Lehmann Type II Frechet Poisson Distribution: Properties, Inference and Applications as a Life Time Distribution

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Abstract

A new generalization of the Frechet distribution called Lehmann Type II Frechet Poisson distribution is defined and studied. Various structural mathematical properties of the proposed model including ordinary moments, incomplete moments, generating functions, order statistics, Renyi entropy, stochastic ordering, Bonferroni and Lorenz curve, mean and median deviation, stress-strength parameter are investigated. The maximum likelihood method is used to estimate the model parameters. We examine the performance of the maximum likelihood method by means of a numerical simulation study. The new distribution is applied for modeling three real data sets to illustrate empirically its flexibility and tractability in modeling life time data.

Keywords: Lehmann Type II Frechet Poisson distribution, stress-strength parameter, generating functions, order statistics, stochastic ordering

1. Introduction

Frechet distribution which is also known as Inverse Weibull distribution belong to the class of Type II extreme value distribution was developed by Frechet (1924) is a very useful distribution for modeling life time data. The Frechet distribution is one of the important distributions in extreme value theory and has several applications which include: floods, horse racing, accelerated life testing, earthquakes, sea waves, rainfall and wind speeds. For more studies on the properties and applications of Frechet distribution, see Kotz and Nadarajah (2000), also Harlow (2002). This distribution can be used to analyse life time data that exhibits decreasing increasing or constant failure rate. However, models with complex hazard rate shapes such as bathtub, unimodal and other shapes are often encountered in real life time data analysis which may include mortality studies, reliability analysis etc., which the Frechet distribution may not provide a reasonable parametric fit when used for modeling complex phenomenon. Several modifications have been made to improve its parametric fits, some of which are: Beta-Exponential Frechet was developed and studied by Mead et al. (2017). The properties of Transmuted Frechet was investigated by Mahmoud and Mandour (2013), Transmuted Exponentiated Frechet was studied by Elbatal et al. (2014), Krishna et al. (2013) developed and studied Marshall-Olkin Frechet distribution, gamma extended Frechet distribution was studied by Silva et al. (2013) and the exponentiated Frechet distribution was studied by Nadarajah and Kotz (2003). The Odd Lindley Frechet Distribution was studied by Korkmaz et al. (2017), alpha power transformed Frechet was studied by Suleman et al. (2019) and Mead and Abd-Eltawab (2014) studied Kumaraswamy Frechet. Afify et al. (2016a) studied Weibull Frechet, Kumaraswamy Marshall-Olkin Frechet distribution was developed by Afify et al. (2016b), Kumaraswamy transmuted Marshall-Olkin Frechet was studied by Yousof et al. (2016), Beta Transmuted Frechet distribution was developed and studied by Afify et al. (2016c). Yousof et al. (2018b) developed and studied the Topp Leone Generated Frechet distribution and Odd log-logistic Frechet was studied by Yousof et al. (2018a).

In recent times, several new families of distribution have been developed by compounding the Poisson distribution with many other univariate continuous distributions to provide a more flexible and tractable distribution for modeling lifetime failure data. Lu and Shi (2012) developed and studied the Weibull Poisson distribution. The exponential Poisson was studied by Francisco et al. (2020). The exponential Weibull-Poisson distribution which generalises the Weibull-Poisson was studied by Mahmoudi and Sepahdar (2013) and the two parameter Poisson-exponential distribution with increasing failure was studied by Cancho et al. (2011). The exponentiated exponential-Poisson

distribution was derived and studied by Barreto-Souza and Cribari-Neto (2009). The Kumaraswamy Lindley-Poisson distribution which generalises the Lindley-Poisson distribution was studied by Pararai et al. (2015), Mohamed and Rezk (2019) developed and studied the properties and applications of the extended Poisson-Frechet distribution.

Suppose that a random variable X follows a Frechet distribution, having a cumulative distribution function (cdf) and probability density function (pdf), respectively given as:

$$G(x; \gamma, \omega) = e^{-\gamma x^{-\omega}} \tag{1}$$

And

$$g(x; \gamma, \omega) = \gamma \omega x^{-\omega-1} e^{-\gamma x^{-\omega}} \tag{2}$$

Where $\gamma > 0$ and $\omega > 0$. γ is a scale parameter and ω is a shape parameter.

2. Frechet Poisson Distribution

Suppose that the failure time of each subsystem has the Frechet model defined by pdf and cdf in (1) and (2). Given N, let Z_j denote the failure time of the jth subsystem which are independently and identically distributed random variable from Frechet distribution. Taking N to be distributed according to the truncated Poisson random variable with probability mass function (pmf)

$$P(N = n) = \frac{v^n}{n! \, (e^v - 1)}; \qquad n = 1, 2, \quad . \quad . \quad ., v > 0 \tag{3}$$

Suppose that the failure time of each subsystem has the Frechet distribution defined by the cdf given in equation (1).

$$X = \min_{j} \{Z_j\}$$

Unconditional cdf of X given N is

$$\bar{H}_{FP}(x; \gamma, \omega, v) = 1 - P(X > x/N) = 1 - P(Z_1 > x)^N = 1 - \left[1 - e^{-\gamma x^{-\omega}}\right]^N \tag{4}$$

The equation (4) above is the exponentiated Frechet distribution.

So, the unconditional cdf of X (for $x > 0$) is given by

$$\bar{H}_{FP}(x; \gamma, \omega, v) = \frac{1}{(e^v - 1)} \sum_{n=1}^{\infty} \frac{v^n \left[1 - \left[1 - e^{-\gamma x^{-\omega}}\right]^n\right]}{n!}$$

The cdf of Frechet Poisson (FP) distribution is given by

$$\bar{H}_{FP}(x; \gamma, \omega, v) = \frac{1 - e^{-v e^{-\gamma x^{-\omega}}}}{1 - e^{-v}}, \qquad x > 0, \gamma, \omega, v > 0 \tag{5}$$

The FP density function is given by

$$\bar{h}_{FP}(x; \gamma, \omega, v) = \frac{\gamma \omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-v e^{-\gamma x^{-\omega}}}}{1 - e^{-v}}, \qquad x > 0, \gamma, \omega, v > 0 \tag{6}$$

Where $v, \gamma > 0$ and $\omega > 0$. γ is a scale parameter, v and ω are shape parameters

2.1 Lehman Type II Frechet Poisson Distribution

In this sub-section, we present the Lehman Type II Frechet-Poisson (LFP) distribution, and derive some of its properties which include: cdf, pdf, hazard function $(h(x))$, reversed hazard function $(H(x))$, quantile function and sub-models.

The Lehman type II distribution is a hybrid of the generalised exponentiated distribution developed by Cordeiro et al. (2013). Given $\bar{H}(x)$ to be an arbitrary baseline cdf in the interval $(0,1)$. The cdf $\bar{H}(x)$, called the Exponentiated-G (EG) distribution has the cdf

$$F(x; \alpha, \beta) = [1 - \{1 - \bar{H}(x)\}^{\alpha}]^{\beta} \tag{7}$$

Where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters which exhibits tractable properties especially for simulations, since the quantile function takes a simple form given by

$$x = Q_{\bar{H}}\left(\left\{1 - \left(1 - u^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right\}\right) \tag{8}$$

Where $Q_{\bar{H}}(u)$ is the baseline quantile function. The two extra shape parameters can control both tail weight and entropy of EG distribution. The expression in (7) can be splitted into two generalised distribution called the Lehman type I and the Lehman type II distribution by respectively taking $\alpha = 1$ and $\beta = 1$. The distribution function of Lehman type I and Lehman type II are given respectively by

$$F(x; \beta) = \{\bar{H}(x)\}^{\beta} \tag{9}$$

and

$$F(x; \alpha) = 1 - \{1 - \bar{H}(x)\}^{\alpha} \tag{10}$$

Where $\bar{H}(x)$ is the baseline distribution. For the purpose of this study $\bar{H}(x)$ is the cdf of FP distribution. Thus, the goal of this study is to develop another generalization of the Frechet Poisson distribution called the Lehman Type II Frechet Poisson distribution with a wider scope of applications that may be used in modeling real life time data which may include applications in medicine, reliability, aeronautical engineering, weather forecasting and other extreme conditions with a better fit than the Frechet Poisson distribution.

