}}{i}\frac{(m+j+1)!}{j!(m+1)!}B(i+j+1,m+1) \quad for, k=z,z-1,\dots,1 \;\; and\; m=z-k \end{cases} \tag{12}$$

where $\mu = E(X)$

Equation (12) expresses the mathematical expectation of uncertainty. Similar to the mean and variance of a random variable, entropy is a derived quantity from probability distribution, but it has a value of its own (Zong, 2006). If we are seeking a pdf subject to certain constraints (r^{th}moment), we choose the density satisfying the constraints and having entropy as large as possible. The principle of maximum entropy states that we should choose the probability distribution that maximizes the uncertainty subject to some constrains about the random variable X. This distribution should be least surprising in terms of the predictions it makes.

Entropy can also be used to produce a model for data-generating distribution in terms of information constraints. Given the prior information about the pdf $g(x,\eta,\theta)$, one may estimate the parameters by maximizing the entropy index $H(X)$ or equivalently, minimizing the log-loss.

In the application examples, we will use the entropy as a statistic in assessment of distribution, by testing the hypothesis that the data follow a specific distribution. Indeed, it measures how well a probability distribution fits a set of observed data (measure of the inherent randomness). Given the estimated parameters $\hat{\eta}$ and $\hat{\theta}$, we can calculate the entropy $H(X)$.

3. Estimation

In this section, we will determine the estimated parameters $\hat{\eta}$ and $\hat{\theta}$ for the EGTG new family of lifetime distributions. Let $(X_1, X_2, \dots, X_n)$ be a random sample with observed values $(x_1, x_2, \dots, x_n)$ from the EGTG distribution with the pdf in equation (4). The log–likelihood function given the observed values, $x_{obs} = (x_1, \dots, x_n)$, is:

$$\ln L(\eta,\theta / x_{obs}) = \begin{cases} n\ln(k) + n\ln(1-\eta) + n\ln(\theta) + (k-1)\sum_{i=1}^{n}\ln(1-e^{-\theta x_i}) \\ -\theta\sum_{i=1}^{n} x_i - (k+1)\sum_{i=1}^{n}\ln(1-\eta e^{-\theta x_i}) \quad k = 1,2,\dots,z \\ \\ n\ln(m+1) + n\ln(1-\eta) + n\ln(\theta) - \theta(m+1)\sum_{i=1}^{n} x_i \\ -(m+2)\sum_{i=1}^{n}\ln[1-\eta(1-e^{-\theta x_i})] \qquad k = z, z-1,\dots,1 \;\; and \; m = z-k \end{cases} \tag{13}$$

We subsequently derive the associated gradients:

$$\frac{\partial}{\partial\eta}\ln L(\eta,\theta / x) = \begin{cases} \dfrac{-n}{(1-\eta)} + (k+1)\sum_{i=1}^{n}\dfrac{e^{-\theta x_i}}{1-\eta e^{-\theta x_i}} \qquad k = 1,2,\dots,z \\ \dfrac{-n}{(1-\eta)} + (m+2)\sum_{i=1}^{n}\dfrac{1-e^{-\theta x_i}}{1-\eta(1-e^{-\theta x_i})} \; ; k = z, z-1,\dots,1 \;\; and \; m = z-k \end{cases} \tag{14}$$

$$\frac{\partial}{\partial\theta}\ln L(\eta,\theta / x) = \begin{cases} \dfrac{n}{\theta} - \sum_{i=1}^{n} x_i + (k-1)\sum_{i=1}^{n}\dfrac{x_i}{e^{\theta x_i}-1} - (k+1)\sum_{i=1}^{n}\dfrac{\eta x_i}{e^{\theta x_i}-\eta} \\ \qquad k = 1,2,\dots,z \\ \dfrac{n}{\theta} - (m+1)\sum_{i=1}^{n} x_i + (m+2)\sum_{i=1}^{n}\dfrac{\eta x_i}{e^{\theta x_i}(1-\eta)+\eta} \\ \qquad k = z, z-1,\dots,1 \;\; and \; m = z-k \end{cases} \tag{15}$$

We need the Fisher information matrix for interval estimation and tests of hypotheses on the parameters. It can be expressed in terms of the second derivatives of the log-likelihood function:

$$I=\begin{pmatrix} I_{11}=E\left(-\frac{\partial^2 lnL}{\partial\eta^2}\right) & I_{12}=E\left(-\frac{\partial^2 lnL}{\partial\eta\,\partial\theta}\right) \\ I_{21}=E\left(-\frac{\partial^2 lnL}{\partial\theta\,\partial\eta}\right) & I_{22}=E\left(-\frac{\partial^2 lnL}{\partial\theta^2}\right)\end{pmatrix}$$

