A New Lindley-Burr XII Distribution: Model, Properties and Applications

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Abstract

A new distribution called the Lindley-Burr XII (LBXII) distribution is proposed and studied. Some structural properties of the new distribution including moments, conditional moments, distribution of the order statistics and Rényi entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. A simulation study to examine the bias and mean square error of the maximum likelihood estimators is presented and applications to real data sets in order to illustrate the usefulness of the new distribution are given.

Keywords: Lindley distribution, Burr XII distribution, generalized distribution, monte Carlo simulations, maximum likelihood estimation

1. Introduction

There are several generalizations of the Lindley distribution that are considered to be useful life models, and are suitable for modeling data with different types of hazard rate functions including increasing, decreasing, bathtub and unimodal. (Lindley, 1958) used a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. This mixture is called the Lindley (L) distribution. There are several generalizations of the Lindley distribution in the literature including the works by (Oluyede, Yang and Omolo, 2015) and (Nadarajah, Bakouch and Tahmasbi, 2011). (Oluyede and Yang, 2015) presented the beta generalized Lindley distribution. (Ghitany, Al-Mutairi, Balakrishnan and Al-Ezeni, 2013) presented a two-parameter power Lindley distribution. (Zakerzadeh and Dolati, 2009) studied an extension of the Lindley distribution. These models constitute flexible family of distributions in terms of the varieties of shapes and hazard functions. The cumulative distribution function (cdf) of the Lindley distribution is given by

$$G_L(x; \lambda) = 1 - \frac{\lambda + 1 + \lambda x}{\lambda + 1} e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{1}$$

The corresponding Lindley probability density function (pdf) is given by

$$g_L(x; \lambda) = \frac{\lambda^2 (1 + x)}{1 + \lambda} e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{2}$$

Lindley distribution is a combination of exponential and gamma distributions, that is $f(x; \lambda) = (1 - p) f_G(x; \lambda) + p f_E(x; \lambda)$ with $p = \frac{1}{1+\lambda}$, where $f_G(x; \lambda) \equiv GAM(2, \lambda)$, and $f_E(x; \lambda) \equiv EXP(\lambda)$. The cdf of the generalized Lindley (GL) distribution (Nadarajah et al., 2011) is given by

$$G_{GL}(x; \alpha, \lambda) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \exp(-\lambda x) \right]^{\alpha}, \tag{3}$$

and the corresponding pdf is given by

$$g_{GL}(x; \alpha, \lambda) = \frac{\alpha \lambda^2}{1 + \lambda} (1 + x) \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \exp(-\lambda x) \right]^{\alpha - 1} \exp(-\lambda x), \tag{4}$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$. This distribution is the exponentiated Lindley distribution.

We conisider the following generalization of the Lindley and Burr XII distributions. Let X_1 and X_2 be independent random variables with Lindley and Burr XII distributions, respectively. That is, $F_{X_1}(x) = 1 - \frac{\lambda + 1 + \lambda x}{\lambda + 1} e^{-\lambda x}$, and $F_{X_2}(x) = 1 - (1 + x^c)^{-k}$,

for $x > 0$, and $c, k > 0$, respectively, then the distribution of $X = \min(X_1, X_2)$ is the Lindley-Burr XII (LBXII) distribution with parameters $\lambda > 0$, $c > 0$, and $k > 0$. This model is good for modeling data with different types of hazard rate functions including increasing, decreasing and bathtub. The hazard rate function is also additive, see (Oluyede, Foya, Warahena-Liyanage and Huang, 2016). This paper employs the competing risk transformation involving both the Lindley and Burr XII distributions to obtain a new more flexible distribution to describe survival and reliability data. This transformation generalizes both the Lindley and Burr XII distribution. The combined distribution of Lindley and Burr XII is obtained from the product of the survival functions of the Lindley and Burr XII distributions via competing risk model. A motivation for developing this model is the advantages presented by this extended distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of the Lindley and Burr XII distributions in modeling lifetime data.

This paper is organized as follows. In section 2, some basic results, the LBXII distribution and its sub-models, hazard rate and quantile functions are presented. The moments, conditional moments, mean and median deviations are given in section 3. Section 4 contain some additional useful results on the distribution of order statistics and Rényi entropy. In section 5, results on the estimation of the parameters of the LBXII distribution via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in section 6. Applications are given in section 7, followed by some concluding remarks in section 8.

2. The Model, Sub-models, Hazard Rate and Quantile Functions

In this section, the LBXII distribution, some sub-models, quantile function, hazard rate function as well some graphs are presented. The cdf and pdf of the LBXII distribution are given by

$$G_{LBXII}(x; \lambda, c, k) = G(x; \lambda, c, k) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)^k}, \tag{5}$$

and

$$g_{LBXII}(x; \lambda, c, k) = \frac{(1 + x^c)^{-k}}{1 + \lambda} e^{-\lambda x} \left[\lambda^2 (1 + x) + \frac{kcx^{c-1}(1 + \lambda + \lambda x)}{1 + x^c} \right], \tag{6}$$

for $\lambda, c, k > 0$. If a random variable X has the LBXII distribution, we write $X \sim LBXII(\lambda, c, k)$. The parameters c and k are shape parameters and λ is a scale parameter. Note that the LBXII pdf can be written as follows:

$$g_{LBXII}(x; \lambda, c, k) = g_L(x; \lambda)\overline{G}_B(x; c, k) + \overline{G}_L(x; \lambda)g_B(x; c, k), \tag{7}$$

where $\overline{G}_B(x; c, k) = 1 - G(x; c, k)$, and $\overline{G}_L(x; \lambda) = 1 - G(x; \lambda)$ are the survival functions of Burr XII and Lindley distributions, respectively, and $g_L(x; \lambda)$, $g_B(x; c, k)$ are the pdf's of Lindley and Burr XII distributions.

Plots of the LBXII pdf shows different shapes including right skewed, almost symmetric and reverse J-shaped as indicated in Figure 1.

2.1 Sub-models of LBXII Distribution

In this subsection, we discuss some special models of the LBXII distribution.

- We obtain Lindley (L) distribution with $k = 0$.

- When $\lambda = 0$, we obtain the Burr XII (BXII) distribution.

- If $k = 1$, we obtain Lindley-Log-logistic (LLLoG) distribution.

- If $k = 1$, and $\lambda = 0$, we obtain Log-logistic (LLoG) distribution.

- If $c = 1$, we obtain Lindley-Lomax (LLomax) distribution.

- If $c = 1$, and $\lambda = 0$, we obtain Lomax (Lomax) distribution.

- If $c = k = 1$, we have the one parameter distribution with cdf

$$G(x; \lambda) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x)}, \tag{8}$$
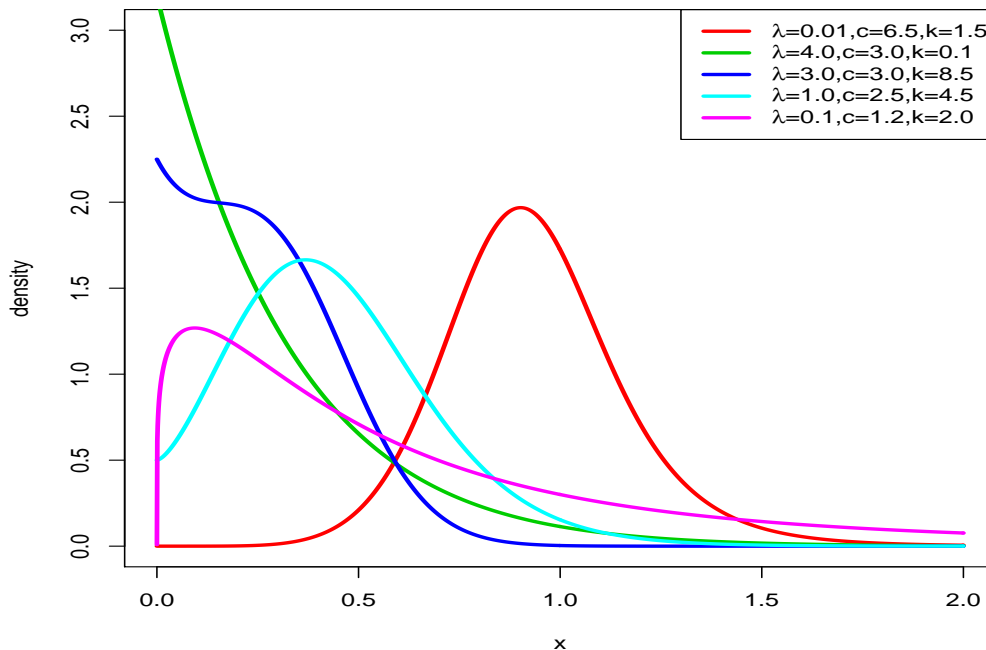
for $x > 0$, and $\lambda > 0$.