By taking $\bar{H}(x)$ as the cdf of FP distribution in equation (10), we obtain the cdf of Lehman Type II Frechet Poisson (LFP) distribution. The cdf of four-parameter LFP distribution is given by

$$F_{LFP}(x) = 1 - \left\{1 - \frac{1 - e^{-ve^{-\gamma x^{-\omega}}}}{1 - e^{-v}}\right\}^{\alpha} \tag{11}$$

for $x > 0, \alpha > 0, v > 0, \omega > 0 \ and \ \gamma > 0$. The corresponding density of LFP distribution is given by

$$f_{LFP}(x) = \frac{\alpha v \gamma \omega x^{-\omega - 1} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-\omega}}}}{1 - e^{-v}}\left\{1 - \frac{1 - e^{-ve^{-\gamma x^{-\omega}}}}{1 - e^{-v}}\right\}^{\alpha - 1} \tag{12}$$

for $x > 0, \alpha > 0, v > 0, \omega > 0, \gamma > 0$.

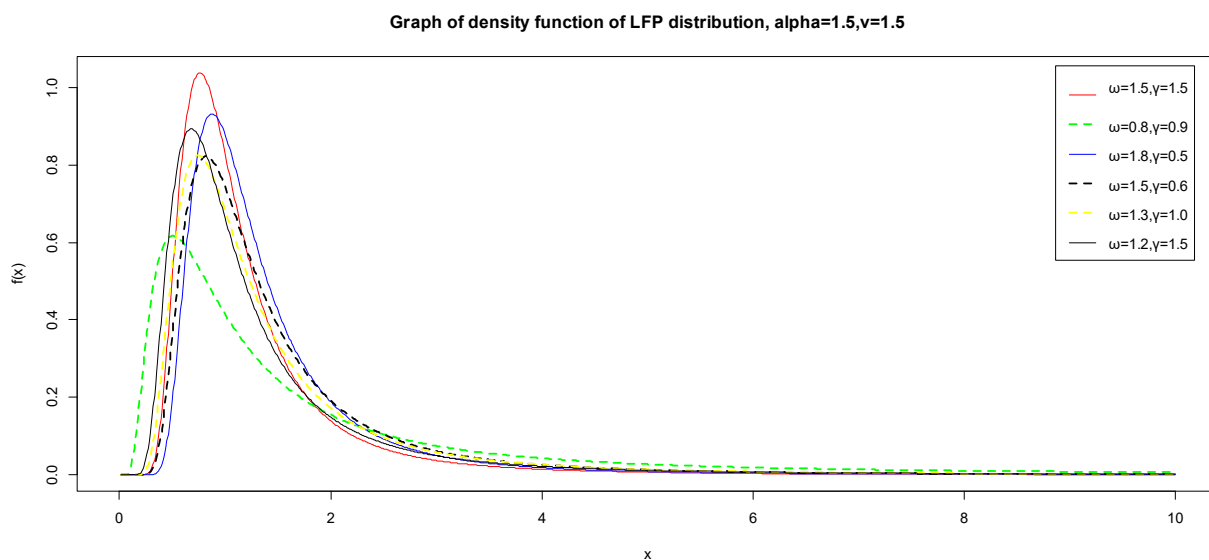Plots of the pdf of LFP distribution are given below in figure 1 and figure 2 for arbitrary values of $, \alpha, v, \omega and \gamma$.

Graph of density function of LFP distribution, alpha=1.5,v=1.5

ω=1.5,γ=1.5
ω=0.8,γ=0.9
ω=1.8,γ=0.5
ω=1.5,γ=0.6
ω=1.3,γ=1.0
ω=1.2,γ=1.5

Figure 1. Plot of the pdf of LFP distribution for different values of ω, γ keeping the values of α, v constant at 1.5

Graph of density function of LFP distribution, alpha=10.5,v=1.5
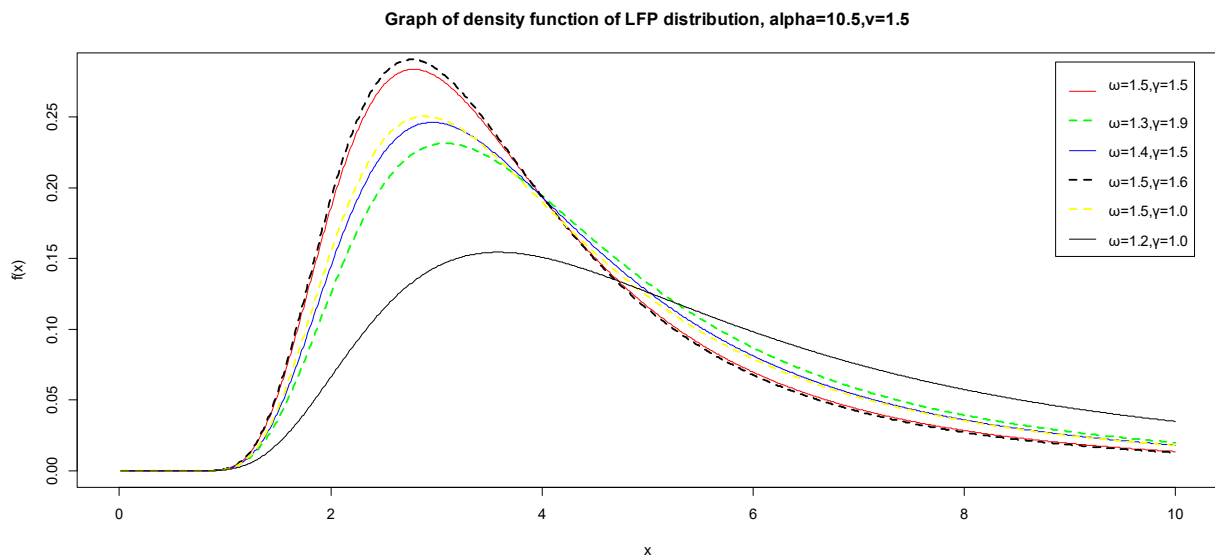
Figure 2. Plot of the pdf of LFP distribution for different values of ω, γ keeping the values of α, v constant at 10.5 and 1.5 respectively

From figure 1 and 2 above, LFP distribution can be viewed as a suitable model for fitting a unimodal and right skewed data

2.2 Survival and the Hazard Function

The survival function of the LFP distribution is given by:

$$S_{LFP}(x) = 1 - F_{LFP}(x) = \left\{ 1 - \frac{1 - e^{-ve^{-\gamma x^{-\omega}}}}{1 - e^{-v}} \right\}^{\alpha} \tag{13}$$

for $x > 0, \alpha > 0, v > 0, \omega > 0$ and $\gamma > 0$. The graph of the survival function for various values of the parameters $\alpha, v, \gamma,$ and ω is given in figure 3.

