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Generalized Mean-Field Fractional BSDEs With Non-Lipschitz Coefficients

Qun Shi

Correspondence: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi, P.R.China

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Abstract

In this paper we consider one dimensional generalized mean-field backward stochastic differential equations (BSDEs) driven by fractional Brownian motion, i.e., the generators of our mean-field FBSDEs depend not only on the solution but also on the law of the solution. We first give a totally new comparison theorem for such type of BSDEs under Lipschitz condition. Furthermore, we study the existence of the solution of such mean-field FBSDEs when the coefficients are only continuous and with a linear growth.

Keywords: backward stochastic differential equation, continuous coefficients, comparison theorem, fractional Brownian motion, mean-field

Mathematical subject classification: 60H05, 60H07.

1. Introduction

General backward stochastic differential equations driven by a Brownian motion were first studied by Pardoux and Peng (1992). Later Pardoux and Zhang (1998) introduced the generalized BSDEs,i.e. BSDEs with an additional term-an integral with respect to an increasing process. Backward stochastic differential equations driven by a fractional Brownian motion with $H \in (1/2, 1)$ were first considered by Biagini,Hu,Øksendal and Sulem (2002), where they studied the stochastic maximal principle in the framework of a fractional Brownian motion. By adapting the four-step scheme introduced by Ma,Protter and Yong(1994) and the so-called S-transform, Bender (2005) stuided BSDEs driven by a fractional Brownian motion with $H \in (0, 1)$. Indeed, throughout a backward parabolic PDE, he constructed an explicit solution of a kind of linear fractional BSDE. Hu and Peng (2009) were the first to study nonlinear BSDEs governed by a fractional Brownian motion.

It is well known that backward stochastic differential equation provided stochastic representation of solution of some classes of partial differential equations of second order. With the help of backward stochastic differential equations with respect to a Brownian motion and a Poisson random measure, some authors generalized this result to integro-partial differential equations. The pioneer result on BSDEs, established by Pardoux and Peng (1990) require Lipschitz condition on the drift of the equation. Sow study on BSDE with jumps, established by Sow (2014) require non-Lipschitz coefficients and application to large deviations.

Mathematical mean-field approaches play an important role in many fields, among them, finance and game theory. Since the pioneering of Lasry and Lions (2007) the research on mean-field has attracted a lot of researchers. Buckdahn, Djehiche, Li and Peng (2009) studied a type of mean field problem by a purely stochastic approach and introduced a new type of BSDE which they called mean-field BSDE. Buckdahn, Li and Peng (2009) obtained the existence and the uniqueness of the solution of the mean-field BSDEs when the coefficient f is Lipschitz, and the terminal condition ξ is a square integrable random variable. They also got a comparison theorem. Later, more and more works have been studied on mean-field SDEs and BSDEs, see Buckdahn, Li and Peng (2009), Buckdahn, Li, Peng and Rainer (2017), Hao and Li (2016), Li (2017), Li and Min (2016). Du, Li and Wei (2011) considered a special type of one dimensional mean-field BSDEs with coefficients which are continuous and have a linear growth. They got the existence of the minimal solution. Recently, Juan (2018) considered general mean-field BSDEs with continuous coefficients. Our aim in the present work is to extend result to generalized mean-field BSDEs driven by fractional Brownian motion with continuous coefficients.

Let us recall that, for $H \in (0, 1)$, a fBm $(B^H(t))_{t\geqslant 0}$ with Hurst parameter H is a continuous and centered Gaussian process with covariance

$$E\left[B^H(t)B^H(s)\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geqslant 0.$$

For $H = 1/2$, the fBm is a standard Brownian motion. If $H > 1/2$, then $B^H(t)$ has a long-range dependence, which means that for $r(n) := cov(B^H(1), B^H(n+1) - B^H(n))$, we have $\sum_{n=1}^{\infty} r(n) = \infty$. Moreover, B^H is self-similar, i.e. $B^H(at)$ has the same law as $a^H B^H(t)$ for any $a > 0$. Since there are many models of physical phenomena and finance which exploit the self-similarily and the long-range dependence, fBm are a very useful tool to characterize such type of problems.

However, since fBm are not semimartingales nor Markov processes when $H \neq 1/2$, we can not use the classical theory of stochastic calculus to define the fractional stochastic integral. In essence, two different integration theories with respect to fractional Brownian motion have been defined and studied. The first one, originally due to Young (1936), concerns the pathwise Riemann-Stieljes integral which exists if the integrand has Hölder continuous paths of order $\alpha > 1 - H$. But it turn out that this integral has the properties comparable to the Stratonovich integral, which leads to difficulties in applications. The second one concerns the divergence operator (Skorohod integral), define as the adjoint of the derivative operator in the framework of the Malliavin calculus. This approach was introduced by Decreusefond and Ušturel (1998).

Concerning the study of BSDEs in the fractional framework, the major problem is the absence of a martingale representation type theorem with respect to fBm. For the first time, Hu and Peng (2009) overcome this problem, in the case $H > 1/2$.

We now introduce a class of reflected diffusion processes with standard Brownian motion. Let G be an open connected subset of R^d , which is such that for some $l \in C^2(\mathbb{R}^d)$, $G = \{x : l(x) > 0\}$, $\partial G = \{x : l(x) = 0\}$ and $|\nabla l(x)| = 1$ for $x \in \partial G$. Note that at any boundary point $x \in \partial G$, $\nabla l(x)$ is a unit normal vector to the boundary, pointing towards to the interior of G. If drift coeffcient and diffusion coeffcient satisfying some Lipschitz , then it follows from the results in Lions and Sznitman (1984) (see also Saisho (1987)) that for each $x \in \partial G$, there exists a unique pair of progressively measurable continuous processes (η_t, Λ_t), such that

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \nabla l(\eta_s)d\Lambda_s, \quad 0 \leqslant t \leqslant T,$$

$$\Lambda_t = \int_0^t \mathbf{1}_{\eta_s \in \partial G} d\Lambda_s, \quad \Lambda_. \quad is \quad a \quad nondecreasing \quad process.$$

the existence of such a problem driven by fBm was shown in Ferrante and Rovira (2013) and a set $D = (0, +\infty)$.

In this paper we study the generalized mean-field BSDEs driven by fBm with Hurst parameter $H > 1/2$. We prove that kind of equation has an adapted solution under continuous coefficients. The paper is organized as follows. In section 2 we give some definitions and results about fractional stochastic integral which will be needed throughout the paper. Section 3 contains the definition of the generalized BSDEs driven by fBm and assumptions. In section 4, we will prove comparison theorem for the generalized mean-field FBSDE. Finally, section 5 is devoted to prove the main theorem of the paper.

2. Fractional Stochastic Calculus

Denote, for given $H \in (1/2, 1)$, $\phi(x) = H(2H-1)|x|^{2H-2}, x \in \mathbb{R}$. Let ξ and η be measurable functions on $[0, T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)dudv$$

and $||\xi||_t^2 = \langle \xi, \xi \rangle_t$.Note that , for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product.

Let $\mathcal{H}$ be the completion of the measurable functions such that $||\xi||_t^2 < \infty$. The elements of $\mathcal{H}$ may be distributions (refer to Pipiras and Taqqu (2000)).

Let $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. By $\mathcal{P}_T$ denote the set of all polynomials of fractional Brownian motion in $[0, T]$, i.e. it contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \cdots, \int_0^T \xi_k(t)dB_t^H\right),$$

where f is a polynomial function of k variables. The Malliavin derivative operator D_s^H of an element $F \in \mathcal{P}_T$ is defined as follows:

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \cdots, \int_0^T \xi_k(t)dB_t^H\right) \cdot \xi_i(s), \quad s \in [0, T].$$

Since the divergence operator D^H is closable from $L^2(\Omega, \mathcal{F}, P)$ to $(\Omega, \mathcal{F}, \mathcal{H})$, By $\mathbb{D}_{1,2}$ denote the Banach space be the a completion of $\mathcal{P}_T$ with the following norm: $||F||_{1,2}^2 = E|F|^2 + E||D_s^H F||_T^2$.