For the first definition,

$$\frac{\partial^2 lnL}{\partial\eta^2}=\frac{-n}{(1-\eta)^2}-(k+1)\sum_{i=1}\frac{e^{-2\theta x_i}}{(1-\eta e^{-\theta x_i})^2}$$

$$\frac{\partial^2 lnL}{\partial\theta^2}=\frac{-n}{\theta^2}+(k-1)\sum_{i=1}\frac{x_i^2 e^{\theta x_i}}{(e^{\theta x_i}-1)^2}-(k+1)\sum_{i=1}\frac{\eta x_i^2 e^{\theta x_i}}{(e^{\theta x_i}-\eta)^2}$$

$$\frac{\partial^2 lnL}{\partial\eta\,\partial\theta}=(k+1)\sum_{i=1}\frac{-x_i e^{-\theta x_i}}{(1-\eta e^{-\theta x_i})^2}$$

For the second definition,

$$\frac{\partial^2 lnL}{\partial\eta^2}=\frac{-n}{(1-\eta)^2}-(m+2)\sum_{i=1}\frac{1-e^{-\theta x_i}}{[1-\eta(1-e^{-\theta x_i})]^2}$$

$$\frac{\partial^2 lnL}{\partial\theta^2}=\frac{-n}{\theta^2}-(m+2)\sum_{i=1}\frac{\eta(1-\eta)x_i^2 e^{\theta x_i}}{[e^{\theta x_i}(1-\eta)+\eta]^2}$$

$$\frac{\partial^2 lnL}{\partial\eta\,\partial\theta}=(m+2)\sum_{i=1}\frac{x_i e^{-\theta x_i}}{[e^{\theta x_i}(1-\eta)+\eta]^2}$$

The maximum likelihood estimates (MLEs) $\hat{\eta}$ and $\hat{\theta}$ of the EGTG parameters η and θ, respectively, can be determined numerically by solving the nonlinear equations (14) and (15) of the associated gradients, using a statistical software (it can be easily done using R, Mathcad and Matlab packages, among others). The choice of a good set of initial values is essential. The MLEs can also be found analytically using the iterative EM algorithm to handle the incomplete data problems (Dempster et al., 1977; McLachlan and Krishnan, 1997). The iterative method consists on repeatedly updating the parameter estimates by replacing the "missing data" with the new estimated values. The standard method used to determine the MLEs is the Newton-Raphson algorithm that requires second derivatives of the log-likelihood function for all iterations. The main drawback of the EM algorithm is its rather slow convergence, compared to the Newton-Raphson method, when the "missing data" contain relatively large amount of information (Little and Rubin, 1983). Recently, several researchers have used the EM method such as Adamidis et al. (2005), Karlis (2003), Ng et al. (2002), Adamidis and Loukas (1998), Adamidis (1999), among others. Newton-Raphson is required for the M-step of the EM algorithm.

To start the EM algorithm, we define a hypothetical distribution of complete-data with the joint density function in equation (3). We drive the conditional mass function as:

$$p(z|x,\eta,\theta)=\begin{cases}\dfrac{\eta^{z-k}\Gamma(z+1)e^{-\theta(z-k)x}(1-\eta e^{-\theta x})^{k+1}}{\Gamma(z-k+1)\Gamma(k+1)} & k=1,2,\ldots,z\\[2ex] \dfrac{\Gamma(z+1)[1-\eta(1-e^{-\theta x})]^{m+2}\eta^{z-m-1}}{\Gamma(z-m)\Gamma(m+2)(1-e^{-\theta x})^{-z+m+1}} & k=z,z-1,\ldots,1 \ \ and\ m=z-k\end{cases} \tag{16}$$

E-step:

$$E(z|x,\eta,\theta)=\begin{cases}\dfrac{k+\eta e^{-\theta x}}{1-\eta e^{-\theta x}} & k=1,2,\ldots,z\\[2ex] m+\dfrac{1}{1-\eta(1-e^{-\theta x})} & k=z,z-1,\ldots,1 \ \ and\ m=z-k\end{cases} \tag{17}$$

M-step:

$$\eta^{(r+1)} = \begin{cases} (k+1)\dfrac{\sum_{i=1}^{n}\eta^{(r)}e^{-\theta^{(r)}x_i}}{\sum_{i=1}^{n}(1-\eta^{(r)}e^{-\theta^{(r)}x_i})} & k=1,2,\ldots,z \\ \dfrac{\sum_{i=1}^{n}\eta^{(r)}(1-e^{-\theta^{(r)}x_i})}{n-(n-1)\sum_{i=1}^{n}\eta^{(r)}(1-e^{-\theta^{(r)}x_i})} & k=z,z-1,\ldots,1 \ \ and\ m=z-k \end{cases} \tag{18}$$