Graph of Survival function of LFP distribution, alpha=2.5,v=1.5   **Graph of Survival function of LFP distribution, alpha=1.5,v=1.5**
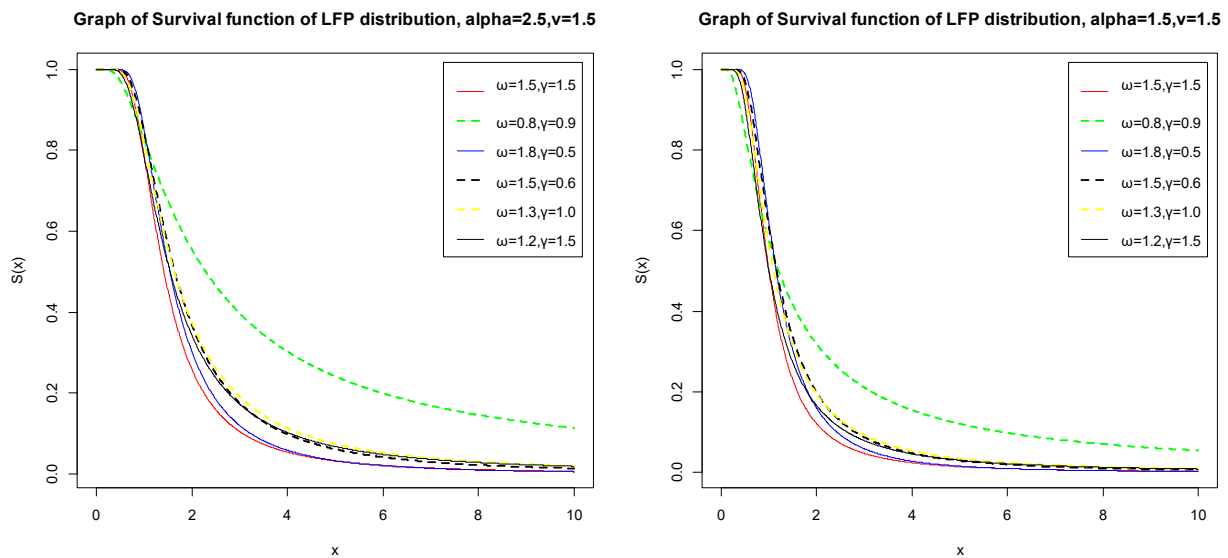
Figure 3. Plot of the survival function of LFP distribution for different values of ω, γ keeping the values of α, v constant

The hazard and the reverse hazard function of the LFP distribution are respectively given by

$$h_{LFP}(x) = \frac{f_{LFP}(x; \alpha, \gamma, \nu, \omega)}{1 - F_{LFP}(x; \alpha, \gamma, \nu, \omega)} \tag{14}$$

Putting equations (12) and (13) in (14), we have

$$h_{LFP}(x) = \frac{\alpha \nu \gamma \omega x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-\nu e^{-\gamma x^{-\omega}}}}{(1 - e^{-\nu}) \left\{ 1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}} \right\}} \tag{15}$$

and

$$\bar{H}_{LFP}(x) = \frac{f_{LFP}(x; \alpha, \gamma, \nu, \omega)}{F_{LFP}(x; \alpha, \gamma, \nu, \omega)} \tag{16}$$

Putting equations (11) and (12) in (16), gives

$$\bar{H}_{LFP}(x) = \frac{\frac{\alpha \nu \gamma \omega}{1 - e^{-\nu}} x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-\nu e^{-\gamma x^{-\omega}}} \left\{ 1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}} \right\}^{\alpha-1}}{1 - \left\{ 1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}} \right\}^{\alpha}} \tag{17}$$

for $x > 0, \alpha > 0, \nu > 0, \omega > 0 \ and \ \gamma > 0$. The graph of the hazard function for various values of the parameters $\alpha, \nu, \gamma,$ and ω is given in figure 4 and 5.
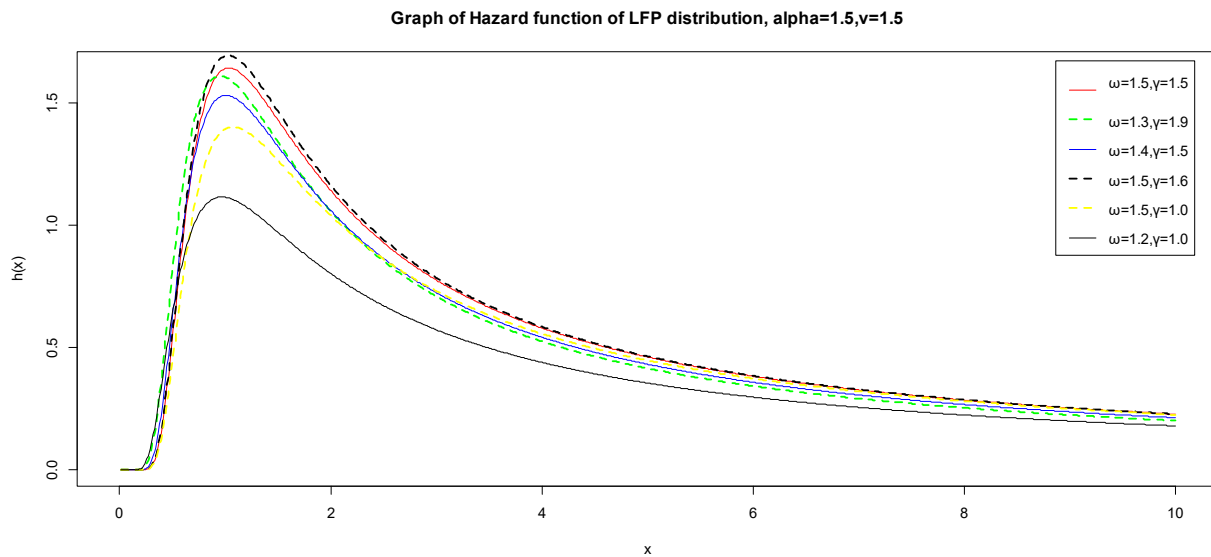
Figure 4. Plot of the hazard function of LFP distribution for different values of ω, γ keeping the values of α, ν constant at 1.5 and 1.5 respectively
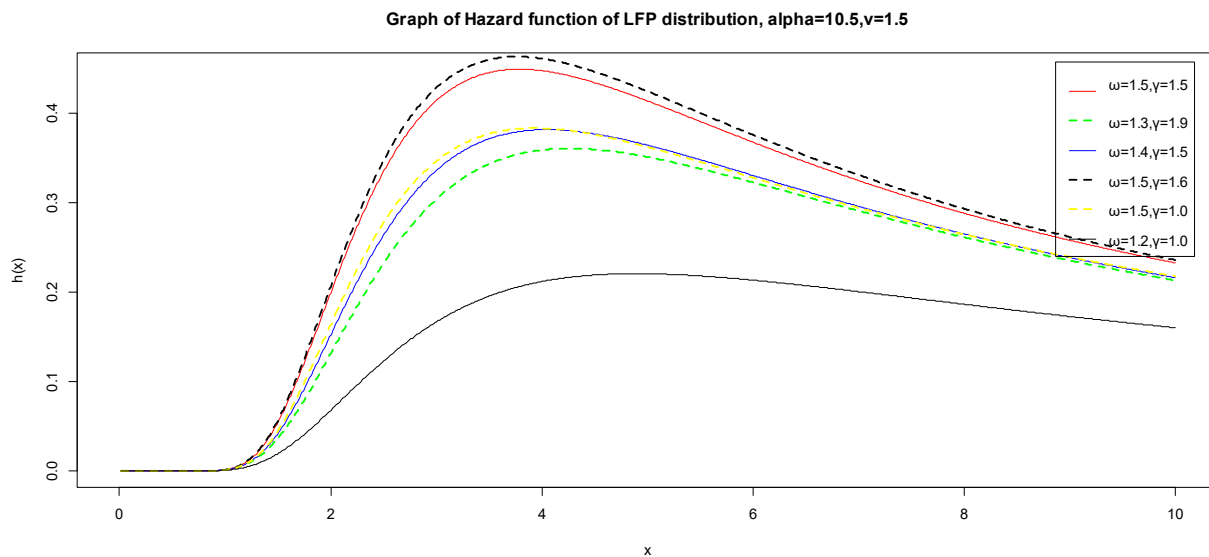
Figure 5. Plot of the survival function of LFP distribution for different values of ω, γ keeping the values of α, ν constant at 10.5 and 1.5 respectively

The graph of the hazard function in figures 4 and 5 for different values of the parameters exhibits various shapes such as monotonically decreasing, increasing, increasing-decreasing and upside down bathtub shapes. This feature indicates the flexibility of LFP distribution and its suitability in modeling monotonic and non-monotonic hazard behaviour which are often encountered in real life situations.