Figure 1. Plots of LBXII Density Function

2.2 Hazard Rate and Quantile Functions

In this section, we present the hazard rate and quantile functions of the LBXII distribution. We note from equation (7) that the hazard rate function of the LBXII distribution is given by

$$h_G(x) = h_{G_L}(x) + h_{G_B}(x), \tag{9}$$

where $h_{G_L}(x)$ and $h_{G_B}(x)$ are the hazard rate functions of the Lindley and Burr XII distributions, respectively, that is,

$$h_G(x) = \frac{g(x)}{\bar{G}(x)} = \frac{\lambda^2(1+x)}{1+\lambda+\lambda x} + (1+x^c)^{-1}kcx^{c-1}. \tag{10}$$

Plots of the LBXII hazard rate function shows different shapes including decreasing, increasing, as well as upside down bathtub shapes as shown in Figure 2. The quantile function of the LBXII distribution is obtained by solving the non-linear equation:

$$G(x; \lambda, c, k) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda} \frac{e^{-\lambda x}}{(1+x^c)^k} = u, \tag{11}$$

$0 \le u \le 1$, that is,

$$\lambda x + k \log(1+x^c) - \log\left(1 + \frac{\lambda x}{1+\lambda}\right) + \log(1-u) = 0. \tag{12}$$

Thus, random numbers can be readily generated from the LBXII distribution by numerically solving the non-linear equation (12). Quantiles of the LBXII distribution are given in Table 1.
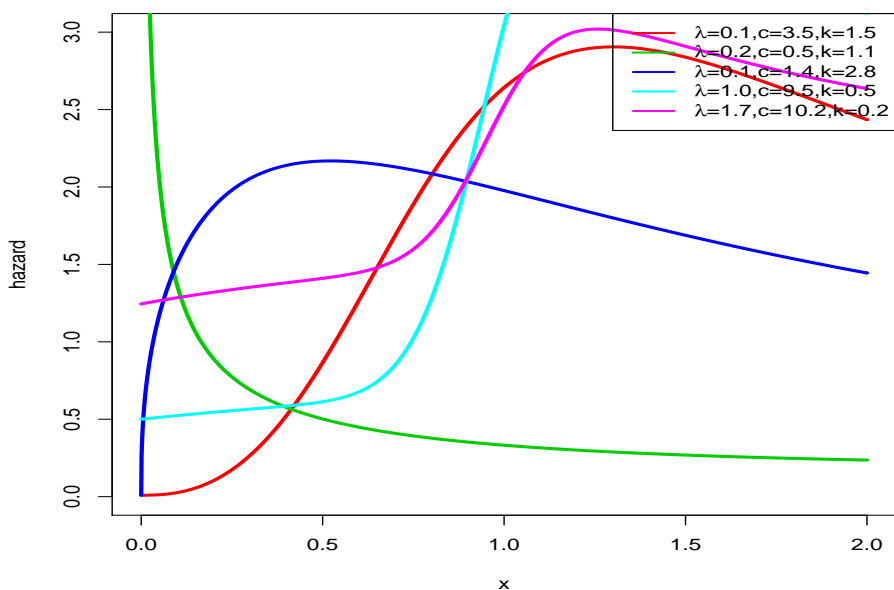
Figure 2. Plots of LBXII Hazard Rate Function

Table 1. Table of Quantile for LBXII Distribution

	(λ, c, k)				
u	$(0.9, 3.5, 5.5)$	$(0.7, 1, 1.5)$	$(1.3, 2, 4.5)$	$(3.1, 4.5, 3.5)$	$(1.2, 2.5, 5.5)$
0.1	0.1953	0.0602	0.0911	0.0447	0.1179
0.2	0.3037	0.1306	0.1544	0.0941	0.1943
0.3	0.3787	0.2143	0.2106	0.1494	0.2563
0.4	0.4406	0.3160	0.2652	0.2116	0.3128
0.5	0.4972	0.4434	0.3214	0.2822	0.3681
0.6	0.5529	0.6093	0.3825	0.3620	0.4256
0.7	0.6121	0.8387	0.4537	0.4519	0.4897
0.8	0.6814	1.1894	0.5448	0.5551	0.5681
0.9	0.7787	1.8529	0.6866	0.6867	0.6839

3. Moments, Conditional Moments and Mean Deviations

In this section, we present the moments, conditional moments and mean deviations of the LBXII distribution.

3.1 Moments

The rth moment of the LBXII distribution is given by

$$
\begin{aligned}
E(X^r) &= r \int_0^\infty x^{r-1} \frac{1 + \lambda + \lambda x}{1 + \lambda} \frac{e^{-\lambda x}}{(1 + x^c)^k} dx \\
&= \sum_{s=0}^\infty \frac{r(-1)^s \lambda^s}{s!} \int_0^\infty x^{r+s-1} \left(1 + \frac{\lambda x}{1 + \lambda}\right)(1 + x^c)^{-k} dx \\
&= \sum_{s=0}^\infty \frac{r(-1)^s \lambda^s}{s!c} \Big[\int_0^1 t^{k-1-\frac{1}{c}-\frac{r+s-1}{c}}(1-t)^{\frac{r+s-1}{c}+\frac{1}{c}-1} dt \\
&\quad + \frac{\lambda}{1+\lambda} \int_0^\infty t^{k-1-\frac{1}{c}-\frac{r+s}{c}}(1-t)^{\frac{r+s+1}{c}-1} dt \Big] \\
&= \sum_{s=0}^\infty \frac{r(-1)^s \lambda^s}{s!c} \Big[B\Big(k - \frac{r+s}{c}, \frac{r+s}{c}\Big) + \frac{\lambda}{1+\lambda} B\Big(k - \frac{r+s+1}{c}, \frac{r+s+1}{c}\Big) \Big],
\end{aligned} \tag{13}
$$

where we have applied the transformation $t = (1 + x^c)^{-1}$, and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the complete beta function. The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively. Some moments for selected parameter values for the LBXII distribution are given in Table 2 and 3D plots for skewness and kurtosis are presented in Figure 3, Figure 4 and Figure 5, respectively. The 3D plots shows the dependence of skewness and kurtosis on the shape parameters.

Table 2. Table of Moments for LBXII Distribution

Moments	(λ, c, k)				
	$(0.5, 0.5, 0.5)$	$(4.5, 2.7, 1.0)$	$(1.5, 1.5, 0.2)$	$(2.0, 2.0, 3.5)$	$(3.4, 4.0, 0.5)$
EX	0.1399	0.2347	0.2942	0.3178	0.2912
EX^2	0.0843	0.0970	0.1789	0.1505	0.1455
EX^3	0.0604	0.0526	0.1266	0.0872	0.0907
EX^4	0.0471	0.0334	0.0973	0.0574	0.0638
EX^5	0.0386	0.0235	0.0787	0.0412	0.0485
EX^6	0.0327	0.0177	0.0660	0.0314	0.0387
SD	0.0284	0.0139	0.0568	0.0251	0.0321
CV	0.0251	0.0115	0.0498	0.0207	0.0273
CS	0.0224	0.0098	0.0443	0.0176	0.0236
CK	0.0203	0.0083	0.0398	0.0152	0.0209
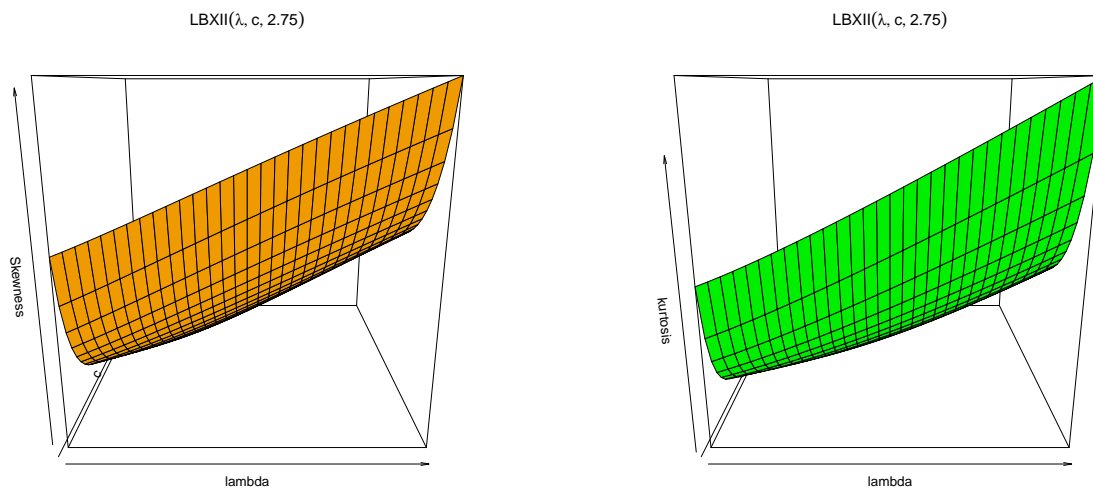
Figure 3. Plots of Skewness and Kurtosis for LBXII distribution