Now we also introduce another derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-s) D_s^H F ds.$$

The following results are well known, refer to Duncan and Hu (2000), Hu (2005).

Theorem 2.1. *(Hu (2005),Proposition 6.25) Let* $F : (\Omega, \mathcal{F}, P) \to \mathcal{H}$ *be a stochastic process such that*

$$E\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 ds dt \right) < \infty.$$

Then, the Itô-type stochastic integral denoted by $\int_0^T F_s dB_s^H$ *exists in* $L^2(\Omega, \mathcal{F}, P)$*. Moreover,* $E\left(\int_0^T F_s dB_s^H\right) = 0$ *and*

$$E\left(\int_0^T F_s dB_s^H\right)^2 = E\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s ds dt \right).$$

Theorem 2.2. *(Hu (2005),Proposition 10.3) Let* $f, g{:}[0,T] \to \mathbb{R}$ *be deterministic continuous functions. If*

$$X_t = X_0 + \int_0^t g(s) ds + \int_0^t f(s) dB_s^H, \quad t \in [0,T],$$

where X_0 *is a constant and* $F \in C^{1,2}([0,T] \times \mathbb{R})$*, then*

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \frac{d}{ds}(\|f\|_s^2) ds, \quad t \in [0,T].$$

Theorem 2.3. *(Hu (2005),Proposition 11.1) Let* $f_i(s), g_i(s), i = 1,2$ *are in* $\mathbb{D}_{1,2}$ *and* $E\int_0^T (|f_i(s)| + |g_i(s)|) ds < \infty$*. Assume that* $\mathbb{D}_t^H f_1(s)$ *and* $\mathbb{D}_t^H f_2(s)$ *are continuously differential with respect to* $(s,t) \in [0,T] \times [0,T]$ *for almost all* $\omega \in \Omega$*. Suppose that*

$$E\left(\int_0^T \int_0^T |\mathbb{D}_t^H f_i(s)|^2 ds dt \right) < \infty.$$

For $i = 1,2$*, denote*

$$X_i(t) = \int_0^t g_i(s) ds + \int_0^t f_i(s) dB_s^H, \quad t \in [0,T],$$

Then

$$\begin{aligned} X_1(t) X_2(t) &= \int_0^t X_1(s) g_2(s) ds + \int_0^t X_1(t) f_2(s) dB_s^H + \int_0^t X_2(s) g_1(s) ds + \int_0^t X_2(t) f_1(s) dB_s^H \\ &+ \int_0^t \mathbb{D}_s^H X_1(s) f_2(s) ds + \int_0^t \mathbb{D}_s^H X_2(s) f_1(s) ds. \end{aligned}$$

3. Generalized Fractional BSDE

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, P)$, $T > 0$, be a complete stochastic basis, and $\mathcal{F}_{t^+} = \cap_{\delta > 0} \mathcal{F}_{t+\delta} = \mathcal{F}_t$. Suppose that the filtration is generated by d-dimensional fractional Brownian motion $(B_t^H)_{0 \leqslant t \leqslant T}$, and $T > 0$ is an arbitrarily fixed time horizon. We suppose that there is a sub-σ-field $\mathcal{F}_0 \subset \mathcal{F}$, $\mathcal{F}_0$ includes all P-null subsets of $\mathcal{F}$, such that

i) the fractional Brownian motion B^H is independent of $\mathcal{F}_0$, and

ii) $\mathcal{F}_0$ is "rich enough", i.e.,$\mathcal{P}_2(\mathbb{R}^k) = \{P_\vartheta, \vartheta \in L^2(\mathcal{F}_0; \mathbb{R}^k)\}$, $k \geqslant 1$.

Recall that $\mathcal{P}_2(\mathbb{R}^k)$ is the set of the probability measures on $(\mathbb{R}^k, B(\mathbb{R}^k))$ with finite second moment. Here $B(\mathbb{R}^k)$ denotes the Borel σ-field over $\mathbb{R}^k$. By $\mathcal{F} = (\mathcal{F}_t), t \in [0,T]$, we denote the filtration generated by B^H, completed and augmented by $\mathcal{F}_0$.

The space $\mathcal{P}_2(\mathbb{R}^d)$ is endowed with the 2-Wasserstein metric

$$W_2(\mu,\nu)=\inf_{\pi\in\prod(\mu,\nu)}\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\pi(dx,dy)\right)^{1/2}\right\},$$

where $\prod(\mu,\nu)$ is the family of all couplings of μ and ν, i.e., $\pi\in\prod(\mu,\nu)$ if and only if π is a measure on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$.

Assume that
- η_0 is a given constant;
- $b,\sigma:[0,T]\to\mathbb{R}$ are continuous deterministic, σ is differentiable and $\sigma_t\neq 0, t\in[0,T]$.

Note that, since $\|\sigma\|_t^2=H(2H-1)\int_0^t\int_0^t|u-v|^{2H-2}\sigma(u)\sigma(v)dudv$, we have

$$\frac{d}{dt}(\|\sigma\|_t^2)=2\sigma(t)\widehat{\sigma}(t)>0,\quad where\quad \widehat{\sigma}(t)=\int_0^t\phi(t-v)\sigma(v)dv,\quad 0\leqslant t\leqslant T.$$

Remark 3.1. *(Remark 6 by Maticiuc and Nie (2015))*
There exists a suitable constant $M>0$ which is only dependent H such that

$$\frac{t^{2H-1}}{M}\leqslant\frac{\widehat{\sigma}(t)}{\sigma(t)}\leqslant Mt^{2H-1},\quad 0\leqslant t\leqslant T.$$

since

$$\begin{aligned}\widehat{\sigma}(t)&=\int_0^t\phi(t-v)\sigma(v)dv=H(2H-1)\int_0^t(t-v)^{2H-2}\sigma(v)dv=H(2H-1)\int_0^1(t(1-u))^{2H-2}\sigma(tu)tdu\\&=H(2H-1)t^{2H-1}\int_0^1(1-u)^{2H-2}\sigma(tu)du,\end{aligned}$$

then by continuity of σ, we get the remark.

We now introduce a class of reflected processes. Let G be an open connected subset of $\mathbb{R}^d$, which is such that for some $l\in C^2(\mathbb{R}^d)$, $G=\{x:l(x)>0\}$, $\partial G=\{x:l(x)=0\}$ and $|\nabla l(x)|=1$ for $x\in\partial G$. Note that at any boundary point $x\in\partial G$, $\nabla l(x)$ is a unit normal vector to the boundary, pointing towards to the interior of G. Let $\eta_0\in G$ and (η_t,Λ_t) be a solution of the following reflected SDE with respect to fractional Brownian motion

$$\eta_t=\eta_0+\int_0^t b(s)ds+\int_0^t\sigma(s)dB_s^H+\int_0^t\nabla l(\eta_s)d\Lambda_s,\quad 0\leqslant t\leqslant T,\tag{1}$$

By a solution of (1), we mean a pair of processes such that $\eta_.\in G$, Λ is a nondecreasing process, $\Lambda_0=0$, and $\int_0^T(\eta_t-a)d\Lambda_s\leqslant 0$ for any $a\in G$,

$$\Lambda_t=\int_0^t\mathbf{1}_{\eta_s\in\partial G}d\Lambda_s.$$

The existence of such a problem was shown in Lions and Sznitman (2007) for a standard Brownian motion.

Remark 3.2. *This problem is solved in Ferrante and Rovira (2009) for a fractional Brownian motion and a set $G=(0,\infty)$.*

Given a final time $T>0$, a final condition ξ, which is a $\mathcal{F}_T$ measurable real valued random variable and the functions

$$f:[0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R},\quad g:[0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})\times\mathbb{R}\to\mathbb{R},$$

we consider the following generalized BSDE with respect to fBm with parameters (ξ,f,g,Λ) (short name GFBSDE) whose generators depend on both the solution (Y,Z) and the law of (Y,Z), the law of Y, respectivity, i.e.

$$Y_t=\xi+\int_t^T f(s,\eta_s,P_{(Y_s,Z_s)},Y_s,Z_s)ds+\int_t^T g(s,\eta_s,P_{(Y_s)},Y_s)d\Lambda_s-\int_t^T Z_sdB_s^H,\quad 0\leqslant t\leqslant T.\tag{2}$$

in order to give a probabilistic formula for the solution of a system of elliptic PDEs, this requires the new term–an integral with respect to a increasing process in this equation (2) which is independent of Z_s, the local time of the diffusion on the boundary.