2.3 Some Sub-models of the LFP Distribution

In this sub-section, we give the sub-models of LFP distribution for selected values of the parameters α, ν, γ and ω are presented

- ✓ When $\gamma = 1$, we obtain the Lehmann Type II Inverted Weibull Poisson distribution.
- ✓ When $\omega = 1$, we obtain the Lehmann Type II Inverse exponential Poisson distribution which is given in equation (49).
- ✓ When $\alpha = 1$, we obtain the Frechet Poisson distribution which is given in equation (50).
- ✓ When $\alpha = \gamma = 1$, we obtain the Inverted Weibull Poisson distribution which pdf is given in equation (51).
- ✓ When $\alpha = \omega = 1$, we obtain the Inverse exponential Poisson distribution which is given in equation (52).
- ✓ When $\nu = \gamma = 1$, we obtain the Lehmann Type II Inverted Weibull distribution.
- ✓ When $\nu = \omega = 1$, we obtain the Lehmann Type II Inverse exponential distribution
- ✓ When $\nu = 1$, we obtain the Lehmann Type II Frechet distribution.
- ✓ When $\alpha = \nu = 1$, we obtain the Frechet distribution.

2.4 Expansion of the Density Function

Considering the binomial series expansion given by

$$(1 - w)^{m-1} = \sum_{j=0}^{\infty} \binom{m-1}{j} (-1)^j w^j; \quad m > 0, |w| < 1 \tag{18}$$

Thus we have:

$$\left\{ 1 - \frac{1 - e^{-\nu e^{-\gamma x^{-\omega}}}}{1 - e^{-\nu}} \right\}^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} \left(\frac{1}{1 - e^{-\nu}} \right)^i \left(1 - e^{-\nu e^{-\gamma x^{-\omega}}} \right)^i$$

Subsequently,

$$\left(1 - e^{-ve^{-\gamma x^{-\omega}}}\right)^i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \left(e^{-ve^{-\gamma x^{-\omega}}}\right)^j$$

Then we have,

$$f_{LFP} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^{i+j} \left(\frac{1}{1-e^{-v}}\right)^{i+1} \alpha\gamma\omega x^{-\omega-1} e^{-\gamma x^{-\omega}} \left(e^{-ve^{-\gamma x^{-\omega}}}\right)^{j+1} \qquad (19)$$

Since,

$$e^m = \sum_{k=0}^{\infty} \frac{m^k}{k!} \qquad (20)$$

Applying equation (20) to equation (19), we obtain

$$f_{LFP} = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \binom{\alpha-1}{i} \binom{i}{j} (-1)^{i+j+k} \left(\frac{1}{1-e^{-v}}\right)^{i+1} (j+1)^k v^{k+1} \alpha\gamma\omega(k+1)x^{-\omega-1} \frac{e^{-\gamma(k+1)x^{-\omega}}}{(k+1)!}$$

Therefore,

$$f_{LFP} = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha,v)\gamma\omega(k+1)x^{-\omega-1} \frac{e^{-\gamma(k+1)x^{-\omega}}}{(k+1)!} \qquad (21)$$

Where,

$$\Gamma_{i,j,k}(\alpha,v) = \binom{\alpha-1}{i} \binom{i}{j} (-1)^{i+j+k} \left(\frac{1}{1-e^{-v}}\right)^{i+1} (j+1)^k v^{k+1} \alpha$$

Finally we have,

$$f_{LFP} = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha,v)\, g(x;\gamma(k+1),\omega)$$

2.5 Quantile Function

The quantile function of the LFP distribution is obtained by solving the equation $F(X_u) = u$, where $0 < u < 1$. Then we obtain

$$X_u = \left[-\frac{1}{\gamma}log\left\{-\frac{1}{v}\left[log(1-(1-e^{-v})\left\{1-\left(1-u\right)^{\frac{1}{\alpha}}\right\})\right]\right\}\right]^{-\frac{1}{\omega}} \qquad (22)$$

Classical measures of skewness and kurtosis may be difficult to obtain due to non-existence of higher moments in several heavy tailed distributions. When such a situation occurs, the quantile measures can be considered. The Bowley (B)skewness; Kenny and Keeping (1962) is one of the foremost measures of skewness that is based on quantile of a distribution. It is given by

$$B = \frac{q_{\frac{3}{4}} - 2q_{\frac{2}{4}} + q_{\frac{1}{4}}}{q_{\frac{3}{4}} - q_{\frac{2}{4}}} \qquad (23)$$

Consequently, the coefficient of Kurtosis can be obtained using Moor's (1988) approach to estimating kurtosis which is based on octiles of a distribution and is given by

$$M = \frac{q_{\frac{7}{8}} - q_{\frac{3}{8}} - q_{\frac{5}{8}} + q_{\frac{1}{8}}}{q_{\frac{6}{8}} - q_{\frac{4}{8}}} \qquad (24)$$

It is of noteworthy that the two measures are more robust to outliers.

Table 1 given below represent the various values of Bowley Skewness and Moors Kurtosis for given values of the parameters taking $\nu = 2.3$ and $\alpha = 1.5$

γ, ω	$q_{\frac{1}{4}}$	$q_{\frac{2}{4}}$	$q_{\frac{3}{4}}$	$q_{\frac{1}{8}}$	$q_{\frac{3}{8}}$	$q_{\frac{7}{8}}$	B	M
0.5,1.2	0.0903	0.3017	1.3621	0.0440	0.1654	5.0189	0.6683	3.3897
1.0,2.1	0.2095	0.5269	1.5341	0.1169	0.3355	3.6548	0.5207	1.9488
1.5,2.1	0.4070	0.9584	2.5329	0.2348	0.6316	5.5003	0.4812	1.6988
2.0,3.5	0.5269	1.0901	2.4018	0.3251	0.7680	4.4039	0.3992	1.2758
3.5,6.2	1.1713	2.1648	4.1056	0.7686	1.6152	6.5881	0.3228	0.9622
10.0,15.5	5.8348	9.3509	14.8654	4.1586	7.4923	20.5708	0.2212	0.6188
20.5,20.5	21.5328	33.4270	51.194	15.6469	27.207	68.7588	0.1980	0.5480

From Table 1, we can conclude that the LFP distribution can be used to model data that skewed to the right (positively skewed) with various degree of kurtosis (Kleptokurtic, mesokurtic and leptokurtic).

3. Moments

In this section, we obtain the moment of the LFP distribution. Momentplays important role in statistical analysis, most especially in determining the structural properties of a distribution such as skewness, kurtosis, dispersion, mean etc.