LBXII(0.05, c, k)

LBXII(0.05, c, k)
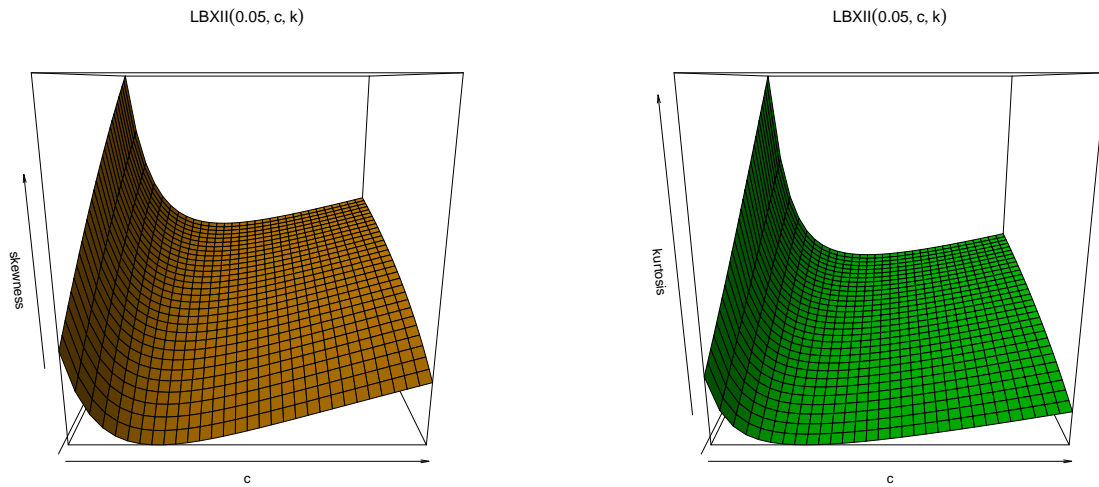
Figure 4. Plots of Skewness and Kurtosis for LBXII distribution

LBXII(λ, 1.5, k)

LBXII(λ, 1, k)

Figure 5. Plots of Skewness and Kurtosis for LBXII distribution

3.2 Conditional Moments

In this subsection, the rth conditional moment is presented. The mean residual life function as well as income inequality measures such as Bonferroni and Lorenz curves as well as mean deviation about the mean and mean deviation about the

median can be readily obtained from the conditional moments. The rth conditional moment is given by

$$
\begin{aligned}
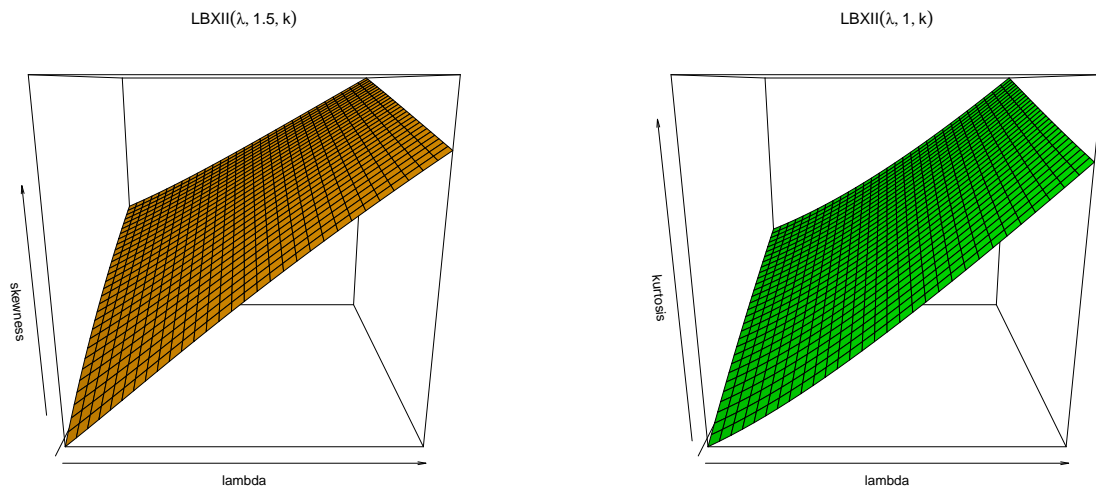E(X^r | X > t) &= \frac{1}{\overline{G}(t)} \int_t^\infty x^r \frac{(1 + x^c)^{-k}}{1 + \lambda} e^{-\lambda x} \left[\lambda^2 (1 + x) + \frac{(1 + \lambda + \lambda x)kcx^{c-1}}{1 + x^c} \right] dx \\
&= \frac{1}{\overline{G}(t)} \sum_{s=0}^\infty \frac{(-1)^s \lambda^s}{s!} \int_t^\infty x^{r+s} \frac{(1 + x^c)^{-k}}{1 + \lambda} \left[\lambda^2 (1 + x) + \frac{(1 + \lambda + \lambda x)kcx^{c-1}}{1 + x^c} \right] dx \\
&= \frac{1}{\overline{G}(t)} \sum_{s=0}^\infty \frac{(-1)^s \lambda^s}{s!} \left(\frac{\lambda^2}{1 + \lambda} \int_t^\infty x^{r+s}(1 + x^c)^{-k}(1 + x)dx \right. \\
&+ kc \int_t^\infty x^{r+s+c-1} \left(1 + \frac{\lambda x}{1 + \lambda} \right)(1 + x^c)^{-k-1}dx \bigg) \\
&= \frac{1}{\overline{G}(t)} \sum_{s=0}^\infty \frac{(-1)^s \lambda^s}{s!c} \left[\frac{\lambda^2}{1 + \lambda} \left(B_{(1+t^c)^{-1}} \left(k - \frac{r + s + 1}{c}, \frac{r + s + 1}{c} \right) \right. \right. \\
&+ B_{(1+t^c)^{-1}} \left(k - \frac{r + s + 2}{c}, \frac{r + s + 2}{c} \right) \bigg) \\
&+ kc \bigg(B_{(1+t^c)^{-1}} \left(k + 1 - \frac{r + s + c}{c}, \frac{r + s + c}{c} \right) \\
&+ \frac{\lambda}{1 + \lambda} B_{(1+t^c)^{-1}} \left(k - \frac{r + s + c + 1}{c}, \frac{r + s + c + 1}{c} \right) \bigg) \bigg],
\end{aligned}
$$

(14)

where $B_y(a, b) = \int_0^y t^{a-1}(1 - t)^{b-1}dt$ is the incomplete beta function.

3.3 Mean Deviation, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni Curves for the LBXII distribution are presented in this subsection. The application of Lorenz and Bonferroni curves is not limited to economics for the study of income and poverty, but they can also be used to other field like reliability, demography, insurance and medicine.