Next we introduce the following sets:
• $C^{1,2}_{pol}([0,T]\times\mathbb{R})$ is the space of all $C^{1,2}$ functions over $[0,T]\times\mathbb{R}$, which together with their derivatives are of polynomial growth,
• $\mathcal{V}_{[0,T]} = \left\{Y = \psi(\cdot,\eta) : \psi \in C^{1,2}_{pol}([0,T]\times\mathbb{R}), \frac{\partial\psi}{\partial t} \quad is \quad bounded \quad , t\in[0,T]\right\}$,
• $\widetilde{\mathcal{V}}^H_{[0,T]}$ the completion of $\mathcal{V}_{[0,T]}$ under the following norm (where $\beta > 0$)

$$||Y||_\beta = \left(\int_0^T t^{2H-1}E[e^{\beta\Lambda_t}|Y_t|^2]dt\right)^{1/2} = \left(\int_0^T t^{2H-1}E[e^{\beta\Lambda_t}|\psi(t,\eta_t)|^2]dt\right)^{1/2},$$

We assume that the coefficients f and g of the GFBSDE are continuous functions and satisfy the following assumption $(H1)$:
$(\boldsymbol{H1.1})$ Linear growth: There exists $K \geqslant 0$, such that

$$|f(t,\eta,\mu,y,z)| \leqslant K(1 + W_2(\mu,\delta_0) + |y| + |\eta| + |z|), dtdP - a.e \quad for \quad all \quad (\eta,\mu,y,z),$$

$$|g(t,\eta,\nu,y)| \leqslant K(1 + W_2(\nu,\delta_0) + |y| + |\eta|), dtdP - a.e \quad for \quad all \quad (\eta,\nu,y).$$

where δ_0 is the Dirac measure with mass at $0\in\mathbb{R}^{1+d}$ or $0\in\mathbb{R}^d$.
$(\boldsymbol{H1.2})$ Lipschitz in (μ,y,z): i.e. there exists a constant $C\in\mathbb{R}^+$ such that for all $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{R}^{1+d})$, $\nu_1,\nu_2\in\mathcal{P}_2(\mathbb{R}^d)$ and all $y_1,y_2\in\mathbb{R}$, $z_1,z_2\in\mathbb{R}^d$,

$$|f(s,\eta,\mu_1,y_1,z_1) - f(s,\eta,\mu_2,y_2,z_2)| \leqslant C(W_2(\mu_1,\mu_2) + |y_1-y_2| + |z_1-z_2|) \;\; dsdP - a.e.$$

$$|g(s,\eta,\nu_1,y_1) - g(s,\eta,\nu_2,y_2)| \leqslant C(W_2(\nu_1,\nu_2) + |y_1-y_2|) \;\; dsdP - a.e.$$

$(\boldsymbol{H1.3})$ A progressively measurable continuous, non-decreasing processes Λ_t has continuous density function.
$(\boldsymbol{H1.4})$ There exists $\beta > 0$ and a function ψ with bounded derivative such that $\xi = \psi(\eta_T)$, $E(e^{\beta\Lambda_T}|\xi|^2) < \infty$ and the integrability condition holds

$$E\left(\int_0^T e^{\beta\Lambda_s}(1 + E[(Y_s,Z_s)^2])ds + \int_0^T e^{\beta\Lambda_s}|\eta_s|^2 ds + \int_0^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right) < \infty.$$

4. Comparison Theorem for General Mean-Field Fractional BSDEs

Definition 4.1. *A binary of processes $(Y_t,Z_t)_{0\leqslant t\leqslant T}$ is called a solution to (2), if $(Y_t,Z_t)\in\widetilde{\mathcal{V}}^{1/2}_{[0,T]}\times\widetilde{\mathcal{V}}^H_{[0,T]}$ and satisfies (2).*

Lemma 4.2. *Assume X is a mean nonzero Gaussian with nonzero covariance, if for two continuous functions $k_1(x), k_2(x)$ such that $k_1(X) = k_2(X)$, then $k_1(x) = k_2(x)$ for all $x\in\mathbb{R}$.*

Proof. Let $f_X(x)$ denote the density function of X, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\theta}e^{-\frac{(x-\alpha)^2}{2\theta^2}},$$

where α denote mean, θ^2 denote variance. Since $k_1(X) = k_2(X)$, take expectation in both sides of this equality, we get

$$\int_{-\infty}^{+\infty}(k_1(x) - k_2(x))f_X(x)dx = 0,$$

by density of $C_0^\infty(\mathbb{R})$ in $C(\mathbb{R})$ and $f_X(x)\geqslant 0$ for all $x\in\mathbb{R}$, consequently $k_1(x) = k_2(x)$ for all $x\in\mathbb{R}$. □

Lemma 4.3. *Assume that h_1, h_2 and $h_3\in C^{0,1}_{pol}([0,T]\times\mathbb{R})$ such that*

$$\int_0^t h_1(s,\eta_s)ds + \int_0^t h_2(s,\eta_s)dB^H_s + \int_0^t h_3(s,\eta_s)d\Lambda_s = 0, \quad 0\leqslant t\leqslant T.$$

Then we have

$$h_1(s,x) = h_2(s,x) = h_3(s,x) = 0, \quad 0\leqslant s\leqslant T, \quad x\in\mathbb{R}.$$

Proof. To simplify notation, we let $\eta_0 = b(t) = 0$ for all $t \in [0, T]$ in (1). Similary to Hu (2005) Theorem 12.3, we have

$$\begin{aligned} h_1(s, \eta_s) &= Eh_1(s, \eta_s) + \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy \right) \sigma(u) dB_u^H \\ &\quad + \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy \right) \nabla l(\eta_u) d\Lambda_u, \end{aligned}$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

and

$$p_{u,s}(x) = p_{\|\sigma\|_s - \|\sigma\|_u}(x).$$

Thus, by stochastic Fubini theorem

$$\begin{aligned} \int_0^t h_1(s, \eta_s) ds &= \int_0^t Eh_1(s, \eta_s) ds + \int_0^t \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy \right) \sigma(u) dB_u^H ds \\ &\quad + \int_0^t \int_0^s \left(\int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy \right) \nabla l(\eta_u) d\Lambda_u ds \\ &= \int_0^t Eh_1(s, \eta_s) ds + \int_0^t \sigma(u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right) dB_u^H \\ &\quad + \int_0^t \nabla l(\eta_u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right) d\Lambda_u \\ &= \int_0^t Eh_1(s, \eta_s) ds + \int_0^t [h_2(u, \eta_u) + \sigma(u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right)] dB_u^H \\ &\quad + \int_0^t [h_3(u, \eta_u) + \nabla l(\eta_u) \left(\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right)] d\Lambda_u \\ &\quad - \int_0^t h_2(u, \eta_u) dB_u^H - \int_0^t h_3(u, \eta_u) d\Lambda_u, \end{aligned}$$

Thus from assumption, we have

$$\int_0^t Eh_1(s, \eta_s) ds = 0,$$

$$\int_0^t \left[h_2(u, \eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right] dB_u^H = 0,$$

$$\int_0^t \left[h_3(u, \eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds \right] d\Lambda_u = 0.$$

But $h_2(u, \eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds$ and $h_3(u, \eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds$ are $\mathcal{F}_u$ adapted (since these are a function of η_u). So from Theorem12.1 of Hu (2005), we see that

$$h_2(u, \eta_u) + \sigma(u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds = 0,$$

$$h_3(u, \eta_u) + \nabla l(\eta_u) \int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s, \eta_s) dy ds = 0.$$