Theorem 1. Let a random variable X follows the Lehmann type II Frechet Poisson distribution, the rth moment of LFP distribution is given by

$$\mu_r' = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, \nu) \frac{\gamma\omega}{(k+1)!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right)$$

Proof: let X be a random variable from LFP distribution, the rth moment is given by

$$E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f_{LFP}(x) dx \qquad (25)$$

Substitute for $f_{LFP}(x)$ in equation (25), we have

$$\mu_r' = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, \nu) \frac{\gamma\omega}{(k+1)!} \int_{-\infty}^{\infty} x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx \qquad (26)$$

By letting

$$B(x) = \int_{-\infty}^{\infty} x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx \qquad (27)$$

Taking, $m = \gamma(k+1)x^{-\omega}, dx = -\frac{1}{\omega}\{\gamma(k+1)\}^{\frac{1}{\omega}} m^{-\frac{1}{\omega}-1} dm$ and putting it in equation (27), we have

$$B(x) = \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right) \qquad (28)$$

It then follows that the rth moment of LFP distribution is given as

$$\mu'_r = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha,\nu) \frac{\gamma\omega}{(k+1)!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right) \tag{29}$$

$$r < \omega, where \; \Gamma(w) = \int_0^{\infty} p^{w-1} e^{-w} dw \; is \; the \; complementary \; incomplete \; gamma \; function$$

Table 2 given below represents the first four moments, Variance(σ^2), the Coefficient of Variation (λ_{cv}), Coefficient of Skewness(λ_{sk}) and Coefficient of Kurtosis (λ_{ku})for arbitrary values of the parameters of LFP distribution taking a fixed value of $\gamma = 3.0$ and $\omega = 5.5$ for Table 2 and for Table 3, we fixed $\alpha = 0.1, \nu = 0.5$.

$$\sigma^2 = \mu'_2 - (\mu'_1)^2$$

$$\lambda_{cv} = \frac{\sqrt{\mu_2}}{\mu'_1} = \frac{\sqrt{\mu'_2 - (\mu'_1)^2}}{\mu'_1}$$

$$\lambda_{sk} = \frac{\mu_3}{(\sigma^2)^{\frac{3}{2}}} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^2}{\sqrt[2]{\mu'_2 - (\mu'_1)^2}^3}$$

$$\lambda_{ku} = \frac{\mu_4}{(\sigma^2)^2} = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'_1 - 3(\mu'_1)^2}{\{\mu'_2 - (\mu'_1)^2\}^2}$$

Table 2 and Table 3 represent the first four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution.

Table 2. First four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution

ν, α	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CV	λ_{sk}	λ_{ku}
2.5,1.5	1.1251	1.2909	1.5117	1.8373	0.0251	0.1408	-78.9055	-78.0228
2.0,1.0	1.2357	1.6062	2.2626	3.7434	0.0793	0.2279	-28.5627	-17.8667
3.0,2.0	1.0705	1.1586	1.2691	1.4087	0.0126	0.1049	-112.9874	-137.3718
5.5,4.0	0.9758	0.9557	0.9396	0.9272	0.0035	0.0606	223.3868	-111.9412
10.5,10.5	0.9076	0.8251	0.7515	0.6855	0.0014	0.0412	2909.2	-10609.4

Table 3. First four moments, σ^2, λ_{cv}, λ_{sk} and λ_{ku} of LFP distribution

γ, ω	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CV	λ_{sk}	λ_{ku}
0.1,4.5	0.5261	0.2822	0.1547	0.0869	0.0054	0.0000	662.4362	-6111.172
0.3,6.5	0.7576	0.5791	0.4470	0.3487	0.0051	0.0943	765.3136	-3668.017
0.8,10.5	0.9236	0.8559	0.7960	0.7429	0.0029	0.0583	835.9249	-1652.537
1.5,15.0	0.9860	0.9738	0.9634	0.9547	0.0016	0.0406	426.4312	-209.6875
5.0,20.5	1.0507	1.1632	1.1632	1.2257	0.0592	0.2316	-20.5070	102.1894

✓ It can be observed from Table 2 and Table 3 that the LFP distribution can be to model data that skewed to the right (positively skewed) or left (negatively skewed) with various degree of kurtosis.

3.1 Moment Generating Function

Moment generating function is a very useful function that can be used to describe certain properties of the distribution. The moment generating function of *LFP* distribution is given in the following theorem.

Theorem 2. Let X follows the *LFP* distribution, the moment generating function, $M_X(t)$ is

$$M_X(t) = \sum_{i=j}^{\infty} \sum_{j,k,r=0}^{\infty} \Gamma_{i,j,k}(\alpha, v) \frac{\gamma\omega}{(k+1)!} \frac{t^r}{r!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right), \quad t \in \mathbb{R}, r < w$$

Proof: The moment generating function of a random variable X is given by

$$M_X(t) = \boldsymbol{E}(e^{Tx}) = \int_{-\infty}^{\infty} e^{tx} f_{LFP}(x) dx, \tag{30}$$

Where $f_{LFP}(x)$ is given in equation (21). Using series expansion in (20), we have

$$M_X(t) = \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{31}$$

Using $E(X^r)$ given in equation (29) in equation (31), we have

$$M_X(t) = \sum_{i=j}^{\infty} \sum_{j,k,r=0}^{\infty} \Gamma_{i,j,k}(\alpha, v) \frac{\gamma\omega}{(k+1)!} \frac{t^r}{r!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left(1 - \frac{r}{\omega}\right), \quad t \in \mathbb{R}, r < w \tag{32}$$

It could be observed from the series expansion of (32) that moments are the coefficient of $\frac{t^r}{r!}$.

3.2 Incomplete Moment

The incomplete moment can be used to estimate the mean deviation, median deviation and the measures of inequalities such as the Bonferroni and Lorenz curves. The incomplete moment of Lehmann type II Frechet Poisson distribution is given in the following theorem.

Theorem 3. Let X follows the *LFP* distribution, the incomplete moment, $\varphi(t)$ is

$$\varphi(t) = \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, v) \frac{\gamma\omega}{(k+1)!} \{\gamma(k+1)\}^{\frac{r}{\omega}} \Gamma\left\{\left(1 - \frac{r}{\omega}\right); \gamma(k+1)t^{-\omega}\right\}, \quad r < \omega$$

Proof: The incomplete moment of Lehmann type II Frechet Poisson distribution is given by

$$\varphi(t) = \int_0^t x^r f_{LFP}(x) dx$$

$$= \sum_{i=j}^{\infty} \sum_{j,k=0}^{\infty} \Gamma_{i,j,k}(\alpha, v) \frac{\gamma\omega}{(k+1)!} \int_0^t x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx$$

By letting

$$C(x) = \int_0^t x^r (k+1) x^{-\omega-1} e^{-\gamma(k+1)x^{-\omega}} dx \tag{33}$$

Taking, $m = \gamma(k+1)x^{-\omega}$, $dx = -\frac{1}{\omega}\{\gamma(k+1)\}^{\frac{1}{\omega}} m^{-\frac{1}{\omega}-1} dm$ and putting it in equation (33), we have

$$C(x) = \{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right); \gamma(k+1)x^{-\omega}\right\} \tag{34}$$

Then we have,

$$\varphi(t) = \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\left\{\left(1-\frac{r}{\omega}\right); \gamma(k+1)t^{-\omega}\right\} \qquad r < \omega \tag{35}$$

$$where \; \Gamma(w,q) = \int_{q}^{\infty}p^{w-1}e^{-w}dw \; is \; the \; complementary \; incomplete \; gamma \; function$$

3.3 Mean Deviation and Median Deviation, Bonferroni and Lorenz Curves

The amount of spread in a population can be obtained using deviation from the mean and median. The mean deviation about the mean and the median of the LFP distribution are expressed as

$$\varphi_1(x) = 2\mu F_{LFP} - 2\mu + 2J(\mu), \text{ and } \; \varphi_1(x) = -\mu + 2J(M),$$

Where

$$J(\mu) = \int_{\mu}^{\infty}xf_{LFP}(x)dx$$

$$= \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right); \gamma(k+1)\mu^{-\omega}\right\}, \quad r < \omega$$

Bonferroni and Lorenz curves are given as

$$B(p) = \frac{1}{p\mu}\int_{0}^{q}xf_{LFP}(x)dx = \{\mu - J(q)\},$$

And

$$L(p) = \frac{1}{\mu}\int_{0}^{q}xf_{LFP}(x)dx = \{\mu - J(q)\},$$

Where

$$J(q) = \sum_{i=j}^{\infty}\sum_{j,k=0}^{\infty}\Gamma_{i,j,k}(\alpha,\nu)\frac{\gamma\omega}{(k+1)!}\{\gamma(k+1)\}^{\frac{r}{\omega}}\Gamma\left\{\left(1-\frac{r}{\omega}\right); \gamma(k+1)q^{-\omega}\right\}, \quad r < \omega$$

3.4 Renyi Entropy

The Renyi Entropy measures the uncertainty in a distribution as defined by Renyi (1961). The Renyi entropy of LFP distribution is given in the following theorem.