3.3.1 Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$
\delta_1(x) = \int_0^\infty |x - \mu| g_{LBXII}(x)dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x - M| g_{LBXII}(x)dx,
$$

(15)

respectively, where $\mu = E[X]$ and $M = Median(X)$ denotes the mean and median, respectively. We note that $\delta_1(x)$ and $\delta_2(x)$ can be expressed as $\delta_1(x) = 2\mu G_{LBXII}(\mu) - 2\mu + 2\int_\mu^\infty g_{LBXII}(x)dx$ and $\delta_2(x) = -\mu + 2\int_M^\infty x g_{LBXII}(x)dx$, respectively, that is,

$$
\delta_1(x) = 2\mu G_{LBXII}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(x) = 2T(M) - \mu,
$$

(16)

where

$$
\begin{aligned}
T(\mu) &= \int_\mu^\infty x g_{LBXII}(x)dx \\
&= \sum_{s=0}^\infty \frac{(-1)^s \lambda^s}{s!c} \left[\frac{\lambda^2}{1 + \lambda} \left(B_{(1+\mu^c)^{-1}} \left(k - \frac{s + 2}{c}, \frac{s + 2}{c} \right) + B_{(1+\mu^c)^{-1}} \left(k - \frac{s + 3}{c}, \frac{s + 3}{c} \right) \right) \right. \\
&+ kc \bigg(B_{(1+\mu^c)^{-1}} \left(k + 1 - \frac{1 + s + c}{c}, \frac{1 + s + c}{c} \right) + \frac{\lambda}{1 + \lambda} B_{(1+\mu^c)^{-1}} \left(k - \frac{s + c + 2}{c}, \frac{s + c + 2}{c} \right) \bigg) \bigg].
\end{aligned}
$$

(17)

3.3.2 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves for the LBXII didtribution are given as

$B(p) = \frac{1}{p\mu} \int_0^q x g_{LBXII}(x)dx = \frac{1}{p\mu}[\mu - T(q)]$, and $L(p) = \frac{1}{\mu} \int_0^q x g_{LBXII}(x)dx = \frac{1}{\mu}[\mu - T(q)]$, respectively,
where $T(q) = \int_q^\infty x g_{LBXII}(x)dx$, $q = G_{LBXII}^{-1}(p)$, $0 \le p \le 1$.

4. Order Statistics and Rényi Entropy

In this section, we present the pdf of the ith order statistic from the LBXII distribution as well as Rényi entropy.

4.1 Order Statistics

Suppose that $X_1, \cdots, X_n$ is a random sample of size n from a continuous pdf, $g(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. If $X_1, \cdots, X_n$ is a random sample from LBXII distribution, it follows from the equations (5) and (6) that the pdf of the ith order statistic from the LBXII distribution is given by

$$
\begin{aligned}
g_{i:n}(x) &= \frac{n!g_{LBXII}(x)}{(i-1)!(n-i)!}[G_{LBXII}(x)]^{i-1}[1 - G_{LBXII}(x)]^{n-i} \\
&= \frac{n!g_{LBXII}(x)}{(i-1)!(n-i)!}\sum_{s=0}^{n-i}\binom{n-i}{s}(-1)^s[G_{LBXII}(x)]^{s+i-1} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{n-i}\binom{n-i}{s}\frac{(-1)^s}{s+i}\frac{d}{dx}[G_{LBXII}(x)]^{s+i},
\end{aligned}
$$

since $[G_{LBXII}(x)]^{s+i-1}g_{LBXII}(x) = \frac{1}{s+i}\frac{d}{dx}[G_{LBXII}(x)]^{s+i}$. Therefore, the cdf of the ith order statistic is given by

$$
G_{i:n}(x; \lambda, c, k) = \frac{n!}{(i-1)!(n-i)!}\sum_{s=0}^{n-i}\binom{n-i}{s}\frac{(-1)^s}{s+i}[G_{LBXII}(x; \lambda, c, k)]^{s+i}, \tag{18}
$$

where

$$
G_{LBXII}(x; \lambda, c, k) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda}\frac{e^{-\lambda x}}{(1 + x^c)^k},
$$

$\lambda, c, k > 0$.

Consequently, the pdf of the ith order statistic from the LBXII distribution is a linear combination of the exponentiated LBXII (ELBXII) pdf's with parameters: $s + i > 0, \lambda > 0, c > 0$, and $k > 0$.

4.2 Rényi Entropy

Rényi entropy (Rényi, 1960) is an extension of Shannon entropy. Rényi entropy is defined to be

$$
I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [g_{LBXII}(x; \lambda, c, k)]^v dx\right), v \neq 1, v > 0. \tag{19}
$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that

$$
\begin{aligned}
\int_0^\infty g_{LBXII}^v(x)dx &= \int_0^\infty \frac{(1 + x^c)^{vk}}{(1 + \lambda)^v}\sum_{s,p=0}^\infty \frac{(-1)^s(v\lambda)^s x^s}{s!}\binom{v}{p}\left(\frac{1 + \lambda + \lambda x}{1 + x^c}kcx^{c-1}\right)^p \\
&\quad \times \left(\lambda^2(1 + x)\right)^{v-p}dx \\
&= \sum_{s,p,t,q=0}^\infty \frac{(-1)^s(v\lambda)^s(kc)^p\lambda^{2(v-p)+t}(1 + \lambda)^{p-t}}{s!(1 + \lambda)^v}\binom{v}{p}\binom{p}{t}\binom{v-p}{q} \\
&\quad \times \int_0^\infty x^{s+cp-p+t+q}(1 + x^c)^{-vk-p}dx \\
&= \sum_{s,p,t,q=0}^\infty \frac{(-1)^s(v\lambda)^s(kc)^p\lambda^{2(v-p)+t}(1 + \lambda)^{p-t}}{s!(1 + \lambda)^v}\binom{v}{p}\binom{p}{t}\binom{v-p}{q} \\
&\quad \times \frac{1}{c}\int_0^\infty y^{vk-\frac{1}{c}(s+t+q+1)-1}(1 - y)^{\frac{1}{c}(s+t+q+1)-1}dy \\
&= \sum_{s,p,t,q=0}^\infty \frac{(-1)^s(v\lambda)^s(kc)^p\lambda^{2(v-p)+t}(1 + \lambda)^{p-t}}{s!(1 + \lambda)^v}\binom{v}{p}\binom{p}{t}\binom{v-p}{q} \\
&\quad \times \frac{1}{c}B\left(vk - \frac{1}{c}(s+t+q+1), \frac{1}{c}(s+t+q+1)\right), \tag{20}
\end{aligned}
$$

where $B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the complete beta function. Consequently, Rényi entropy for the LBXII distribution is given by

$$
\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\left[\sum_{s,p,t,q=0}^{\infty}\frac{(-1)^s(v\lambda)^s(kc)^p\lambda^{2(v-p)+t}(1+\lambda)^{p-t}}{s!(1+\lambda)^v}\binom{v}{p}\binom{p}{t}\binom{v-p}{q}\right. \\
&\times \left.\frac{1}{c}B\left(vk - \frac{(s+t+q+1)}{c}, \frac{(s+t+q+1)}{c}\right)\right], \quad v \neq 1, v > 0.
\end{aligned}
\tag{21}
$$

5. Maximum Likelihood Estimates

The maximum likelihood estimates (MLEs) of the parameters of the LBXII distribution are presented in this section. Let $X_i \sim LBXII(\lambda, c, k)$ and $\Delta = (\lambda, c, k)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$
\ell = \ell(\Delta) = -k\sum_{i=1}^{n}\ln(1+x_i^c) - n\ln(1+\lambda) - \lambda\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln\left[\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)kcx_i^{c-1}}{1+x_i^c}\right].
\tag{22}
$$

Elements of the score vector $U = (\frac{\partial\ell}{\partial k}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial c})$ are given by:

$$
\frac{\partial\ell}{\partial k} = -\sum_{i=1}^{n}\ln(1+x_i^c) + \sum_{i=1}^{n}\left(\frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)kcx_i^{c-1}}{1+x_i^c}}\right),
\tag{23}
$$

$$
\frac{\partial\ell}{\partial\lambda} = -\frac{1}{1+\lambda} - \sum_{i=1}^{\infty}x_i + \sum_{i=1}^{n}\left(\frac{2\lambda(1+x_i) + \frac{(1+x_i)kcx_i^{c-1}}{1+x_i}}{\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)cx_i^{c-1}}{1+x_i^c}}\right),
\tag{24}
$$

and

$$
\begin{aligned}
\frac{\partial\ell}{\partial c} &= -k\sum_{i=1}^{n}\frac{x_i^c\ln(x_i)}{1+x_i^c} + \sum_{i=1}^{n}\left(\frac{[k(1+\lambda+\lambda x_i)[cx_i^{c-1}(-1)(1+x_i^c)^{-2}x_i^c\ln(x_i)]}{\left(\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)kcx_i^{c-1}}{1+x_i^c}\right)}\right. \\
&+ \left.\frac{k(1+\lambda+\lambda x_i)(1+x_i^c)^{-1}[x_i^{c-1} + cx_i^{c-1}\ln(x_i)]}{\left(\lambda^2(1+x_i) + \frac{(1+\lambda+\lambda x_i)kcx_i^{c-1}}{1+x_i^c}\right)}\right),
\end{aligned}
\tag{25}
$$

respectively. The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters k, λ, c must be found via iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the non-linear equation $(\frac{\partial\ell}{\partial k}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial c})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\Delta) = [\mathbf{I}_{\theta_i,\theta_j}]_{3X3} = E(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j})$, $i, j = 1, 2, 3$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}(\Delta)$ can be approximated by

$$
\mathbf{J}_n(\hat{\Delta}) \approx \left[-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\Big|_{\Delta=\hat{\Delta}}\right]_{3X3}, \quad i, j = 1, 2, 3.
\tag{26}
$$

For a given set of observations, the matrix given in equation (26) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software.