In our situation, (η_u, Λ_u) is a solution of reflected SDE with respect to fractional Brownian motion

$$\eta_u = \int_0^s \sigma(u) dB_u^H + \int_0^s \nabla l(\eta_u) d\Lambda_u, \quad 0 \leqslant u \leqslant s,$$

where Λ is a nondecreasing process, and

$$\Lambda_u = \int_0^s \mathbf{1}_{\eta_u \in \partial G} d\Lambda_u.$$

Although η_u is not center Gaussian process, but by Lemma 4.2, we have

$$h_2(u,z) + \sigma(u)\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,z) dy ds = 0, \tag{3}$$

$$h_3(u,z) + \nabla l(z)\int_u^t \int_R \frac{\partial}{\partial x} p_{u,s}(\eta_u - y) h_1(s,z) dy ds = 0. \tag{4}$$

for all $z \in \mathbb{R}$. Next, the step as same as Lemma 4.2 of Hu (2005),and consequently $h_1(s,z) = 0$ for all $0 \leqslant s \leqslant T$, $z \in \mathbb{R}$. Finally, Bringing $h_1(s,z) = 0$ into the formulas (3) and (4), $h_2(u,z) = 0, h_3(u,z) = 0$ are then an immediate consequence for all $0 \leqslant s \leqslant T$, $z \in \mathbb{R}$. □

It is well known following Lemma (refer to Hu (2005)).

Lemma 4.4. *Let $(Y_t, Z_t)_{0\leqslant t\leqslant T}$ be a solution of the GFBSDE (2). Then we have the stochastic representation*

$$\mathbb{D}_t^H Y_t = \frac{\widehat{\sigma}(t)}{\sigma(t)} Z_t, \quad 0 \leqslant t \leqslant T,$$

Proposition 4.5. *Let $\xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$. Assume* $(H1)$ *holds. Then there exists a unique solution of (2). Moreover, for all $t \in [0,T]$,*

$$E\left(e^{\beta\Lambda_s}|Y_t|^2 + \int_t^T e^{\beta\Lambda_s} s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \leqslant K\Theta(t,T),$$

where

$$\Theta(t,T) := E\left(e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s,Z_s)^2])ds + \int_t^T e^{\beta\Lambda_s}|\eta_s|^2 ds + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right).$$

Proof. First we will show the second part of the above theorem. Assume that (Y,Z) is a solution of (5). By K we will denote a constant which may vary from line to line. From the Itô formula

$$\begin{aligned}
& e^{\beta\Lambda_t}|Y_t|^2 + 2\int_t^T e^{\beta\Lambda_s}(\mathbb{D}_s^H Y_s) Z_s ds + \beta\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s \\
= \;& e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}|Y_s| f(s,\eta_s, P_{(Y_s,Z_s)}, Y_s, Z_s) ds + 2\int_t^T e^{\beta\Lambda_s}|Y_s| g(s,\eta_s, P_{(Y_s)}, Y_s) d\Lambda_s \\
& +2\int_t^T e^{\beta\Lambda_s}|Y_s| Z_s dB_s^H.
\end{aligned}$$

By linear growth of f and g, for all $\mu \in \mathcal{P}_2(\mathbb{R}^{1+d})$, $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\begin{aligned}
2|yf(t,\eta,\mu,y,z)| &\leqslant 2K|y|(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|) \\
&\leqslant (2K^2 + 2K + \frac{MK^2}{s^{2H-1}})|y|^2 + |\eta|^2 + \frac{1}{M}s^{2H-1}|z|^2 + (1 + W_2(\mu,\delta_0))^2
\end{aligned}$$

$$2|yg(t,\eta,\nu,y)| \leqslant 2K|y|(1 + W_2(\nu,\delta_0) + |\eta| + |y|) \leqslant (2K + 2K^2)|y|^2 + |\eta|^2 + (1 + W_2(\nu,\delta_0))^2$$

There, we can write

$$\begin{aligned}
& E\left(e^{\beta\Lambda_t}|Y_t|^2 + \frac{2}{M}\int_t^T e^{\beta\Lambda_s} s^{2H-1}|Z_s|^2 ds + \beta\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \\
\leqslant \;& E(e^{\beta\Lambda_T}|\xi|^2) + 2E\int_t^T e^{\beta\Lambda_s}|Y_s| f(s,\eta_s, P_{(Y_s,Z_s)}, Y_s, Z_s) ds + 2E\int_t^T e^{\beta\Lambda_s}|Y_s| g(s,\eta_s, P_{(Y_s)}, Y_s) d\Lambda_s
\end{aligned}$$

$$\begin{aligned}
&\leqslant E(e^{\beta\Lambda_T}|\xi|^2) + E\int_t^T (2K^2+2K+\frac{MK^2}{s^{2H-1}}+1)e^{\beta\Lambda_s}|Y_s|^2 ds + (2K+2K^2)E\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s \\
&+ E\int_t^T e^{\beta\Lambda_s}(|\eta_s|^2)ds + \frac{1}{M}E\int_t^T s^{2H-1}e^{\beta\Lambda_s}|Z_s|^2 ds \\
&+ E\int_t^T e^{\beta\Lambda_s}(1+W_2(P_{(Y_s,Z_s)},\delta_0))^2 ds + E\int_t^T e^{\beta\Lambda_s}(1+W_2(P_{(Y_s)},\delta_0))^2 d\Lambda_s \\
&\leqslant \Theta(t,T) + E\int_t^T (2K^2+2K+\frac{MK^2}{s^{2H-1}})e^{\beta\Lambda_s}|Y_s|^2 ds + (2K+2K^2)E\int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s \\
&+ \frac{1}{M}E\int_t^T s^{2H-1}e^{\beta\Lambda_s}|Z_s|^2 ds
\end{aligned}$$

Choosing $\beta \geqslant (2K+2K^2+1)$, we get

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|Y_t|^2 + \frac{1}{M}\int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \\
\leqslant\ &\Theta(t,T) + E\int_t^T (2K^2+2K+\frac{MK^2}{s^{2H-1}})e^{\beta\Lambda_s}|Y_s|^2 ds.
\end{aligned}$$

By Gronwall's inequality,

$$Ee^{\beta\Lambda_t}|Y_t|^2 \leqslant \Theta(t,T)\exp\left\{(2K^2+2K)(T-t) + MK^2\frac{T^{2H-1}-t^{2H-1}}{2-2H}\right\}$$

and also get

$$E\left(\int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \leqslant C\Theta(t,T).$$

Now we will prove the existence and uniqueness of the solution of (5). The method used here is the fixed point theorem. We will show that the mapping $\Gamma : \widetilde{\mathcal{V}}_{[0,T]}^{1/2}\times\widetilde{\mathcal{V}}_{[0,T]}^{H} \to \widetilde{\mathcal{V}}_{[0,T]}^{1/2}\times\widetilde{\mathcal{V}}_{[0,T]}^{H}$ given by $(X,W)\to\Gamma(X,W)=(Y,Z)$ is a contraction, where (Y,Z) is a solution of the following generalized BSDE:

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{(X_s,W_s)},X_s,W_s)ds + \int_t^T g(s,\eta_s,P_{(X_s)},X_s)d\Lambda_s - \int_t^T Z_s dB_s^H$$

Let $k\in\mathbb{N}$ and $t_i = \frac{i-1}{k}T, i=1,\cdots,k+1$. First we will show that Γ is a contraction on $\widetilde{\mathcal{V}}_{[t_k,T]}^{1/2}\times\widetilde{\mathcal{V}}_{[t_k,T]}^{H}$. Take $X,X'\in\widetilde{\mathcal{V}}_{[t_k,T]}^{1/2}$, $W,W'\in\widetilde{\mathcal{V}}_{[t_k,T]}^{H}$, let $\Gamma(X,W)=(Y,Z)$, $\Gamma(X',W')=(Y',Z')$ and let $\overline{Y}=Y-Y'$, $\overline{Z}=Z-Z'$, $\overline{X}=X-X'$, $\overline{W}=W-W'$. From Itô formula, for $t\in[t_k,T]$, we have

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|\overline{Y}_t|^2 + 2\int_t^T e^{\beta\Lambda_s}(\mathbb{D}_s^H\overline{Y}_s)\overline{Z}_s ds + \beta\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|^2 d\Lambda_s\right) \\
=\ &2E\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(f(s,\eta_s,P_{(X_s,W_s)},X_s,W_s) - f(s,\eta_s,P_{(X'_s,W'_s)},X'_s,W'_s))ds \\
+\ &2E\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(g(s,\eta_s,P_{(X_s)},X_s) - g(s,\eta_s,P_{(X'_s)},X'_s))d\Lambda_s
\end{aligned}$$

Note that $2|\overline{y}_s||f(s,\eta_s,\mu_1,x_s,w_s) - f(s,\eta_s,\mu_2,x'_s,w'_s)| \leqslant 2C|\overline{y}_s|(|\overline{x}_s| + |\overline{w}_s| + W_2(\mu_1,\mu_2))$.