Theorem 4. Let X follows the LFP distribution, the Renyi Entropy, $I_\theta(x)$ is

$$I_\theta(x) = \frac{1}{1-\theta}\left\{log\left(H^l\omega^{\theta-1}\{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}}\Gamma\left\{1+\frac{(\theta-1)(\omega+1)}{\omega}\right\}\right)\right\}, \theta > 0, \theta \neq 1$$

Proof: By definition

$$I_\theta(x) = \frac{1}{1-\theta}\left\{log\left(\int_{-\infty}^{\infty}f^\theta_{LFP}(x)dx\right)\right\}, \quad \theta > 0, \theta \neq 1 \tag{36}$$

From equation (11)

$$f^\theta{}_{LFP}(x) = \left[\frac{\alpha v \gamma \omega x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}} \left\{ 1 - \frac{1-e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}} \right\}^{\alpha-1} \right]^\theta$$

Applying binomial series expansion given in equation (18), we have

$$f^\theta{}_{LFP}(x) = H^l \omega^\theta x^{\theta(-\omega-1)} e^{-\gamma x^{-\omega}(k+\theta)}$$

Where

$$H^l = (v\alpha\gamma)^\theta \sum_{i=j}^{\infty} \sum_{j,k}^{\infty} \binom{\theta(\alpha+1)}{i} \binom{i}{j} (-1)^{i+j+k+1} \left(\frac{1}{1-e^{-v}} \right)^{i+1} \frac{v^{k+\theta}}{k!} (i+\theta)^k$$

Consequently,

$$\int_0^\infty f^\theta{}_{LFP}(x)dx = H^l \omega^\theta \int_{-\infty}^\infty x^{\theta(-\omega-1)} e^{-\gamma x^{-\omega}(k+\theta)} dx \tag{37}$$

By letting $c = \gamma x^{-\omega}(k+\theta)$ and $dx = -\frac{1}{\omega} c^{-\frac{1}{\omega}-1} \{\gamma(k+\theta)\}^{\frac{1}{\omega}} dc$ and putting it in equation (37), we have

$$\int_0^\infty f^\theta{}_{LFP}(x)dx = H^l \omega^{\theta-1} \{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}} \Gamma\left\{ 1 + \frac{(\theta-1)(\omega+1)}{\omega} \right\} \tag{38}$$

Putting equation (38) in (36), we have

$$I_\theta(x) = \frac{1}{1-\theta} \left\{ \log\left(H^l \omega^{\theta-1} \{\gamma(k+\theta)\}^{\frac{1-\theta(\omega+1)}{\omega}} \Gamma\left\{ 1 + \frac{(\theta-1)(\omega+1)}{\omega} \right\} \right) \right\} \tag{39}$$

It should be noted that Renyi entropy tends to Shannon entropy as $\theta \to 1$.

3.5 Stress-strength Parameter

Suppose X_1 and X_2 are two continuous and independent random variables, where $X_1 \sim LFP(\alpha_1, v_1, \gamma, \omega)$ and $X_2 \sim LFP(\alpha_2, v_2, \gamma, \omega)$, then an expression for the stress-strength parameter can be obtained using the relation given by

$$\acute{K} = \int_{-\infty}^\infty f_1(x; \alpha_1, v_1, \gamma, \omega) F_2(x; \alpha_2, v_2, \gamma, \omega)\, dx \tag{40}$$

Using the pdf and the cdf of LFP in the expression above, the strength-stress parameter, $\acute{K}$, can be obtained as

$$\acute{K} = q_1 - q_2$$

Where

$$q_1 = \alpha_1 v_1 \gamma \omega \int_{-\infty}^\infty \frac{\omega x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}} \left\{ 1 - \frac{1-e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}} \right\}^{\alpha_1-1} dx$$

$$= F_1(x; \alpha_1, v_1, \gamma, \omega)$$

and

$$q_2 = \alpha_1 v_1 \gamma \omega \int_{-\infty}^\infty \frac{\omega x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}} \left\{ 1 - \frac{1-e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}} \right\}^{\alpha_1-1} \left\{ 1 - \frac{1-e^{-v_2 e^{-\gamma x^{-\omega}}}}{1-e^{-v_2}} \right\}^{\alpha_2} \tag{41}$$

Using equations (18) and (20) in (41)

$$q_2 = \alpha_1 v_1 \gamma \omega \sum_{i=j}^{\infty} \sum_{k=l}^{\infty} \sum_{j,l,m=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{i}{j}\binom{\alpha_2}{k}\binom{k}{l}(-1)^{i+j+k+l+m} \frac{(v_1 + v_{1j} + v_{2l})^m}{m!(1 - e^{-v})^{i+k+1}}$$

Finally, we have

$$\acute{K} = F_1(x) - \alpha_1 v_1 \gamma \omega \sum_{i=j}^{\infty} \sum_{k=l}^{\infty} \sum_{j,l,m=0}^{\infty} \binom{\alpha_1 - 1}{i}\binom{i}{j}\binom{\alpha_2}{k}\binom{k}{l}(-1)^{i+j+k+l+m} \frac{(v_1 + v_{1j} + v_{2l})^m}{m!(1 - e^{-v})^{i+k+1}}$$

3.6 Order Statistics

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $x_{1:n} < x_{2:n} < \ldots x_{n:n}$ represent the corresponding order statistics. If $X_1, X_2, \ldots, X_n$ is a random sample from LFP distribution, it then follows from equations (11) and (12) that the *pdf* of the mth order statistic, say $Z_m = X_{m:n}$ is given by

$$f_m(z_m) = \frac{n! f_{LFP}(x)}{(m-1)!(n-m)!} \sum_{p=0}^{n-m} \binom{n-m}{p}(-1)^p [F_{LFP}(x)]^{p+m-1}$$

$$= \frac{n!\,\alpha}{(m-1)!(n-m)!} \sum_{p=0}^{n-m} \sum_{q,s,u=0}^{\infty} \sum_{r=s}^{\infty} \binom{n-m}{p}\binom{p+m-1}{q}\binom{\alpha(p+1)-1}{r}\binom{r}{s}$$

$$\times (-1)^{p+q+r+s+u}\left(\frac{1}{1-e^{-v}}\right)^{r+1} v^{u+1}(s+1)^u(u+1)\gamma\omega x^{-\omega-1}\frac{e^{-\gamma x^{-\omega}(u+1)}}{(u+1)!}$$

$$= \sum_{p,q,r,s,u=0}^{\infty} \psi_{p,q,r,s,u}(\alpha, v)\, g(x; \omega, \gamma(u+1))$$

Where

$$\psi_{p,q,r,s,u}(\alpha, v) = \sum_{p,q,s,u}^{\infty} \sum_{r=s}^{\infty} \binom{n-m}{p}\binom{p+m-1}{q}\binom{\alpha(p+1)-1}{r}\binom{r}{s}\frac{1}{(u+1)!}$$

$$\times (-1)^{p+q+r+s+u}\left(\frac{1}{1-e^{-v}}\right)^{r+1} v^{u+1}(s+1)^u(u+1)$$

And $g(x; \omega, \gamma(u+1))$ is the Frechet *pdf* with parameters $\omega > 0$ and $\gamma(u+1) > 0$. Thus, we can define the distribution of the mth order statistics as a linear combination of the Frechet distribution.

3.7 Stochastic Ordering

In this section, we examine the stochastic and reliability properties of LFP distribution. Stochastic ordering has applications in many field of study such as survival analysis, insurance, actuarial and management sciences, finance, reliability and survival analysis Shaked and Shanthikumar (2007).