6. Monte Carlo Simulations

In this section, the performance of the maximum likelihood estimates is examined by conducting simulation studies for different sample sizes. We conducted various simulations for different sizes sample (n=25, 50, 100, 200, 400, 800) via the R package. We simulate $N = 1000$ samples for the true parameters values given in the Table 3 and Table 4. The Average

Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The tables list the mean MLEs of the parameters along with the respective root mean squared errors (RMSEs) and Average Bias. We note that as the sample size n increase the mean estimates of the parameters tend to be closer to the true parameter value and the RMSEs and Average Bias decay towards zero.

Table 3. Monte Carlo Simulation Results: Mean, Average Bias and RMSE

Parameter	n	I (0.4 ,1.0 ,2.0)			II (0.7, 1.0, 2.0)		
		Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
λ	25	0.8527	0.8414	0.4527	1.1077	0.9361	0.4077
	50	0.7165	0.6452	0.3165	1.0017	0.7799	0.3017
	100	0.5987	0.4564	0.1987	0.8866	0.6092	0.1866
	200	0.5292	0.3269	0.1292	0.8137	0.4638	0.1137
	400	0.4845	0.2004	0.0845	0.7517	0.3008	0.0517
	800	0.4727	0.1479	0.0727	0.7393	0.2120	0.0393
c	25	1.3605	7.7538	0.3604	1.3025	3.0454	0.3025
	50	1.2548	5.0278	0.2548	1.0854	2.4853	0.0854
	100	0.9743	0.1172	-0.0257	0.9695	0.1460	-0.0304
	200	0.9861	0.0789	-0.0139	0.9846	0.0997	-0.0153
	400	0.9894	0.0551	-0.0106	0.9916	0.0656	-0.0083
	800	0.9931	0.0374	-0.0069	0.9945	0.0442	-0.0054
k	25	1.5643	0.9454	-0.4357	1.6391	1.1509	-0.3608
	50	1.6554	0.7641	-0.3446	1.6532	0.9343	-0.3467
	100	1.7864	0.5426	-0.2136	1.7736	0.7349	-0.2263
	200	1.8623	0.3875	-0.1376	1.8607	0.5682	-0.1392
	400	1.9086	0.2499	-0.0914	1.9267	0.3765	-0.0732
	800	1.9302	0.1787	-0.0698	1.9515	0.2660	-0.0484

7. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the LBXII distribution for data modeling. We compared the LBXII distribution with the power generalized Weibull (PGW) distribution (Bagdonavicious and Nikulin, 2002), modified weibull (MW) distribution (Lai et al., 2003), the Lindley-Weibull (LW) distribution (Asgharzadeh, Nadarajah and Sharafi, 2016), and the exponentiated power Lindley (EPL) distribution (Ghitany et al., 2013). The new model was also compared to the Weibull Exponential (WE) distribution by (Oguntunde et al., 2015) and the Marshall-Olkin Log-logistic (MOLLD) distribution (Wenhao, 2013).

The pdf of the power generalized Weibull (PGW) distribution (Bagdonavicious & Nikulin, 2002) is given by

$$g_{PGW}(x; \lambda, \alpha, \beta) = \lambda \alpha \beta x^{\alpha-1} (1 + \lambda x^\alpha)^{\beta-1} \exp\left(1 - [1 + \lambda x^\alpha]^\beta\right),$$

for $\lambda, \alpha, \beta > 0$. The pdf of the MW distribution is given by

$$g_{MW}(x; \theta, \beta\alpha) = \theta(\beta + \alpha x)x^{\beta-1}e^{\alpha x} \exp\{-\theta x^\beta e^{\alpha x}\}$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$. The LW distribution (see Asgharzadeh et al., 2016), has the pdf given by

$$g_{LW}(x; \lambda, \alpha, \beta) = \frac{e^{-\lambda x - \alpha x^\beta}}{1 + \lambda}[\lambda^2(1 + x) + (1 + \lambda + \lambda x)\alpha\beta x^{\beta-1}],$$

Table 4. Monte Carlo Simulation Results: Mean, Average Bias and RMSE

Parameter	n	III (1.5, 0.7, 0.3)			IV (1.5,0.5,1.3)		
		Mean	RMSE	Average Bias	Mean	RMSE	Average Bias
λ	25	1.2528	0.5830	-0.2472	1.2576	0.9319	-0.2424
	50	1.3205	0.4658	-0.1795	1.3234	0.8359	-0.1765
	100	1.3837	0.3420	-0.1163	1.3553	0.7269	-0.1447
	200	1.4238	0.2415	-0.0762	1.3928	0.6168	-0.1072
	400	1.4540	0.1664	-0.0459	1.4595	0.4449	-0.0405
	800	1.4754	0.1148	-0.0246	1.4849	0.3179	-0.0151
c	25	1.3224	3.4527	0.6224	0.6279	0.6398	0.1279
	50	1.0004	1.4109	0.3004	0.5459	0.1577	0.0459
	100	1.0021	2.8167	0.3021	0.5434	0.1015	0.0434
	200	0.8422	1.2885	0.1422	0.5435	0.0796	0.0435
	400	0.7846	1.3368	0.0846	0.5404	0.0612	0.0404
	800	0.7394	0.4856	0.0394	0.5362	0.0460	0.0362
k	25	0.7159	0.7882	0.4159	1.1118	0.6807	-0.1882
	50	0.5865	0.6222	0.2865	1.1066	0.5895	-0.1934
	100	0.4888	0.4591	0.1888	1.1747	0.4652	-0.1253
	200	0.4253	0.3317	0.1253	1.2121	0.3596	-0.0879
	400	0.3679	0.2267	0.0679	1.2350	0.2417	-0.0649
	800	0.3399	0.1524	0.0398	1.2564	0.1710	-0.0436

for $\lambda, \alpha, \beta > 0$. The EPL pdf is given by

$$g_{EPL}(x; \alpha, \beta, \delta) \quad = \quad \alpha\beta\delta x^{\alpha-1}(1+x^{\alpha})e^{-\beta x^{\alpha}}\left[1 - \left(1 + \frac{\beta x^{\alpha}}{1+\beta}\right)e^{-\beta x^{\alpha}}\right]^{\delta-1},$$

for $\alpha, \beta, \delta > 0$, and $x > 0$. The pdf of the WE is given by

$$g_{WE}(x; \alpha, \beta, \lambda) = \alpha\beta\left(\lambda e^{-\lambda x}\right)\left[\frac{(1-e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}}\right]e^{-\alpha\left[\frac{(1-e^{-\lambda x})}{e^{-\lambda x}}\right]^{\beta}},$$

for $\alpha, \beta, \lambda > 0$, and the pdf of the MOLLD distribution is given by

$$g_{MOLLD}(x; \alpha, \beta, \lambda) = \frac{\alpha^{\beta}\beta\lambda x^{\beta-1}}{(x^{\beta} + \alpha^{\beta}\lambda)^2},$$

for $\alpha, \beta, \lambda > 0$ and $x > 0$.