$2|\overline{y}_s||g(s,\eta_s,\nu_1,x_s) - g(s,\eta_s,\nu_2,x'_s)| \leqslant 2C|\overline{y}_s|(W_2(\nu_1,\nu_2)+|\overline{x}_s|) \leqslant \frac{C^2}{\alpha}|\overline{y}_s|^2 + 2\alpha|\overline{x}_s|^2 + 2\alpha W_2^2(\nu_1,\nu_2)$ for some $\alpha>0$.

Choose $\beta = \frac{C^2}{\alpha}+1$. Then by the Schwartz inequality we obtain

$$\begin{aligned}
&E\left(e^{\beta\Lambda_t}|\overline{Y}_t|^2 + \frac{2}{M}\int_t^T e^{\beta\Lambda_s}s^{2H-1}|\overline{Z}_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|^2 d\Lambda_s\right) \\
=\ &2KE\int_t^T e^{\beta\Lambda_s}|\overline{Y}_s|(|\overline{X}_s| + |\overline{W}_s|)ds + \alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s
\end{aligned}$$

$$\leqslant \quad 2K\int_t^T \left(Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\right)^{1/2}\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s.$$

Denote $\varphi(t)=\left(Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\right)^{1/2}$ and $\psi(t)=\alpha E\int_t^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s$ which is nonincrease. Then by above

$$\varphi(t)^2\leqslant 2K\int_t^T \varphi(t)\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\psi(t),\quad t\in[t_k,T].$$

Applying Lemma 20 in Maticiuc and Nie (1994) to the above inequality we get

$$\varphi(t)\leqslant \sqrt{2}K\int_t^T \left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds+\sqrt{\psi(t)},\quad t\in[t_k,T].$$

and therefore for $t\in[t_k,T]$

$$Ee^{\beta\Lambda_s}|\overline{Y}_s|^2\leqslant 4K^2\left(\int_t^T \left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds\right)^2+2\psi(t),$$

Integrate of both sides on $[t_k,T]$ of above inequality, we can compute

$$\begin{aligned}
\int_{t_k}^T \varphi(s)^2 ds \quad &\leqslant \quad 2\psi(t_k)(T-t_k)+4K^2\int_{t_k}^T\left(\int_t^T\left(Ee^{\beta\Lambda_s}(|\overline{X}_s|+|\overline{W}_s|)^2\right)^{1/2}ds\right)^2 dt\\
&\leqslant \quad 2\psi(t_k)(T-t_k)+8K^2(T-t_k)\left(\int_{t_k}^T (Ee^{\beta\Lambda_s}|\overline{X}_s|^2)^{1/2}ds\right)^2\\
&+ \quad 8K^2(T-t_k)\left(\int_{t_k}^T\left(\frac{1}{s^{2H-1}}Ee^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2\right)^{1/2}ds\right)^2\\
&\leqslant \quad 2\psi(t_k)(T-t_k)+8K^2(T-t_k)^2E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2 ds\\
&+ \quad 8K^2(T-t_k)\int_{t_k}^T\frac{1}{s^{2H-1}}dsE\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2 ds\\
&:= \quad C\cdot(T-t_k)\widetilde{\Theta}(t_k,T),
\end{aligned}$$

and similarly

$$\int_{t_k}^T\frac{1}{s^{2H-1}}\varphi(s)^2 ds \quad\leqslant\quad \frac{C}{2-2H}\cdot(T^{2-2H}-t_k^{2-2H})\cdot\widetilde{\Theta}(t_k,T),$$

where

$$\widetilde{\Theta}(t_k,T)=E\left(\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{W}_s|^2 ds+\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2(ds+d\Lambda_s)\right).$$

Using above inequalities, we deduce

$$\begin{aligned}
& E\left(\int_{t_k}^T e^{\beta\Lambda_s}s^{2H-1}|\overline{Z}_s|^2 ds+\int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2(ds+d\Lambda_s)\right)\\
\leqslant\quad & E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2 ds+C\alpha E\int_{t_k}^T e^{\beta\Lambda_s}|\overline{X}_s|^2 d\Lambda_s+CE\int_{t_k}^T e^{\beta\Lambda_s}\frac{1}{\alpha}|\overline{Y}_s|^2(2+\frac{1}{s^{2H-1}})ds\\
+\quad & CE\int_{t_k}^T e^{\beta\Lambda_s}\alpha(|\overline{X}_s|^2+s^{2H-1}|\overline{W}_s|^2)ds\\
\leqslant\quad & C\cdot(T-t_k)\widetilde{\Theta}(t_k,T)+\frac{C}{\alpha}\int_{t_k}^T\varphi(s)(2+\frac{1}{s^{2H-1}})ds+C\alpha\widetilde{\Theta}(t_k,T)\\
\leqslant\quad & C\left(\alpha+(2+\frac{1}{\alpha})(T-t_k))+\frac{1}{\alpha}(T^{2-2H}-t_k^{2-2H})\right)\widetilde{\Theta}(t_k,T)
\end{aligned}$$

Choosing α such that $C\alpha \leqslant 1/4$ and taking k large enough that $C(\alpha+2)(T-t_k)/\alpha \leqslant 1/4$ and $C(T^{2-2H}-t_k^{2-2H})/\alpha \leqslant 1/4$, we obtain

$$\begin{aligned} & E\left(\int_{t_k}^T e^{\beta\Lambda_s} s^{2H-1}|\overline{Z}_s|^2 ds + \int_{t_k}^T e^{\beta\Lambda_s}|\overline{Y}_s|^2(ds+d\Lambda_s)\right) \\ \leqslant\ & \frac{3}{4}\widetilde{\Theta}(t_k,T) \end{aligned}$$

Thus Γ is contraction operator in $\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]}\times\widetilde{\mathcal{V}}^H_{[t_k,T]}$, and (Y^m,Z^m) is a Cauchy sequence in $\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]}\times\widetilde{\mathcal{V}}^H_{[t_k,T]}$, where $(Y^0,Z^0)\in\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]}\times\widetilde{\mathcal{V}}^H_{[t_k,T]}$, and for $m\geqslant 0$

$$\begin{aligned} Y_t^{m+1} &:= \xi + \int_t^T f(s,\eta_s,P_{(Y_s^m,Z_s^m)},Y_s^m,Z_s^m)ds + \int_t^T g(s,\eta_s,P_{(Y_s^m)},Y_s^m)d\Lambda_s \\ &- \int_t^T Z_s^{m+1}dB_s^H. \end{aligned}$$

Then there exists $(Y,Z)\in\widetilde{\mathcal{V}}^{1/2}_{[t_k,T]}\times\widetilde{\mathcal{V}}^H_{[t_k,T]}$ being a limit of (Y^m,Z^m), i.e.

$$\lim_{m\to+\infty} E\left(e^{\beta\Lambda_t}|Y_t^m-Y_t|^2 + \int_{t_k}^T e^{\beta\Lambda_s}(|Y_s^m-Y_s|^2 + s^{2H-1}|Z_s^m-Z_s|^2)ds\right) = 0,$$

$$\lim_{m\to+\infty} E\left(\int_{t_k}^T e^{\beta\Lambda_s}|Y_s^m-Y_s|^2 d\Lambda_s\right) = 0,$$