Suppose X and Z are two random variables with *cdf*'s G and F respectively. Survival functions $\bar{G} = 1 - G$ and $\bar{F} = 1 - F$ and their corresponding densities g and f. Then X is said to be smaller than Z in stochastic order $(X \leq_{st} Z)$ if $\bar{G}(x) \leq \bar{F}(x)$ for all $x \geq 0$; in likelihood ratio order $(X \leq_{lr} Z)$ if $g(x)/f(x)$ is decreasing in all $x \geq 0$; hazard rate order $(X \leq_{hr} Z)$ if $\bar{G}(x)/\bar{F}(x)$ is decreasing in all $x \geq 0$; reversed hazard rate order $(X \leq_{rhr} Z)$ if $G(x)/F(x)$ is decreasing in all $x \geq 0$. These four stochastic orders are related to each other as

$$X \leq_{rhr} Z \Longleftarrow X \leq_{lr} Z \Longrightarrow X \leq_{hr} Z \Longrightarrow X \leq_{st} Z$$

Theorem: let $X \sim LFP(\alpha_1, v_1, \gamma, \omega)$ and $Z \sim LFP(\alpha_2, v_2, \gamma, \omega)$. If $\alpha_1 < \alpha_2$ and $v_1 < v_2$, then

$$X \leq_{rhr} Z \Longleftarrow X \leq_{lr} Z \Longrightarrow X \leq_{hr} Z \Longrightarrow X \leq_{st} Z$$

$$\text{Proof:} \frac{g(x)}{f(x)} = \frac{\alpha_1 v_1 e^{-v_1 e^{-\gamma t^{-\omega}}}}{1-e^{-v_1}} \left\{ 1 - \frac{1-e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}} \right\}^{\alpha_1-1} \div \frac{\alpha_2 v_2 e^{-v_2 e^{-\gamma x^{-\omega}}}}{1-e^{-v_2}} \left\{ 1 - \frac{1-e^{-v_2 e^{-\gamma x^{-\omega}}}}{1-e^{-v_2}} \right\}^{\alpha_2-1}$$

$$\boldsymbol{log}\left[\frac{g(x)}{f(x)}\right] = log\left(\frac{\alpha_1}{\alpha_2}\right) + log\left(\frac{v_1}{v_2}\right) + log\left(\frac{1-e^{-v_2}}{1-e^{-v_1}}\right) + (v_2 - v_1)e^{-\gamma x^{-\omega}} +$$

$$(\alpha_1 - 1)log\left(\left\{1 - \frac{1-e^{-v_1 e^{-\gamma x^{-\omega}}}}{1-e^{-v_1}}\right\}\right) - (\alpha_2 - 1)log\left(\left\{1 - \frac{1-e^{-v_2 e^{-\gamma x^{-\omega}}}}{1-e^{-v_2}}\right\}\right)$$

$$\frac{d}{dx}\left(log\left[\frac{g(x)}{f(x)}\right]\right) = x^{-\omega-1}e^{-\gamma x^{-\omega}}\left\{(v_2-v_1)\gamma\omega - \frac{v_1 e^{-v_1 e^{-\gamma x^{-\omega}}}}{\left(e^{-v_1 e^{-\gamma x^{-\omega}}}-e^{-v_1}\right)} + \frac{v_2 e^{-v_2 e^{-\gamma x^{-\omega}}}}{\left(e^{-v_2 e^{-\gamma x^{-\omega}}}-e^{-v_2}\right)}\right\}$$

Hence, for

$$\alpha_1 < \alpha_2 \text{ and } v_1 < v_2$$

It follows that

$$X \leq_{rhr} Z \Longleftarrow X \leq_{lr} Z \Longrightarrow X \leq_{hr} Z \Longrightarrow X \leq_{st} Z$$

4. Maximum Likelihood Estimation

The log-likelihood function $l(\underline{x}/\omega) = log\left(L(\underline{x}/\omega)\right)$ of the *LPF* distribution is given by

$$l(\underline{x}/\omega) = nlog(\alpha v \gamma \omega) - (\omega + 1)\sum_{i=1}^{n} log(x_i) - \gamma \sum_{i}^{n} x_i^{-\omega} - v \sum_{i=1}^{n} e^{-\gamma x_i^{-\omega}}$$

$$(\alpha - 1)\sum_{i=1}^{n}\left[1 - \frac{1-e^{-ve^{-\gamma x_i^{-\omega}}}}{1-e^{-v}}\right] \tag{43}$$

The partial derivatives of the log-likelihood function with respect to the model parameters $(\alpha, \gamma, v, \omega)$ yield the score vector and are obtained as

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\left[1 - \frac{1-e^{-ve^{-\gamma x_i^{-\omega}}}}{1-e^{-v}}\right] \tag{44}$$

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} x_i^{-\omega} + v\gamma \sum_{i=1}^{n} x_i^{-\omega} e^{-\gamma x_i^{-\omega}} + (\alpha - 1)v \sum_{i=1}^{n} \frac{\left(x_i^{-\omega}e^{-\gamma x_i^{-\omega}}e^{-ve^{-\gamma x_i^{-\omega}}}\right)}{\left(e^{-ve^{-\gamma x_i^{-\omega}}}-e^{-v}\right)} \tag{45}$$

$$\frac{\partial l}{\partial v} = \frac{n}{v} - \sum_{i=1}^{n} e^{-\gamma x_i^{-\omega}} + (\alpha - 1)\sum_{i=1}^{n}\frac{e^{-v}\left(1-e^{-ve^{-\gamma x_i^{-\omega}}}\right)+ve^{-\gamma x_i^{-\omega}}e^{-ve^{-\gamma x_i^{-\omega}}}(1-e^{-v})}{\left(e^{-ve^{-\gamma x_i^{-\omega}}}-e^{-v}\right)(1-e^{-v})} \tag{46}$$

And

$$\frac{\partial l}{\partial \omega} = \frac{n}{\omega} - \sum_{i=1}^{n} log(x_i) + \gamma \sum_{i}^{n} x_i^{-\omega} log(x_i) + (\alpha - 1)v\gamma \sum_{i=1}^{n} \frac{x_i^{\omega}log(x_i)e^{-\gamma x_i^{-\omega}}e^{-ve^{-\gamma x_i^{-\omega}}}}{\left(e^{-ve^{-\gamma x_i^{-\omega}}}-e^{-v}\right)} \tag{47}$$

The equations (44), (45), (46) and (47) are non-normal equations which cannot be solved by setting the above partial derivatives to zero, therefore the parameters $\alpha, \gamma, v, \omega$ must be found using the iterative methods. The maximum likelihood estimate of the parameters, denoted by $\widehat{\omega}$ is obtained by solving the nonlinear equation $\left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \gamma}, \frac{\partial l}{\partial v}, \frac{\partial l}{\partial \omega}\right)^{T} = 0$, using a numerical method such as Newton-Raphson procedure, Trapezoidal techniques etc. The Fisher information is given by $I(\omega) = \left[I_{\theta_i, \theta_j}\right]_{4 \times 4} = E\left(-\frac{\partial^2 l}{\partial \theta_i \partial \theta_j}\right), i, j = 1,2,3,4$ can be numerically obtained by using R or MATLLAB software. For the purpose of this study we make use of Adequacy model in R, the Fisher information matrix $nI(\omega)$ can be approximated by

$$\Omega(\widehat{\omega}) \approx \left[-\frac{\partial^2 l}{\partial \theta_i \partial \theta_j}\bigg|_{\omega = \widehat{\omega}}\right]_{4 \times 4}, \quad i, j = 1,2,3,4 \qquad (48)$$

For a given set of observations, the matrix given in equation (48) is obtained after convergence of the Newton-Raphson procedure in R or MATLAB.