$$\theta^{(r+1)} = \begin{cases} \dfrac{n}{(1-k)\left(\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\dfrac{\theta^{(r+1)}}{e^{\theta^{(r+1)}x_i}-1}\right)+\sum_{i=1}^{n}x_i\left(\dfrac{k+\eta^{(r)}e^{-\theta^{(r)}x_i}}{1-\eta^{(r)}e^{-\theta^{(r)}x_i}}\right)} & k=1,2,\ldots,z \\ \dfrac{n}{(m+1)\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left(\dfrac{\eta^{(r)}(1-e^{-\theta^{(r)}x_i})}{1-\eta^{(r)}(1-e^{-\theta^{(r)}x_i})}\dfrac{\theta^{(r+1)}e^{-\theta^{(r+1)}x_i}}{1-e^{-\theta^{(r+1)}x_i}}\right)} & k=z,z-1,\ldots,1 \ \ and\ m=z-k \end{cases} \tag{19}$$

4. Simulation Study

As an illustration of the MLEs, numerical computations have been performed using the steps presented in section 2.4 for random number generation. The numerical study was based on 1000 random samples of the sizes 20, 50 and 100 from the EGTG distribution for the ascending and descending order statistics with cases $k = (1;2)$ and $m = (0;1)$, respectively. We have considered the initial values of (η,θ): $(0.1\,,0.5)$, $(0.3\,,0.5)$, $(0.5\,,1)$, $(0.1\,,1.5)$, $(0.7\,,1.5)$. After determining the parameter estimates $\hat{\lambda} = (\hat{\eta},\hat{\theta})$ we compute the biases, variances and mean square errors (MSEs), where $MSE(\hat{\lambda}) = E(\hat{\lambda}-\lambda)^2 = Bias^2(\hat{\lambda}) + var(\hat{\lambda})$ and $Bias(\hat{\lambda}) = E(\hat{\lambda}) - \lambda$. An estimator $\hat{\lambda}$ is said to be efficient if its mean square error (MSE) is minimum among all competitors. In fact, $\hat{\lambda}_1$ is more efficient than $\hat{\lambda}_2$ if $MSE(\hat{\lambda}_1) < MSE(\hat{\lambda}_2)$.

Table 4 reports the results from the simulated data where the variances and the MSEs of the estimated parameters are given. The results show that, for each case, the convergence has been achieved. Indeed, the estimated parameters $\hat{\lambda} = (\hat{\eta},\hat{\theta})$ approach to their real values when the size of the sample increases. the variances and the MSEs decrease and converge to zero when the sample size increases, which may suggest that the MLEs are performed consistently.