Therefore for any $t\in[t_k,T]$,

$$\begin{aligned} & \lim_{m\to\infty}\left(-Y_t^{m+1} + \xi + \int_t^T f(s,\eta_s,P_{(Y_s^m,Z_s^m)},Y_s^m,Z_s^m)ds + \int_t^T g(s,\eta_s,P_{(Y_s^m)},Y_s^m)d\Lambda_s\right) \\ =\ & -Y_t + \xi + \int_t^T f(s,\eta_s,P_{(Y_s,Z_s)},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{(Y_s)},Y_s)d\Lambda_s \end{aligned}$$

in $L^2(\Omega,\mathcal{F},P)$ and $Z^m\mathbf{1}_{[t,T]}\to Z\mathbf{1}_{[t,T]}$ in $L^2(\Omega,\mathcal{F},\mathcal{H})$. We show (Y,Z) that satisfies (5) on $[t_k,T]$. The next step is to solve the equation on $[t_{k-1},t_k]$. With the same arguments, repeating the above technique we obtain a uniqueness of the solution of generalized BSDE with respect to fBm on the whole interval $[0,T]$. □

Now we would like to study the comparison theorem. From the counter examples in Borkowska (2013) (see the example 3.1 and 3.2 therein) and example 2.1 in Juan, Hao and Zhang (2018) (only need to simple modify), we know that if the driver f depends on the law of Z or is not increasing with respect to the law of Y, we usually do not have the comparison theorem. Now we give two examples here.

Example: Let $d=1$. We consider

$$Y_t^i = \xi^i + \int_t^T E[|Z_s^i|]ds - \int_t^T Z_s^i dB_s^H,\ \ i=1,2.\quad 0\leqslant t\leqslant T.$$

For $\xi^2=0,(Y^2,Z^2)=(0,0)$, in particular, $Y_0^2=0$. We consider two cases for ξ^1.
(i) For $\xi^1 := -((B_T^H)^+)^2 \leqslant 0$, $Z_t^1 := E[D_t^H[\xi^1]|\mathcal{F}_t] = -2E[(B_T^H)^+|\mathcal{F}_t] \leqslant 0$. Thus $E[|Z_t^1|] = E[-Z_t^1] = 2E[(B_T^H)^+] = 2\int_0^\infty x\frac{1}{\sqrt{2\pi}T^H}e^{-\frac{x^2}{2T^{2H}}}dx = \frac{2T^H}{\sqrt{2\pi}}$, $t\in[0,T]$. And $Y_0^1 = E[\xi^1]+\int_t^T E[|Z_s^1|]ds = -\frac{T^{2H}}{2}+\frac{2T^{H+1}}{\sqrt{2\pi}} > 0$, for $T > (\frac{\sqrt{2\pi}}{4})^{\frac{1}{1-H}}$, i.e. for $T > (\frac{\sqrt{2\pi}}{4})^{\frac{1}{1-H}}$, $Y_0^1 > 0 = Y_0^2$, although $\xi^1\leqslant 0=\xi^2, P-a.s.$
(ii) For $\xi^1 := -e^{-B_T^H} < 0$, $Z_t^1 := E[D_t^H[\xi^1]|\mathcal{F}_t] = E[e^{-B_T^H}|\mathcal{F}_t] > 0, t\in[0,T]$. Thus $E[|Z_t^1|] = E[Z_t^1] = E[e^{-B_T^H}] = \int_R e^{-x}\frac{1}{\sqrt{2\pi}T^H}e^{-\frac{x^2}{2T^{2H}}}dx = \frac{1}{\sqrt{2\pi}T^H}\int_R e^{-\frac{1}{2T^{2H}}(x+T^{2H})^2}dxe^{\frac{T^{2H}}{2}} = e^{\frac{T^{2H}}{2}}$, $t\in[0,T]$, and $Y_0^1 = E[\xi^1]+\int_t^T E[|Z_s^1|]ds = -e^{\frac{T^{2H}}{2}}+Te^{\frac{T^{2H}}{2}} > 0$, for $T>1$, i.e. for $T>1$, $Y_0^1 > 0 = Y_0^2$, although $\xi^1 < 0 = \xi^2, P-a.s.$

We consider now the mean-field BSDE as follows

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{Y_s},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{Y_s},Y_s)d\Lambda_s - \int_t^T Z_s dB_s^H,\quad 0\leqslant t\leqslant T. \tag{5}$$

Theorem 4.6. *(Comparison theorem) Let* $(f_i, g_i) = (f_i(s,\omega,\eta,\mu,y,z), g_i(s,\omega,\eta,\nu,y))$, $i = 1,2$, *be two pair drivers satisfying the assumption (H1.4). Moreover, we suppose*

(i) One of the both coefficients pairs satisfies Lipschitz in (μ, y, z) *and* (ν, y).

(ii) One of the both coefficients pairs satisfies: for all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}; \mathbb{R})$, *and all* $(s, \eta, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, $f_i(s, \eta, P_{\theta_1}, y, z) - f_i(s, \eta, P_{\theta_2}, y, z) \leqslant L(E[((\theta_1 - \theta_2)^+)^2])^{1/2}$,

$g_i(s, \eta, P_{\theta_1}, y) - g_i(s, \eta, P_{\theta_2}, y) \leqslant L(E[((\theta_1 - \theta_2)^+)^2])^{1/2}$.

Let $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$ *and denote by* (Y^1, Z^1) *and* (Y^2, Z^2) *the solution of the mean-field BSDE (6) with data* (ξ_1, f_1, g_1) *and* (ξ_2, f_2, g_2), *respectively. Then, if* $\xi_1 \leqslant \xi_2, P-a.s.$, $f_1(s,\eta,\mu,y,z) \leqslant f_2(s,\eta,\mu,y,z), dsdP-a.e.$, *and* $g_1(s,\eta,\nu,y) \leqslant g_2(s,\eta,\nu,y), dsdP-a.e.$ *for all* (η,μ,ν,y,z), *it holds that also* $Y_s^1 \leqslant Y_s^2$, *for all* $s \in [0,T], P-a.s.$

Proof. Without loss of generality, we assume that (i) and (ii) are satisfied by (f_1, g_1). Let us put $\overline{f_s} := f_1(s, \eta, P_{Y_s^1}, Y_s^1, Z_s^1) - f_2(s, \eta, P_{Y_s^2}, Y_s^2, Z_s^2)$, $\overline{g_s} := g_1(s, \eta, P_{Y_s^1}, Y_s^1) - g_2(s, \eta, P_{Y_s^2}, Y_s^2)$, and $\overline{Z_s} := Z_s^1 - Z_s^2$, $\overline{Y_s} := Y_s^1 - Y_s^2$. From Itô-Tanakas formula applied to $(\overline{Y_s}^+)^2$, we have

$$E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}\|\overline{Z_r}\|_s^2 1_{(\overline{Y_s}>0)}ds = 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\overline{f_s}ds + 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\overline{g_s}d\Lambda_s,$$

Notice that, since (f_1, g_1) is Lipschitz continuous and $f_1 \leqslant f_2, g_1 \leqslant g_2$, we have

$$\begin{aligned} & E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}(\|\overline{Z_r}\|_s^2)1_{(\overline{Y_s}>0)}ds \\ \leqslant\ & 2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left(f_1(s,\eta,P_{Y_s^1},Y_s^1,Z_s^1) - f_2(s,\eta,P_{Y_s^2},Y_s^1,Z_s^1) + C|\overline{Y_s}| + C|\overline{Z_s}|\right)ds \\ & +2E\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left(g_1(s,\eta,P_{Y_s^1},Y_s^1) - g_2(s,\eta,P_{Y_s^2},Y_s^1) + C|\overline{Y_s}|\right)d\Lambda_s, \end{aligned}$$