The maximum likelihood estimates (MLEs) of the LBXII parameters and its sub-models are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R. The estimated values of the parameters (standard errors in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion ($AIC = 2p - 2\ln(L)$) Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), and Consistent Akaike Information Criterion ($AICC = AIC + 2\frac{p(p+1)}{n-p-1}$), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. Tables 6, 8 and 10 shows results for the data sets for LBXII distribution and several non-nested models.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers, Cleveland, Kleiner and Turkey, 1983) are given in Figure 6, Figure 7, and Figure 8, 9, 10 and 11, respectively. For the probability plot, we plotted $G_{LBXII}(x_{(j)}; \hat{\lambda}, \hat{c}, \hat{k})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures

of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[G_{LBXII}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

The goodness-of-fit statistics W^* and A^*, described by (Chen and Balakrishnan, 1995) are presented in the tables. The Kolmogorov-Smirnov (K-S) statistic and its p-value are also presented. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^*, A^* and K-S, the better the fit.

7.1 Failure Times of 50 Components Data

The first data set consists of failure times of 50 components (per 1000h) taken from (Murthy, Xie and Jiang 2004).

Table 5. Failure times of 50 components data

0.036	0.148	0.590	3.076	6.816	0.058	0.183	0.618	3.147	7.896	0.061	0.192	0.645
3.625	7.904	0.074	0.254	0.961	3.704	8.022	0.078	0.262	1.228	3.931	9.337	0.086
0.379	1.600	4.073	10.940	0.102	0.381	2.006	4.393	11.020	0.103	0.538	2.054	4.534
13.880	0.114	0.570	2.804	4.893	14.73	0.116	0.574	3.058	6.274	15.08		

The estimated asymptotic variance-covariance matrix of the maximum likelihood estimates for LBXII distribution for the failure times of 50 components data are as follows:

$$\begin{bmatrix} 0.0035 & -2.6786 \times 10^{-3} & -4.5626 \times 10^{-3} \\ -0.0027 & 1.6200 \times 10^{-2} & -3.6119 \times 10^{-6} \\ -0.0046 & -3.6119 \times 10^{-6} & 2.7161 \times 10^{-2} \end{bmatrix} .$$

The approximate 95% confidence intervals for the parameters, λ, c and k are ($0.2094 \pm 1.96 \times \sqrt{0.0035}$), ($0.7389 \pm 1.96 \times \sqrt{1.6200 \times 10^{-2}}$) and ($0.7305 \pm 1.96 \times \sqrt{2.7161 \times 10^{-2}}$), respectively.

For the failure times of 50 components data, the values of the statistic AIC, AICC and BIC are the smallest for the L-BXII distribution. Also the goodness of fit statistics W^*, A^* and $K - S$, are the lowest. The P-value and the SS values from the probability plots indicate that indeed our model performs better than the non-nested models that were considered.
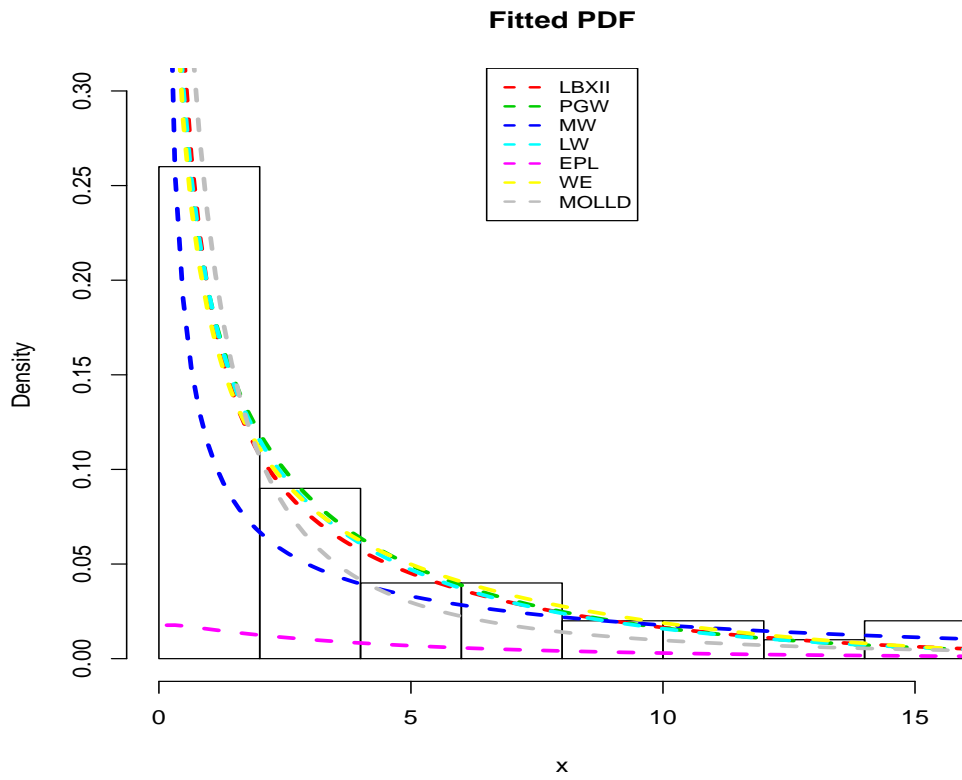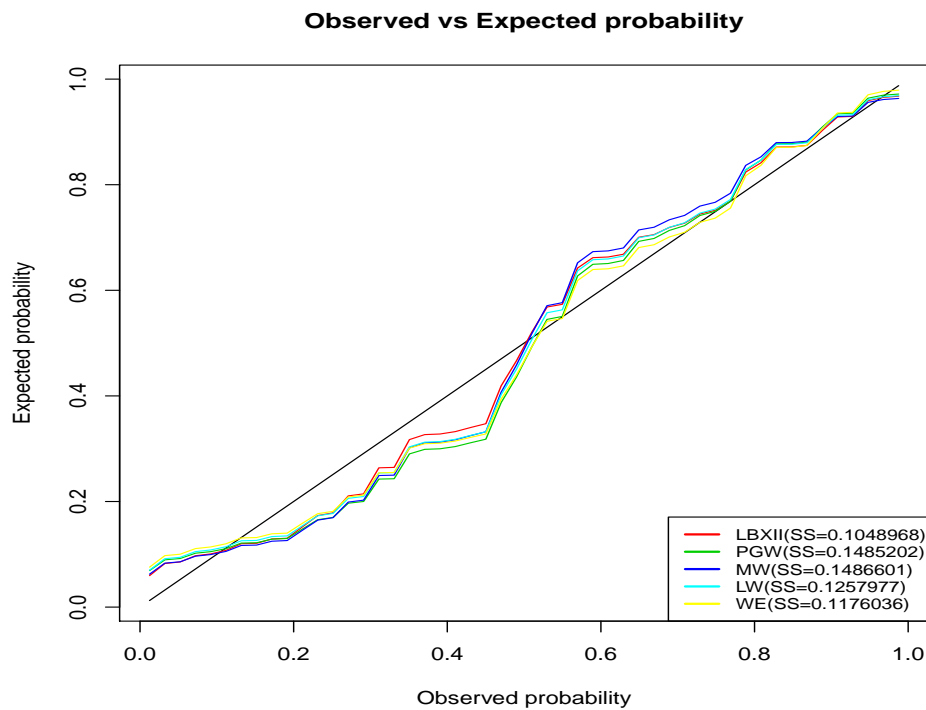
Fitted PDF

Figure 6. Fitted pdf's for failure times data

Observed vs Expected probability

Figure 7. Probability plots for failure times data

Table 6. Failure times of 50 components data

Model	Estimates			Statistics								
	λ	c	k	$-2logL$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	$P-value$	SS
LBXII	0.2094	0.7388	0.7305	200.7	206.7	207.2	212.4	0.1153	0.7431	0.1126	0.5141	0.1049
	(0.0595)	(0.1272)	(0.1648)									
	λ	α	β									
PGW	0.1261	0.5477	3.4268	204.4	210.9	210.9	216.2	0.1446	0.9276	0.1419	0.2418	0.1485
	(0.5100)	(0.1841)	(12.7544)									
	θ	β	α									
MW	0.0439	0.6195	0.4925	204.6	210.6	211.2	216.4	0.1502	0.9452	0.1279	0.3564	0.1487
	(0.1315)	(0.1545)	(0.1808)									
	λ	α	β									
LW	0.1325	0.5055	0.5863	203.6	209.6	210.2	215.4	0.1385	0.8862	0.1278	0.3563	0.1258
	(0.0950)	(0.1089)	(0.1068)									
	α	β	δ									
EPL	0.5632	0.5290	0.5094	462.5	468.5	469.0	474.2	0.4229	2.6352	0.58409	8.8×10^{-16}	3.6252
	(0.0598)	(0.0682)	(0.0499)									
	α	β	δ									
WE	1.6476	0.5544	0.1149	202.4	308.8	309.3	314.7	0.16408	1.0574	0.3672	1.49×10^{-06}	0.1176
	(1.0097)	(0.0942)	(0.0807)									
	α	β	λ									
MOLLD	1.0619	0.9108	1.0603	210.9	234.3	234.8	240.1	0.2480	1.5451	0.3109	8.607×10^{-05}	6.9697
	(131.8401)	(0.1033)	(119.8942)									

7.2 Epoxy Strands Failure at 90% Stress Level Data

The second data set represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure, given by (Barlow, Toland and Freeman, 1984) and (Andrews and Herzberg, 1985).