Table 4. Results from the simulation study

n	(η, θ)	Order statistics							Reverse order statistics						
		k	$\hat{\eta}$	$\hat{\theta}$	$var(\hat{\eta})$	$var(\hat{\theta})$	$MSE(\hat{\eta})$	$MSE(\hat{\theta})$	m	$\hat{\eta}$	$\hat{\theta}$	$var(\hat{\eta})$	$var(\hat{\theta})$	$MSE(\hat{\eta})$	$MSE(\hat{\theta})$
20	**(0.1, 0.5)**		0.1678	0.5035	0.0587	0.0233	0.0633	0.0233	**m=0**	0.3142	0.6515	0.0875	0.0511	0.1334	0.074
	(0.3, 0.5)	**k=1**	0.2422	0.5472	0.0754	0.0365	0.0788	0.0387		0.4017	0.6195	0.0909	0.0449	0.1013	0.0592
	(0.5, 1)		0.3500	1.2167	0.0915	0.2634	0.1140	0.3103		0.5248	1.1845	0.0811	0.1515	0.0817	0.1855
	(0.1, 1.5)		0.1835	1.4525	0.0670	0.2053	0.0740	0.2075		0.3036	1.9180	0.0891	0.5230	0.1305	0.6977
	(0.7, 1.5)		0.5175	2.0487	0.0985	1.4007	0.1318	1.7018		0.6864	1.6633	0.0560	0.3062	0.0562	0.3328
50	**(0.1, 0.5)**	**k=1**	0.1638	0.4824	0.0441	0.0106	0.0482	0.0109	**m=0**	0.2239	0.5629	0.0526	0.0171	0.0679	0.0211
	(0.3, 0.5)		0.2532	0.5217	0.0561	0.017	0.0583	0.0175		0.3338	0.5414	0.0606	0.0165	0.0618	0.0183
	(0.5, 1)		0.4076	1.1051	0.0609	0.1228	0.0694	0.1339		0.4890	1.0550	0.0534	0.0601	0.0535	0.0631
	(0.1, 1.5)		0.1545	1.4721	0.0407	0.0949	0.0436	0.0957		0.2272	1.6973	0.0526	0.1543	0.0688	0.1932
	(0.7, 1.5)		0.6043	1.7898	0.046	0.5401	0.0552	0.6241		0.6839	1.5597	0.0266	0.1067	0.0269	0.1102
100	**(0.1, 0.5)**	**k=1**	0.1341	0.4917	0.0270	0.0052	0.0282	0.0053	**m=0**	0.1769	0.5363	0.0304	0.0064	0.0363	0.0077
	(0.3, 0.5)		0.2657	0.5109	0.0419	0.0098	0.0431	0.0099		0.3128	0.5214	0.0369	0.0069	0.0371	0.0074
	(0.5, 1)		0.4412	1.0558	0.0418	0.0680	0.0452	0.0711		0.4954	1.0306	0.0266	0.0252	0.0267	0.0262
	(0.1, 1.5)		0.1437	1.4636	0.028	0.0474	0.0300	0.0488		0.1792	1.6108	0.0349	0.0653	0.0412	0.0776
	(0.7, 1.5)		0.6543	1.6344	0.0191	0.2342	0.0212	0.2523		0.6960	1.5368	0.0108	0.0480	0.0108	0.0493
20	**(0.1, 0.5)**	**k=2**	0.1573	0.4945	0.0500	0.0134	0.0533	0.0134	**m=1**	0.3455	0.7500	0.0964	0.138	0.1566	0.2005
	(0.3, 0.5)		0.2359	0.5293	0.0649	0.0210	0.0690	0.0218		0.4212	0.6930	0.0971	0.1098	0.1117	0.1471
	(0.5, 1)		0.3637	1.1446	0.0785	0.1533	0.0971	0.1742		0.5310	1.2839	0.0868	0.3306	0.0878	0.4112
	(0.1, 1.5)		0.1526	1.4825	0.0476	0.1151	0.0504	0.1154		0.3342	2.1769	0.0923	1.0073	0.1472	1.4654
	(0.7, 1.5)		0.5510	1.8997	0.0732	0.7110	0.0954	0.8708		0.6845	1.7542	0.0525	0.4238	0.0528	0.4884
50	**(0.1, 0.5)**	**k=2**	0.1444	0.4900	0.0336	0.0065	0.0355	0.0066	**m=1**	0.2496	0.6056	0.0606	0.0306	0.083	0.0418
	(0.3, 0.5)		0.2592	0.5133	0.0482	0.0107	0.0499	0.0108		0.3500	0.5713	0.0659	0.0284	0.0684	0.0335
	(0.5, 1)		0.4270	1.0669	0.0489	0.0735	0.0543	0.0780		0.4958	1.0940	0.0569	0.0970	0.0569	0.1059
	(0.1, 1.5)		0.148	1.4667	0.0335	0.0554	0.0358	0.0565		0.2496	1.8167	0.0606	0.2756	0.0830	0.3758
	(0.7, 1.5)		0.6354	1.6692	0.0286	0.282	0.0328	0.3106		0.6896	1.5972	0.0251	0.1537	0.0252	0.1632
100	**(0.1, 0.5)**	**k=2**	0.1256	0.4936	0.0223	0.0034	0.0230	0.0034	**m=1**	0.1962	0.5605	0.0380	0.0135	0.0472	0.0172
	(0.3, 0.5)		0.2783	0.5040	0.0340	0.0065	0.0345	0.0066		0.3180	0.5352	0.0441	0.0138	0.0444	0.0150
	(0.5, 1)		0.4695	1.0223	0.0262	0.0393	0.0271	0.0398		0.4917	1.0449	0.0334	0.0475	0.0334	0.0495
	(0.1, 1.5)		0.1402	1.4633	0.0256	0.0335	0.0272	0.0348		0.1962	1.6815	0.0380	0.1215	0.0472	0.1545
	(0.7, 1.5)		0.6699	1.5811	0.0113	0.1229	0.0122	0.1294		0.6943	1.5485	0.0117	0.0735	0.0118	0.0759

4. Application Examples

In this section, we fit the EGTG distribution to two real data sets using the MLEs determined numerically by direct integration using Mathcad 14.0. The first set (table A in appendix) consists of "107 failure times for right rear brakes on D9G-66A caterpillar tractors", reproduced from Barlow and Campo (1975) and used also by Chang and Rao (1993). These data are used in many applications in reliability (Adamidis et al., 2005; Tsokos, 2012; Shahsanaei et al., 2012).

The second set of data involves 100 observations (table B in appendix) of the results from an experiment concerning "the tensile fatigue characteristics of a polyester/viscose yarn". These data were presented by Picciotto (1970) to study the problem of warp breakage during weaving. The "observations were obtained on the cycles to failure of a 100 cm yarn sample put to test under 2.3% strain level". The sample is used in Quesenberry and Kent (1982) as an example to illustrate selection procedure among probability distributions used in reliability. The reliability function of these two data sets belongs to the increasing failure rate class (Doksum and Yandell, 1984; Adamidis et al., 2005).