Moreover, as for all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}; \mathbb{R})$ and $(s, \eta, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$f_1(s,\eta,P_{\theta_1},y,z) - f_1(s,\eta,P_{\theta_2},y,z) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2},$$

$$g_1(s,\eta,P_{\theta_1},y) - g_1(s,\eta,P_{\theta_2},y) \leqslant L(E[((\theta_1-\theta_2)^+)^2])^{1/2}.$$

we have

$$\begin{aligned} & E[(\overline{Y_s}^+)^2] + E\int_t^T \frac{d}{ds}(\|\overline{Z_r}\|_s^2)1_{(\overline{Y_s}>0)}ds \\ \leqslant\ & CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}| + |\overline{Z_s}|\right)ds \\ & +CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}|\right)d\Lambda_s, \end{aligned}$$

by Remark 3.1, we obtain, there exists a suitable constant $M > 0$,

$$\frac{2}{M}s^{2H-1}|\overline{Z_s}|^2 \leqslant \frac{d}{ds}(\|\overline{Z_r}\|_s^2) \leqslant 2Ms^{2H-1}|\overline{Z_s}|^2,$$

Thus

$$\begin{aligned} & E[(\overline{Y_s}^+)^2] + \frac{2}{M}E\int_t^T s^{2H-1}|\overline{Z_s}|^2 1_{(\overline{Y_s}>0)}ds \\ \leqslant\ & CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}| + |\overline{Z_s}|\right)ds \\ & +CE\int_t^T \overline{Y_s}^+ 1_{(\overline{Y_s}>0)}\left((E[(\overline{Y_s}^+)^2])^{1/2} + |\overline{Y_s}|\right)d\Lambda_s \\ \leqslant\ & CE\int_t^T (\overline{Y_s}^+)^2 ds + CE\int_t^T (\overline{Y_s}^+)^2 s^{1-2H}ds + CE\int_t^T |\overline{Z_s}|^2 s^{2H-1}1_{(\overline{Y_s}>0)}ds + CE\int_t^T (\overline{Y_s}^+)^2 d\Lambda_s \end{aligned}$$

$$\leqslant \quad CE\int_t^T (\overline{Y_s}^+)^2(1+p(s)+s^{1-2H})ds + CE\int_t^T |\overline{Z_s}|^2 s^{2H-1} 1_{(\overline{Y_s}>0)}ds,$$

the last inequality applies assumption ($H1.4$). Choose suitable M, such that $\frac{2}{M} - C > 0$, then we have

$$E[(\overline{Y_s}^+)^2] \leqslant CE\int_t^T (\overline{Y_s}^+)^2(1+p(s)+s^{1-2H})ds,$$

From Gronwall's inequality, $E(\overline{Y_s}^+)^2 = 0$, $s \in [0,T]$, i.e. $Y_s^1 \leqslant Y_s^2$, $P-a.s$, $s \in [0,T]$. □

5. General Mean-Field Fractional BSDEs Under Continuous Coefficients

We assume that the coefficients f and g of the GFBSDE are continuous functions and satisfy the following assumption ($H2$):

(***H*2.1**) Linear growth: There exists $K \geqslant 0$, such that

$$|f(t,\eta,\mu,y,z)| \leqslant K(1+W_2(\mu,\delta_0)+|y|+|\eta|+|z|), dtdP-a.e \quad for \quad all \quad (\eta,\mu,y,z),$$

$$|g(t,\eta,\nu,y)| \leqslant K(1+W_2(\nu,\delta_0)+|y|+|\eta|), dtdP-a.e \quad for \quad all \quad (\eta,\nu,y).$$

where δ_0 is the Dirac measure with mass at $0 \in \mathbb{R}^{1+d}$ or $0 \in \mathbb{R}^d$.

(***H*2.2**) Monotonicity in μ: for all $\theta_1, \theta_2 \in L^2(\Omega, \mathcal{F}; \mathbb{R})$, and all $(\eta, y, z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$,

$$f(s,\eta,P_{\theta_2},y,z) \leqslant f(s,\eta,P_{\theta_1},y,z), \ \ dtdP-a.e, whenever \quad \theta_2 \leqslant \theta_1,$$

$$g(s,\eta,P_{\theta_2},y) \leqslant g(s,\eta,P_{\theta_1},y), \ \ dtdP-a.e, whenever \quad \theta_2 \leqslant \theta_1.$$

(***H*2.3**) For a.e.$(s,\omega) \in [0,T]\times\Omega$, $f(s,\omega,\cdot,\cdot,\cdot,\cdot)$, $g(s,\omega,\cdot,\cdot,\cdot)$ are continuous with a continuity modulus $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ for μ:

$$|f(s,\omega,\eta,\mu_1,y,z) - f(s,\omega,\eta,\mu_2,y,z)| + |g(s,\omega,\eta,\nu_1,y) - g(s,\omega,\eta,\nu_2,y)| \leqslant \rho(W_2(\mu_1,\mu_2)).$$

Here ρ is supposed to be increasing and such that $\rho(0+) = 0$.

Remark 5.1. *(**H2.2**) is equivalent to the following condition:*

*(**H2.2′**): For all $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$, $(s,\eta,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$, it holds $f(s,\eta,\mu_2,y,z) \leqslant f(s,\eta,\mu_1,y,z)$, whenever the distribution functions F_{μ_1}, F_{μ_2} satisfy $F_{\mu_1} \leqslant F_{\mu_2}$. Recall that $F_\mu(x) = \mu((-\infty,x])$, $x\in\mathbb{R}$, $\mu \in \mathcal{P}_2(\mathbb{R})$.*

*Indeed, if we let $\mu_1 = P_{\theta_1}$,$\mu_2 = P_{\theta_2}$, then from $\theta_2 \leqslant \theta_1$, P-a.s., we get $F_{\mu_1} \leqslant F_{\mu_2}$, and (**H2.2′**) implies $f(s,\eta,P_{\theta_2},y,z) \leqslant f(s,\eta,P_{\theta_1},y,z)$. This shows that (**H2.2′**) ⇒ (**H2.2**).*

*In order to show that (**H2.2**)⇒(**H2.2′**): We consider $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$, with $F_{\mu_1} \leqslant F_{\mu_2}$. Let ξ be a random variable uniformly distributed on $[0,1]$, and let $F_{\mu_i}^{-1}$ be the left inverse function of F_{μ_i} . Then $\theta_2 := F_{\mu_2}^{-1}(\xi) \leqslant F_{\mu_1}^{-1}(\xi) =: \theta_1$, and $P_{\theta_1} = \mu_1$, $P_{\theta_2} = \mu_2$. From (**H2.2**) we get $f(s,\eta,P_{\theta_2},y,z) \leqslant f(s,\eta,P_{\theta_1},y,z)$.*

Before proving the main theorem in this paper, we need the following lemma which gives the approximation of continuous functions by the Lipschitz functions and it was presented by Lepeltier and Martin (1997). we have to introduce a new method to study the relationship between two measures, we define

$$W_{2,+}(\mu,\nu) = \inf_{\pi\in\prod(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |(x-y)^+|^2 \pi(dx,dy) \right)^{1/2} \right\},$$

where $\prod(\mu,\nu)$ is the family of all couplings of μ and ν, i.e., $\pi \in \prod(\mu,\nu)$ if and only if π is a measure on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals $\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d)$.

The following Lemma is a modified based on Lemma 3.1 in Li, Liang and Zhang (2018).

Lemma 5.2. *Let $f : [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ be a continuous function in (η,μ,y,z) and satistying $(H2)$, Then the sequence of functions*

$$f_n(s,\omega,\eta,\mu,y,z) := ess \inf_{(\zeta,\nu,r,b)\in\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d}\{f(s,\omega,\zeta,\nu,r,b) + nW_{2,+}(\mu,\nu) + n|\eta-\zeta| + n|y-r| + n|z-b|\}$$

is well defined for $n \geqslant K$ and has the following properties

(i) Linear growth: for all $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $|f_n(s,\omega,\eta,\mu,y,z)| \leqslant C(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|)$;

(ii) Monotonicity in μ: $f_n(s,\omega,\eta,\mu_2,y,z) \leqslant f_n(s,\omega,\eta,\mu_1,y,z)$, for $\mu_1,\mu_2 \in \mathcal{P}_2(\mathbb{R}^{1+d})$ with $F_{\mu_2} \geqslant F_{\mu_1}$, for all $(s,\omega,\eta,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d, n \geqslant 1$;

(iii) Monotonicity in n: for any $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $n \leqslant m$, $f_n(s,\omega,\eta,\mu,y,z) \leqslant f_m(s,\omega,\eta,\mu,y,z)$;

(iv) Lipschtiz condition: for any $(s,\omega,\eta,\mu,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$, $|f_n(s,\omega,\eta,\mu,y,z) - f_n(s,\omega,\eta_1,\mu_1,y_1,z_1)| \leqslant n(W_2(\mu,\mu_1) + |\eta-\eta_1| + |y-y_1| + |z-z_1|)$;

(v) Strong convergence: If $(\eta_n,\mu_n,y_n,z_n) \to (\eta,\mu,y,z)$ in $\mathbb{R}\times\mathcal{P}_2(\mathbb{R}^{1+d})\times\mathbb{R}\times\mathbb{R}^d$ as $n\to\infty$, then $f_n(s,\omega,\eta_n,\mu_n,y_n,z_n) \to f(s,\omega,\eta,\mu,y,z)$ as $n\to\infty$.