Table 7. Epoxy strands failure at 90% stress level data

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09	0.09
0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24	0.29	0.34	0.35
0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72
0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33	1.34	1.40	1.43
1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69	7.89				

The estimated variance-covariance matrix for the LBXII of the Epoxy strands failure at 90% stress level data is given by

$$\begin{bmatrix} 0.0401 & -0.0236 & -0.0459 \\ -0.0236 & 0.0446 & 0.0357 \\ -0.0459 & 0.0357 & 0.0749 \end{bmatrix}.$$

The 95% confidence intervals for the parameters λ, c and k of LBXII (Epoxy strands failure at 90% stress level data) are: $(1.1187 \pm 1.96 \times \sqrt{0.0401}), (0.7219 \pm 1.96 \times \sqrt{0.0446})$ and $(0.3952 \pm 1.96 \times \sqrt{0.0749})$, respectively.

For epoxy strands failure at 90% stress level data, the values of the AIC, AICC and BIC are smaller for the LBXII distribution also the values of the statistics W^*, A^*, $K - S$ and its P-value, and the SS value indicate that the LBXII distribution performs better than all the models as indicated in Table 8.

Table 8. Epoxy strands failure at 90% stress level data

Model	Estimates			Statistics								
	λ	c	k	$-2logL$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	$P-value$	SS
LBXII	1.1186	0.7218	0.3952	205.2	211.2	211.5	219.1	0.1414	0.8518	0.0770	0.5871	0.1377
	(0.2002)	(0.2112)	(0.2737)									
	λ	α	β									
PGW	0.6925	0.8617	1.3082	205.6	211.6	211.8	219.4	0.1697	0.9785	0.0858	0.4468	0.1683
	(0.4646)	(0.1253)	(0.6453)									
	θ	β	α									
MW	0.7229	0.7419	0.2797	205.8	211.8	212.1	219.7	0.1842	1.0451	0.0886	0.4059	0.1683
	(0.5807)	(0.4507)	(0.6049)									
	λ	α	β									
LW	0.8890	0.4716	0.7481	205.3	211.3	211.5	219.1	0.1479	0.8797	0.0785	0.5617	0.1441
	(0.5528)	(0.4810)	(0.2544)									
	α	β	δ									
EPL	0.6292	0.8701	0.5088	808.6	814.6	814.9	822.4	0.3157	2.4794	0.6779	2.2×10^{-16}	11.8025
	(0.0521)	(0.0662)	(0.0352)									
	α	β	δ									
WE	$1.3380 \times 10^{+02}$	9.2270×10^{-01}	4.9882×10^{-03}	205.9	211.9	212.2	219.8	0.1972	1.1047	0.0903	0.3821	0.1936
	(2.1542×10^{-06})	(7.0985×10^{-02})	(1.9289×10^{-03})									
	α	β	λ									
MOLLD	0.9556	1.2705	0.5819	225.4	231.4	231.6	239.2	0.5654	3.0709	0.1113	0.1637	0.3893
	(316.0467)	(0.1069)	(244.5294)									
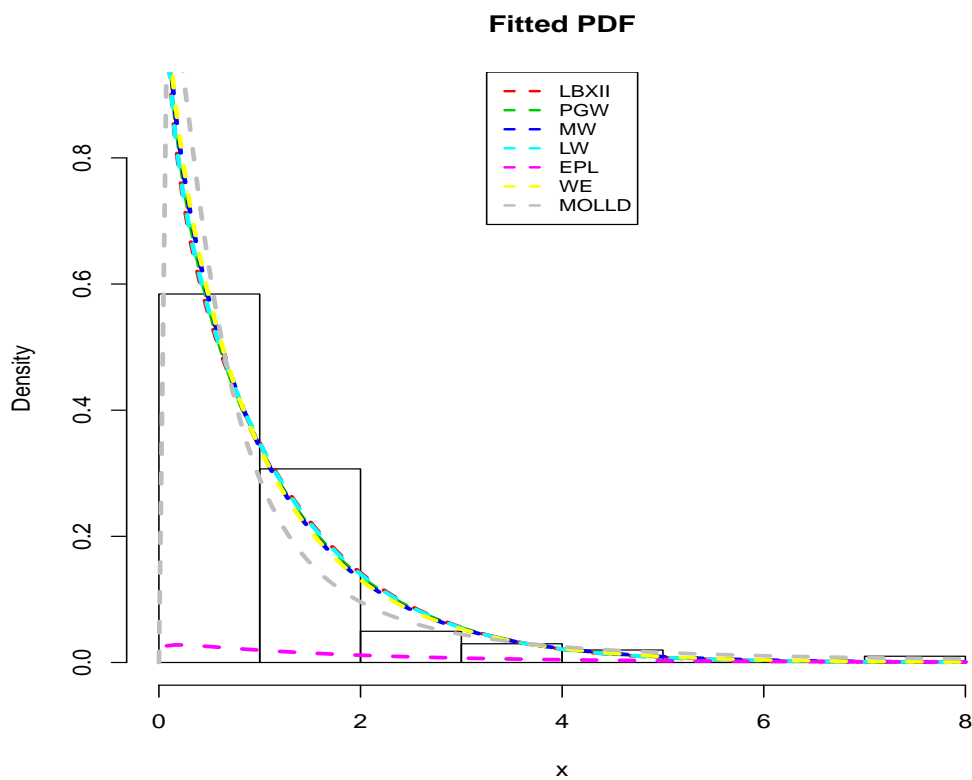
Figure 8. Fitted pdf's for epoxy strands data

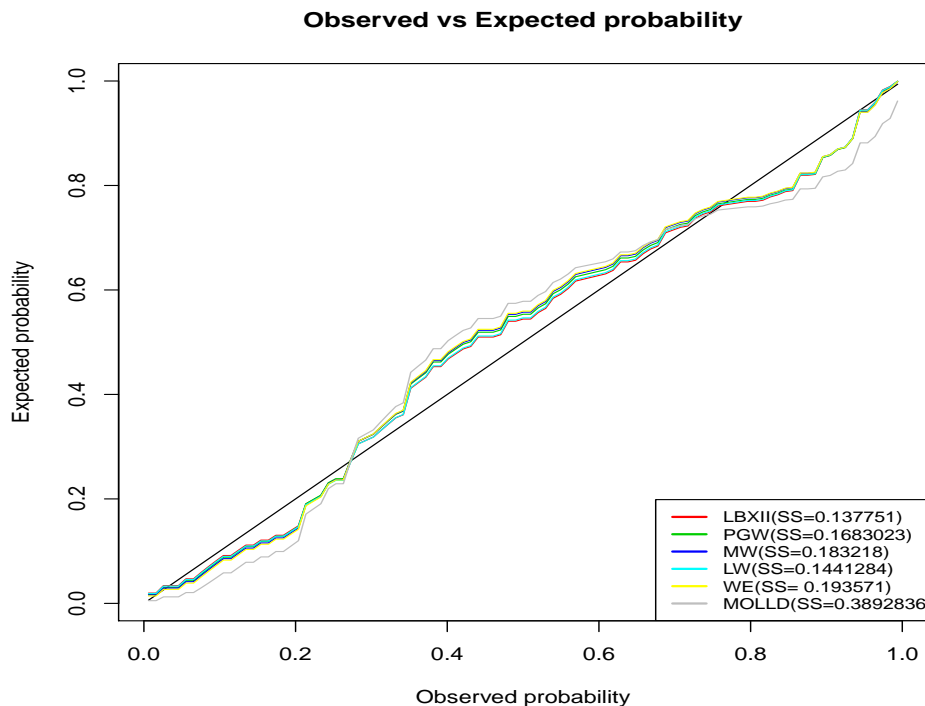**Observed vs Expected probability**

Figure 9. Probability plots for epoxy strands data

7.3 Wheaton River Data

The third data set are the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984. These data were also analyzed by (Akinsete, Famoye and Lee, 2008).