We use different statistical tests to assess the agreement between the EGTG distribution and the data sets. In addition to our class of distributions, the gamma and Weibull distributions fitted these data sets. The respective densities of gamma and Weibull distributions are:
$f_1(x,\lambda_1,\beta_1)={\lambda_1}^{\beta_1}x^{\beta_1-1}\exp(-\lambda_1 x)\Gamma(\beta_1)^{-1}$ and $f_2(x,\lambda_2,\beta_2)=\beta_2{\lambda_2}^{\beta_2}x^{\beta_2-1}\exp(-\lambda_2 x)^{\beta_2}$.
Tables 5 and 6 show the fitted parameters, calculated values of Kolmogorov–Smirnov (K-S) and their respective p-values for the two data sets. It should be noted that the K-S test compares an empirical distribution with a known (not estimated) one. It is used to decide if a sample comes from a population with a specific distribution (H0: the data follow a specified distribution). We estimate some special cases of the EGTG distribution at 5% significant level. The tables report also the AIC and BIC information criteria and the Shannon's entropy (H) for model selection. Table (7) gives the means and the standard errors for some special cases of the EGTG distribution, compared to their empirical values.

In order to identify empirical behaviors that the failure rate function can take, we shall consider the graphical method based on the total test on time (TTT-plot) proposed by Aarset (1987). In its empirical version the TTT-plot is constructed by values $r|n$ and $G(r|n)$, where

$$G(r|n)=\frac{\sum_{i=1}^{r}X_{i:n}+(n-r)X_{r:n}}{\sum_{i=1}^{n}X_{i:n}}$$

where $r=1,\dots,n$ and $X_{i:n}$ represents the order statistics of the sample. The graphic TTT may have various forms. It resembles to the Gini index and it is used as a crude indicative of the shape of the failure rate function. Indeed, when the curve approaches a straight diagonal function, constant failure rate is adequate and the data are from an exponential distribution. When the curve is approximately concave or convex, the data are from IFR distribution or DFR distribution, respectively.

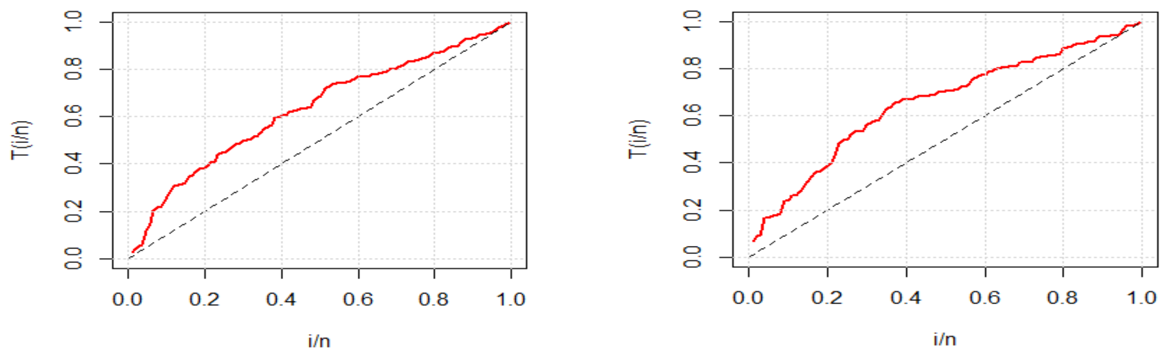
Figure (3) shows concave tendencies indicating that the two data sets exhibit IFR distributions. This result is in agreement with the MLEs of the shape parameters and the K-S test. The p-values are only significant for the case $k=1$ for the two data sets. As shown in section 2.3, If k = 1, the EGTG is DFR, but here the data exhibit an increasing hazard rate. Tables (5) and (6) show smallest values of K-S statistics for the last order statistics ($k=z$) with largest associated *p-values* equal to 0.9899 and 0.9538, respectively. The K-S distances between the empirical distribution function of the two samples and the cdf of the corresponding distribution are respectively 0.0426 and 0.0515. We can see that the new lifetime family provides good fit to the data sets. The K-S test shows that the EGTG distribution is an attractive alternative to the popular gamma and Weibull distributions.

Table 5. The goodness of fit for some special cases, for the first data set (107 obs.)