From Lemma 5.2, for fixed s, we consider the sequence $f_n(s,\omega,\eta,\mu,y,z)$, and $g_n(s,\omega,\nu,\mu,y)$ $n \geqslant 1$, related to f and g, respectively. Also consider $h(s,\omega,\eta,\mu,y,z) = K(1 + W_2(\mu,\delta_0) + |\eta| + |y| + |z|)$. It is obvious now that f_n and h are F-progressively measurable functions which are Lipschitz in (μ,y,z), uniformly in (s,ω). For $\xi \in L^2(\Omega,\mathcal{F}_T;\mathbb{R})$ we know from Proposition 4.5, for $n \geqslant K$, that the following mean-field BSDEs have a unique adapted solution

$$Y_t^n = \xi + \int_t^T f_n(s,\eta_s,P_{Y_s^n},Y_s^n,Z_s^n)ds + \int_t^T g_n(s,\eta_s,P_{Y_s^n},Y_s^n)d\Lambda_s - \int_t^T Z_s^n dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{6}$$

$$U_t = |\xi| + \int_t^T h(s,\eta_s,P_{U_s},U_s,V_s)ds + \int_t^T q(s,\eta_s,P_{U_s},U_s)d\Lambda_s - \int_t^T V_s dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{7}$$

From Lemma 5.2, we know that (f_n, g_n) and (h, q) satisfy the assumptions of Proposition 4.5, therefore we have

$$-U_s \leqslant Y_s^m \leqslant Y_s^n \leqslant U_s, P-a.s, \quad s\in[0,T], \quad for \quad all \quad n \geqslant m \geqslant K. \tag{8}$$

The following two Lemmas have been implied in Proposition 4.5.

Lemma 5.3. *There exists a conatant C which depends on K, T and $E[e^{\beta\Lambda_T}\xi^2]$, such that*

$$E\left(e^{\beta\Lambda_s}|Y_t^n|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s^n|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s^n|^2 d\Lambda_s\right) \leqslant C,$$

$$E\left(e^{\beta\Lambda_s}|U_t|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|V_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|U_s|^2 d\Lambda_s\right) \leqslant C.$$

Lemma 5.4. $(Y^n, Z^n), n \geqslant 1$*, converges in* $\mathcal{V}_{[0,T]}^{1/2}\times\mathcal{V}_{[0,T]}^H$.

Now ,we give the main result of this paper:

Theorem 5.5. *Let $\xi \in L^2(\Omega,\mathcal{F}_T;\mathbb{R})$. Assume $(H2)$ holds. Then equation*

$$Y_t = \xi + \int_t^T f(s,\eta_s,P_{Y_s},Y_s,Z_s)ds + \int_t^T g(s,\eta_s,P_{Y_s},Y_s)d\Lambda_s - \int_t^T Z_s^n dB_s^H, \quad 0 \leqslant t \leqslant T. \tag{9}$$

has an adapted solution (Y,Z). Also, there is a minimal solution (Y^*,Z^*) of (9), in the sense that for any other solution (Y,Z) of (9), we have $Y_s^* \leqslant Y_s, s\in[0,T]$, P-a.s. Moreover, for all $t\in[0,T]$,

$$E\left(e^{\beta\Lambda_s}|Y_t|^2 + \int_t^T e^{\beta\Lambda_s}s^{2H-1}|Z_s|^2 ds + \int_t^T e^{\beta\Lambda_s}|Y_s|^2 d\Lambda_s\right) \leqslant C\Theta(t,T),$$

where

$$\Theta(t,T) := E\left(e^{\beta\Lambda_T}|\xi|^2 + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s,Z_s)^2])ds + \int_t^T e^{\beta\Lambda_s}|\eta_s|^2 ds + 2\int_t^T e^{\beta\Lambda_s}(1 + E[(Y_s)^2])d\Lambda_s\right).$$

Proof. From (8) we have $Y^{n_0} \leqslant Y^n \leqslant U$ for all $n \geqslant n_0 \geqslant K$. Moreover, $Y^n \to Y$ converges in $\mathcal{V}^{1/2}_{[0,T]}$, On the other hand, also $Z^n \to Z$ in $\mathcal{V}^H_{[0,T]}$.

Hence, thanks to (i) and (v) in Lemma 5.2, we get

$$f_n(s,\eta_s,P_{Y^n_s},Y^n_s,Z^n_s) \to f(s,\eta_s,P_{Y_s},Y_s,Z_s), \quad n \to \infty,$$

$$g_n(s,\eta_s,P_{Y^n_s},Y^n_s) \to g(s,\eta_s,P_{Y_s},Y_s). \quad n \to \infty.$$

Thus

$$E\left(\int_t^T e^{\beta\Lambda_s}|f_n(s,\eta_s,P_{Y^n_s},Y^n_s,Z^n_s) - f(s,\eta_s,P_{Y_s},Y_s,Z_s)|^2 ds\right) \to 0, as \quad n \to \infty,$$

and

$$E\left(\int_t^T e^{\beta\Lambda_s}(g_n(s,\eta_s,P_{Y^n_s},Y^n_s) - g(s,\eta_s,P_{Y_s},Y_s))d\Lambda_s\right)^2 \to 0, as \quad n \to \infty.$$

From Theorem 2.1, Lemma 4.5 and remark 3.2, we can get

$$\begin{aligned}
E(\int_t^T e^{\beta\Lambda_s}(Z^n_s - Z_s)dB^H_s)^2 &= E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + \int_t^T \int_t^T \mathbb{D}^H_r(Z^n_s - Z_s)\mathbb{D}^H_s(Z^n_r - Z_r)drds\right)\\
&= E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + 2\int_t^T \int_s^T \mathbb{D}^H_r(Z^n_s - Z_s)\mathbb{D}^H_s(Z^n_r - Z_r)drds\right)\\
\leqslant\quad & E\left(\int_t^T e^{2\beta\Lambda_s}(Z^n_s - Z_s)^2 ds + 2M^2\int_t^T \int_s^T (sr)^{2H-1}(Z^n_s - Z_s)(Z^n_r - Z_r)drds\right), \quad as \quad n \to \infty.
\end{aligned}$$

On the other hand, from the BSDE (6) we can prove similarly that $E[\int_0^T |Y^n_t - Y^m_t|dt] \to 0$ as $n,m \to \infty$. Therefore, Y has a continuous version, i.e. $Y \in \mathcal{V}^{1/2}_{[0,T]}$ and $E[\int_0^T |Y^n_t - Y_t|dt] \to 0$ as $n \to \infty$. Thus, taking the limit in (6), we get that (Y,Z) solves (9).

Let $(\widehat{Y},\widehat{Z}) \in \mathcal{V}^{1/2}_{[0,T]} \times \mathcal{V}^H_{[0,T]}$ be any solution of (9). From the comparison theorem we get that $Y^n_s \leqslant \widehat{Y}_s$, $s \in [0,T], P-a.s.$, for all $n \geqslant 1$, and therefore $Y_s \leqslant \widehat{Y}_s$, $s \in [0,T]$, P-a.s., that is, Y is the minimal solution of (9). □

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