Table 9. Wheaton River data set

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3	1.4	18.7	8.5	25.5
11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9
1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0	1.5	2.5	27.4	
1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0							

The estimated asymptotic variance-covariance matrix of the maximum likelihood estimates for LBXII distribution for Wheaton River data set are as follows:

$$\begin{bmatrix} 0.0002 & -0.0009 & -0.0002 \\ -0.0009 & 0.1567 & -0.0125 \\ -0.0002 & -0.0125 & 0.0024 \end{bmatrix}.$$

The approximate 95% confidence intervals for the parameters, λ, c and k are $(0.0923 \pm 1.96 \times \sqrt{0.0002})$, $(1.4905 \pm 1.96 \times \sqrt{0.1567})$ and $(0.1565 \pm 1.96 \times \sqrt{0.0024})$, respectively.

Table 10. Wheaton River data set

| Model	Estimates			Statistics								
	λ	c	k	$-2logL$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	$P-value$	SS
LBXII	0.0923	1.4905	0.1565	497.5	503.5	503.8	510.3	0.0771	0.4348	0.0757	0.8039	0.0779
	(0.0146)	(0.3959)	(0.0494)									
	λ	α	β									
PGW	0.0365	0.7441	3.2248	501.8	507.8	508.2	514.7	0.1041	0.6326	0.1063	0.3903	0.1059
	(0.0607)	(0.1472)	(5.5776)									
	θ	β	α									
MW	0.0412	0.8120	0.0704	502.9	507.3	507.74	514.2	0.0969	0.5889	0.0994	0.4752	0.14500
	(0.1016)	(0.4037)	(0.1046)									
	α	β	δ									
LW	0.0468	0.1176	0.8027	502.2	508.2	508.6	515.1	0.1203	0.7038	0.1017	0.4458	0.1177
	(0.0398)	(0.0342)	(0.1557)									
	α	β	δ									
EPL	1.3251	0.0301	0.5102	665.3	671.3	671.6	678.1	0.4971	3.1751	0.1843	0.0150	0.8055
	(0.0938)	(0.0091)	(0.0417)									
	α	β	δ									
WE	2.3899	0.7808	0.0227	501.3	507.3	507.7	514.2	0.0969	0.5889	0.0994	0.4752	0.0946
	(2.1029)	(0.1181)	(0.0181)									
	α	β	λ									
MOLLD	2.3411	1.2127	3.6168	515.7	521.8	522.0	528.5	0.2961	1.6484	0.1138	0.309	0.2328
	(0.3769)	(0.1167)	(0.2012)									
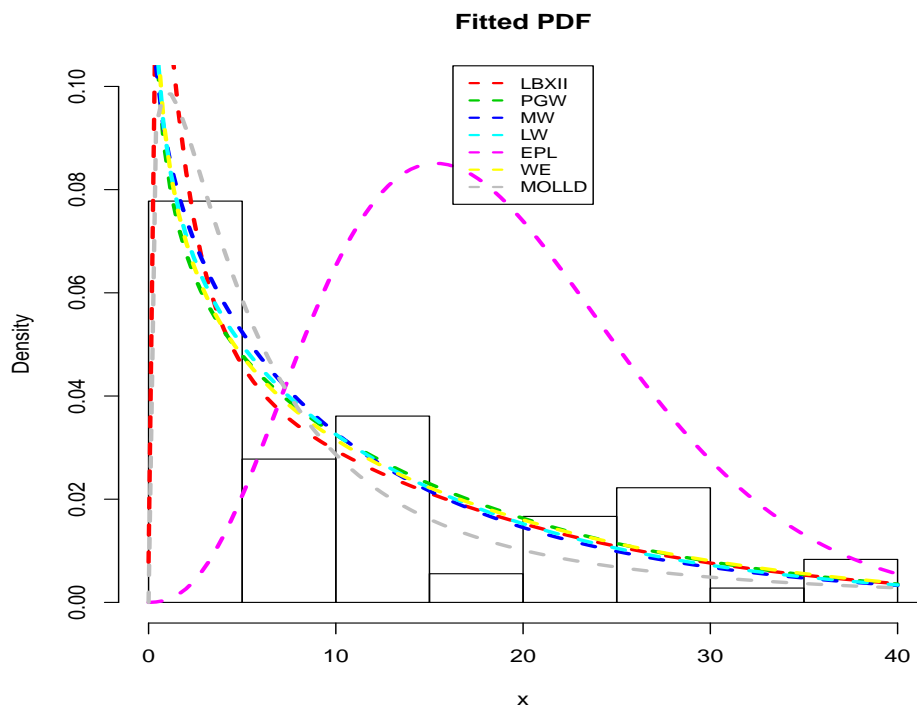
Figure 10. Fitted pdf's for Wheaton River data

Similarly for Wheaton River data , the values of the statistics $K - S$, P-value, and SS from the probability plots shows that the LBXII distribution performs better for this data set. In overall the LBXII distribution performs better PGW, MW, EPL, LW, WE and the MOLLD distributions for the Wheaton River data.
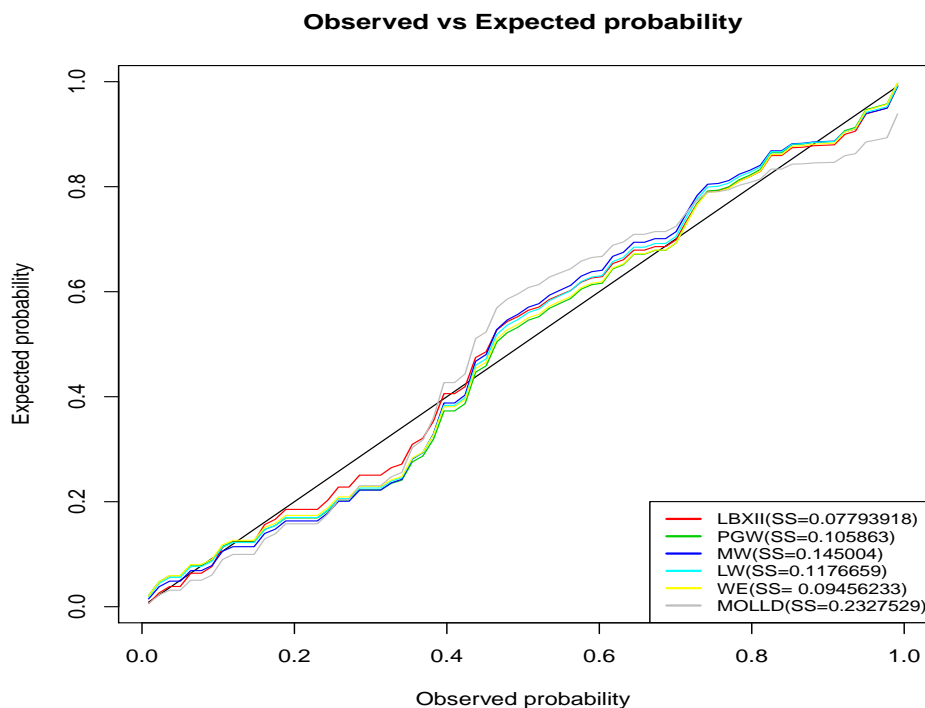
Figure 11. Probability plots for Wheaton River data

8. Concluding Remarks

A new three parameter distribution called the Lindley-Burr XII (LBXII) distribution is presented, which extends the Burr XII (BXII) distribution. The LBXII distribution has two components namely the Lindley and the Burr XII distributions. The LBXII distribution possesses hazard function with flexible behaviour. We derive expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the LBXII distribution was examined by conducting various simulations for different sizes. Finally, the distribution was fitted to real data sets to illustrate empirically that the LBXII provides consistently better fits to two real data sets than other six non-nested distributions with same number of parameters. The two examples proves that the LBXII is a desired alternative for modelling survival data. The LBXII include several other distributions, hence, it is our belief that the LBXII will attract wider applications in several different areas of research.

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