Distributions	**ML Estimates** $\hat{\theta}$	$\hat{\eta}$	**K-S**	**p-value**	**Log-lik**	**AIC**	**BIC**	**H**
EGTG:								
• First order $(k=1)$	5.00×10^{-4}	0.0001	0.1572	0.0101	-921.5867	1847.1734	1847.2321	8.601
• Second order $(k=2)$	6.69×10^{-4}	0.1738	0.0662	0.6850	-911.7965	1827.593	1827.6517	8.550
• Third order $(k=3)$	7.56×10^{-4}	0.3485	0.0707	0.6582	-913.9317	1831.8634	1831.9221	8.467
• Fourth order $(k=4)$	8.57×10^{-4}	0.3970	0.0765	0.5575	-918.7038	1841.4076	1841.4663	8.390
• Last order $(\mathrm{k}=\mathrm{z})$	1.03×10^{-3}	0.8146	0.0426	0.9899	-909.7043	1823.4086	1823.4673	8.495
• Last order-1 $(\mathrm{k}=\mathrm{z}-1)$	6.56×10^{-4}	0.8137	0.0537	0.5559	-911.2673	1826.5346	1826.5933	8.507
• Last order-2 $(\mathrm{k}=\mathrm{z}-2)$	5.25×10^{-4}	0.8310	0.0625	0.7971	-912.0977	1828.1954	1828.2541	8.516
• Last order-3 $(\mathrm{k}=\mathrm{z}-3)$	4.54×10^{-4}	0.8461	0.0680	0.7046	-912.5824	1829.1648	1829.2235	8.524
	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{\beta}}$						
Gamma	9.43×10^{-4}	1.9084	0.0680	0.7343	-910.6056	1825.211	1830.557	8.510
Weibull	4.32×10^{-4}	1.5006	0.0599	0.8363	-910.1225	1824.245	1829.591	8.534

Table 6. The goodness of fit for some special cases, for the second data set (100 obs).

Distributions	**ML Estimates** $\hat{\theta}$	$\hat{\eta}$	**K-S**	**p-value**	**Log-lik**	**AIC**	**BIC**	**H**
EGTG:								
• First order $(k=1)$	4.51×10^{-3}	0.0002	0.1905	0.0000	-640.2614	1284.5228	1284.5815	6.401
• Second order $(k=2)$	6.47×10^{-3}	0.0521	0.1050	0.2197	-626.6579	1257.3158	1257.3745	6.329
• Third order $(k=3)$	7.86×10^{-3}	0.1124	0.0856	0.4557	-626.5988	1257.1976	1257.2563	6.216
• Fourth order $(k=4)$	9.09×10^{-3}	0.1134	0.0785	0.5680	-630.6498	1265.2996	1265.3583	6.123
• Last order $(k=z)$	10.59×10^{-3}	0.8657	0.0515	0.9538	-625.1210	1254.242	1254.3007	6.236
• Last order-1 $(k=z-1)$	6.92×10^{-3}	0.8627	0.0669	0.7610	-627.0645	1258.129	1258.1877	6.257
• Last order-2 $(k=z-2)$	5.52×10^{-3}	0.8702	0.0742	0.6398	-628.1569	1260.3138	1260.3725	6.273
• Last order-3 $(k=z-3)$	4.78×10^{-3}	0.8797	0.0779	0.5787	-628.8036	1261.6072	1261.6659	6.281
	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{\beta}}$						
Gamma	10.8×10^{-3}	2.2383	0.1326	0.0594	-625.2443	1254.4890	1259.6990	6.184
Weibull	4.02×10^{-3}	1.6060	0.0738	0.6468	-625.2000	1254.4000	1259.7460	2.260



Barlow and Campo (1975) data set Quesenberry and Kent (1982) data set

Figure 3. Empirical TTT-plot

For models' comparison, we compute the Akaike's information criterion ($AIC = -2lnL + 2\,p$) and Schawarz's Bayesian information criterion ($BIC = -2lnL + p\,log(n)$), where n is the size of the sample and p is the number of parameters. The results indicate that the last order statistic $(k = z)$ has the smallest AIC and BIC values. Then, the maximum lifetime distribution could be commonly chosen as the preferred model for describing the two data sets. The entropy index shows that our distribution is a good alternative in estimating lifetime data.

Figure (4) illustrates the fitted models and the observed histograms and figure (5) shows the probability-probability plots for the two data sets. The plots corroborate the previous results and confirm the good performance of our distribution. The diagonal is the reference line in the PP-plot.

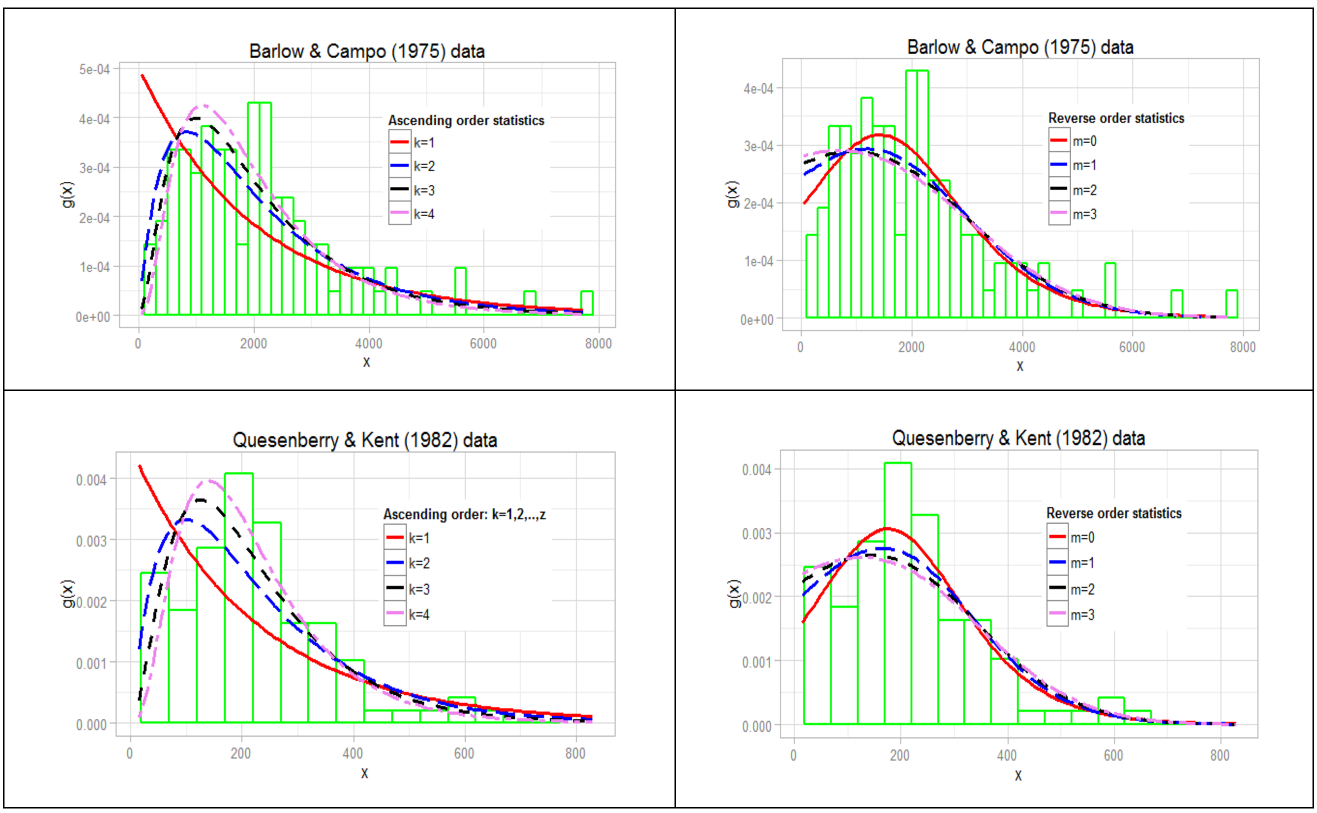


Figure 4. Fitted models and observed histograms

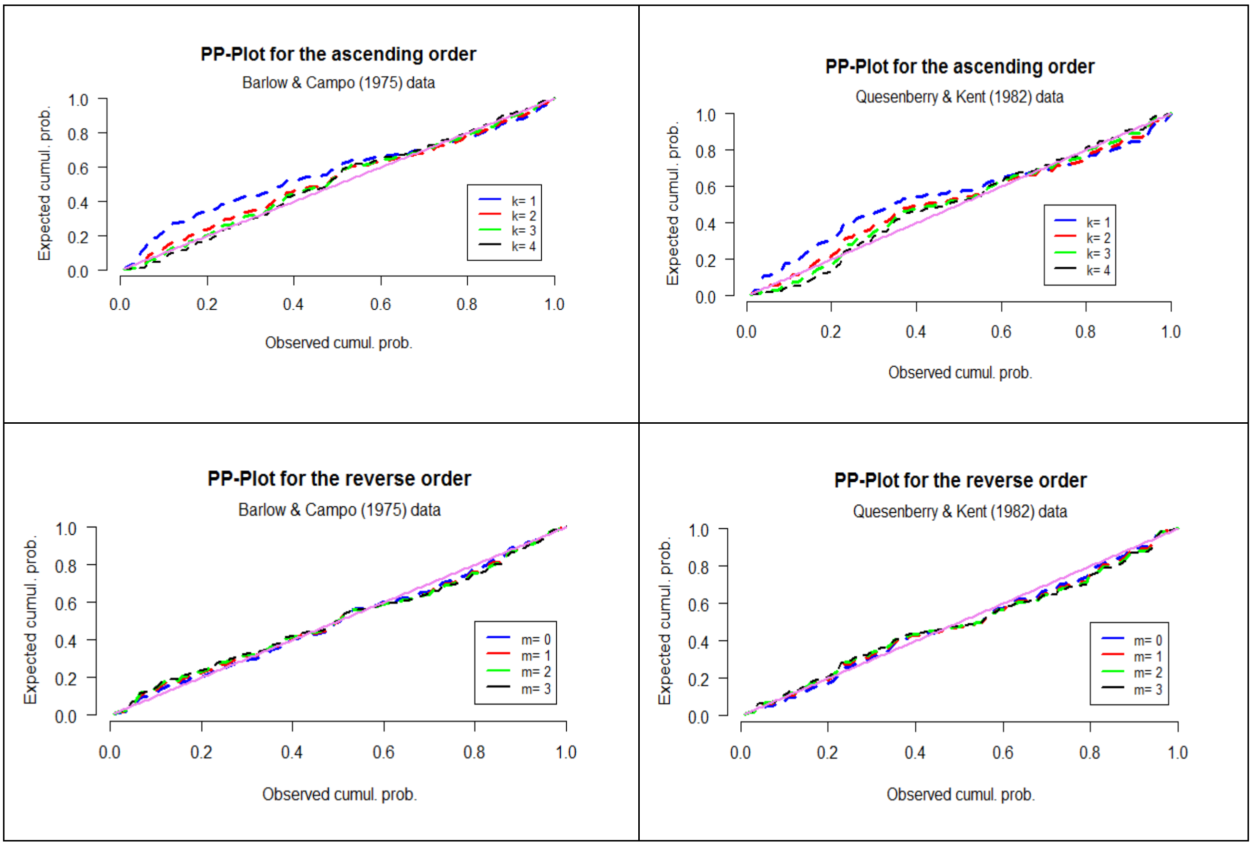


Figure 5. The probability-probability plots for the data sets

Table 7. Means and standard errors for some special cases

	Barlow & Campo (1975) data set (107 obs.)		**Quesenberry & Kent (1982) data set (100 obs.)**	
	$E(X)$	$\sigma(X)$	$E(X)$	$\sigma(X)$
EGTG distribution				
• First order $(k = 1)$	1999.9	1.99×10^3	221.7	221.7×10^3
• Second order $(k = 2)$	2056.5	1.61×10^3	226.3	171.4×10^3
• Third order $(k = 3)$	2019.0	1.44×10^3	222.0	145.9×10^3
• Fourth order $(k = 4)$	1981.0	1.31×10^3	218.7	129.4×10^3
• Last order $(k = z)$	2008.5	1.35×10^3	219.0	136.4×10^3
• Last order-1 $(k = z - 1)$	1994.3	1.37×10^3	218.0	139.6×10^3
• Last order-2 $(k = z - 2)$	1993.9	1.39×10^3	218.0	143.0×10^3
• Last order-3 $(k = z - 3)$	1994.5	1.41×10^3	213.5	151.1×10^3
Empirical values	2024.3	1.39×10^3	221.9	143.89×10^3

6. Conclusion

In this paper we proposed the EGTG distribution, that generalizes the exponential-geometric (Adamidis and Loukas, 1998) and the extended (or complementary) exponential-geometric distribution (Adamidis et al., 2005; Louzada et al., 2011) in the minimum and maximum cases, respectively. The application study was illustrated based on two sets of real data used in many applications of reliability. We have shown that our proposed distribution is suitable for modelling the lifetime of any order statistics. Future research, that should be considered, includes the Bayesian approach with censored data.

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Appendix

Table (A): "Ordered Failure Times (in hours) of 107 Right Rear Brakes on D9G-66A Caterpillar Tractors" (Barlow and Campo, 1975; Chang and Rao, 1993)

56	753	1153	1586	2150	2624	3826	83	763	1154	1599	2156	2675	3995	104
806	1193	1608	2160	2701	4007	116	834	1201	1723	2190	2755	4159	244	838
1253	1769	2210	2877	4300	305	862	1313	1795	2220	2879	4487	429	897	1329
1927	2248	2922	5074	452	904	1347	1957	2285	2986	5579	453	981	1454	2005
2325	3092	5623	503	1007	1464	2010	2337	3160	6869	552	1008	1490	2016	2351
3185	7739	614	1049	1491	2022	2437	3191	661	1069	1532	2037	2454	3439	673
1107	1549	2065	2546	3617	683	1125	1568	2096	2565	3685	685	1141	1574	2139
2584	3756													

Table (B): "Results of Model Selection Program on Yarn Data" (Quesenberry and Kent, 1982)

86	146	251	653	98	249	400	292	131	169
175	176	76	264	15	364	195	262	88	264
157	220	42	321	180	198	38	20	61	121
282	224	149	180	325	250	196	90	229	166
38	337	65	151	341	40	40	135	597	246
211	180	93	315	353	571	124	279	81	186
497	182	423	185	229	400	338	290	398	71
246	185	188	568	55	55	61	244	20	284
393	396	203	829	239	286	194	277	143	198
264	105	203	124	137	135	350	193	188	236