The multivariate Normal distribution $N_4\left(\underline{0}, \Omega((\widehat{\omega})^{-1})\right)$, with mean vector $\underline{0} = (0,0,0,0)^T$, can be used to construct the confidence interval and the confidence regions for the model parameters and for the hazard and the survival functions. The approximate $100(1 - \Delta)\%$ two-sided confidence intervals for α, γ, v and ω are given by:

$$\hat{\alpha} \pm Z_{\frac{\Delta}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\widehat{\omega})}, \qquad \hat{v} \pm Z_{\frac{\Delta}{2}}\sqrt{I_{vv}^{-1}(\widehat{\omega})}, \quad \hat{\gamma} \pm Z_{\frac{\Delta}{2}}\sqrt{I_{\gamma\gamma}^{-1}(\widehat{\omega})}, \quad and \quad \hat{\omega} \pm Z_{\frac{\Delta}{2}}\sqrt{I_{\omega\omega}^{-1}(\widehat{\omega})},$$

Respectively, where $I_{\alpha\alpha}^{-1}(\widehat{\omega}), I_{vv}^{-1}(\widehat{\omega}), I_{\gamma\gamma}^{-1}(\widehat{\omega})$ and $I_{\omega\omega}^{-1}(\widehat{\omega})$ are diagonal elements of $I_n^{-1}(\widehat{\omega})$, and $Z_{\frac{\Delta}{2}}$ is the upper $\frac{\Delta}{2}^{th}$ percentile of the distribution of the standard normal.

5. Application

In this section, we demonstrate the applicability and flexibility of the LFP distribution in modeling real life data using three life data sets. The method of maximum likelihood is used to estimate the model parameters; also, we carried out a Monte Carlo simulation for different parameter values coupled with different sample sizes.

5.1 Monte Carlo Simulation

A simulation study is carried out in order to test the performance of the MLEs for estimating *LFP* model parameters. We consider two different sets of parameters $\alpha = 0.4, v = 0.6, \gamma = 0.3, \omega = 0.5$ and $\alpha = 0.5, v = 1.6, \gamma = 0.5, \omega = 0.5$. For each parameter combination, we simulate data from *LFP* model with different sample sizes $n = 50, n = 100, n = 150$ and $n = 200$, taking from a population size $N = 1000$. Table 4 list the Absolute bias (AB), standard error (SE) and the mean square error (MSE). According to the simulation result the mean square error decay to zero as the sample size increases as expected.

Table 4. Monte Carlo Simulation Results for LFP distribution

Par	n	AB	SE	MSE	Par	AB	SE	MSE
$\alpha = 0.4$	50	1.2931	3.0523	10.9886	$\alpha = 0.5$	1.2338	3.0033	10.5421
	100	1.0742	2.0149	5.2137		1.1873	2.0772	5.7244
	150	1.3460	1.6685	4.5956		1.4545	1.7065	5.0277
	200	1.1821	1.3739	3.2850		1.2836	1.3856	3.5675
$v = 0.6$	50	0.0049	0.2922	0.0854	$v = 1.6$	1.0050	0.2834	1.0903
	100	0.3409	0.1450	0.1372		1.3486	0.1453	1.8345
	150	0.2981	0.1257	0.1047		1.3053	0.1255	1.7196
	200	0.2705	0.1079	0.0848		1.231	0.1445	1.5362
$\gamma = 0.3$	50	0.4086	0.9149	1.0040	$\gamma = 1.5$	0.9433	1.3902	2.7948
	100	0.2486	0.1197	0.0761		0.7897	0.8137	1.2857
	150	0.2454	0.0926	0.0688		0.8244	0.5973	1.0364
	200	0.1846	0.1523	0.0573		0.5952	0.6236	0.7431
$\omega = 0.5$	50	0.9691	0.2707	1.0125	$\omega = 0.5$	0.7709	0.7458	1.1505
	100	0.5074	0.4870	0.4946		2.2749	1.3907	7.1092
	150	0.4625	0.3345	0.3257		2.1381	0.94`12	4.5715
	200	0.2879	0.2531	0.1469		1.6586	0.69871	3.2370

5.2 Application to Real Life Data

In this section, the LFP distribution is applied to three real life data sets in order to demonstrate the usefulness, tractability and applicability of the model. We fit the density of the LFP distribution and compare it performance with its sub-models which includes: Lehmann type II Inverse Exponential Poisson ($LIEP$) distribution, Frechet Poisson (FP) distribution, Inverted Weibull Poisson (IWP) distribution, Inverse Exponential Poisson (IEP) distribution, Frechet (F) distribution. The density of the competing models is given by:

$$f_{LIEP}(x) = \frac{\alpha v \gamma x^{-2} e^{-\gamma x^{-1}} e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}} \left\{ 1 - \frac{1-e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}} \right\}^{\alpha-1} , x > 0; \alpha, v, \gamma > 0 \qquad (49)$$

$$f_{FP}(x; \gamma, \omega, v) = \frac{\gamma \omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-\omega}}}}{1-e^{-v}}, \qquad x > 0; \gamma, \omega, v > 0 \qquad (50)$$

$$f_{IWP}(x; \omega, v) = \frac{\omega v x^{-\omega-1} e^{-\gamma x^{-\omega}} e^{-ve^{-x^{-\omega}}}}{1-e^{-v}}, \qquad x > 0; \omega, v > 0 \qquad (51)$$

$$f_{IEP}(x; \gamma, v) = \frac{\gamma v x^{-2} e^{-\gamma x^{-\omega}} e^{-ve^{-\gamma x^{-1}}}}{1-e^{-v}}, \qquad x > 0; \gamma, v > 0 \qquad (52)$$

We consider various measures of the goodness-of-fit including the Akaike Information Criterion ($AIC = 2k - 2l$), Consistent Akaike Information Criterion ($CAIC = AIC + \frac{2k(k+1)}{n-k-1}$), Bayesian Information Criterion ($BIC = klog(n) - 2l$), Hannan-Quinn information criterion ($HQIC = -2l + 2klog\{log(n)\}$), Anderson Darling Statistic ($A^* = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum_{j=1}^{n}(2j-1)log[z_i(1 - z_n - j + 1)]\right\}$), Kolmogorov Smirnoff ($KS$),Crammer Von-Misses ($W^* = \left(\frac{1}{2n} + 1\right)\left\{\sum_{j}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\}$) statistic and the Probability value (PV), where n is the number of observations, $z_i = F(y_i)$, k is the is the number of estimated parameters and $y_i's$ are the ordered observations. The smaller these statistics are, the better the fit of the model to the data except for PV which must be the largest among the competing models. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics were tabulated in Nichols and Padgett (2006). We provide the estimates of the parameters of the distributions, standard errors (in parenthesis), confidence interval (in curly bracket).

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. For previous study see Lee and Wang (2003). The second data consists of 101 observations obtained from a failure time in

hours of Kevlar 49/epoxy strands with pressure at 90% and had been studied by Andrews and Herzberg (2012). For the third data set, we consider the number of failures for the air conditioning system of jet airplanes which were reported by Cordeiro and Lemonte (2011) and Huang and Oluyede (2014). The three data sets are given in Table 5 given below.

Table 5. Cancer data, Kevlar 49/Epoxy data, and Air condition failure data

Data 1	Data 2	Data 3
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70,5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41,7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66,11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69	0.01, 0.01, 0.02, 0.02,0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09,0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20,0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42,0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68,0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85,0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10,1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40,1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58,1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17,2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89	194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18,209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220,120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182,71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46,230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71

The Exploratory data analysis of the three sets of data is given in Table 6. From this table, we can conclude that the three sets of data are over-dispersed, positively skewed and kleptokurtic. From the Total Time on Test (TTT) plot, figure 4 (Diagram 1) indicates that the first data set exhibits unimodal failure rate, figure 5 (Diagram 1) indicates that the second data set exhibits a bathtub failure rate and figure 6 (Diagram 1) indicates that the third data set exhibits decreasing failure rate.

Table 6. Exploratory data Analysis of the tree failure data

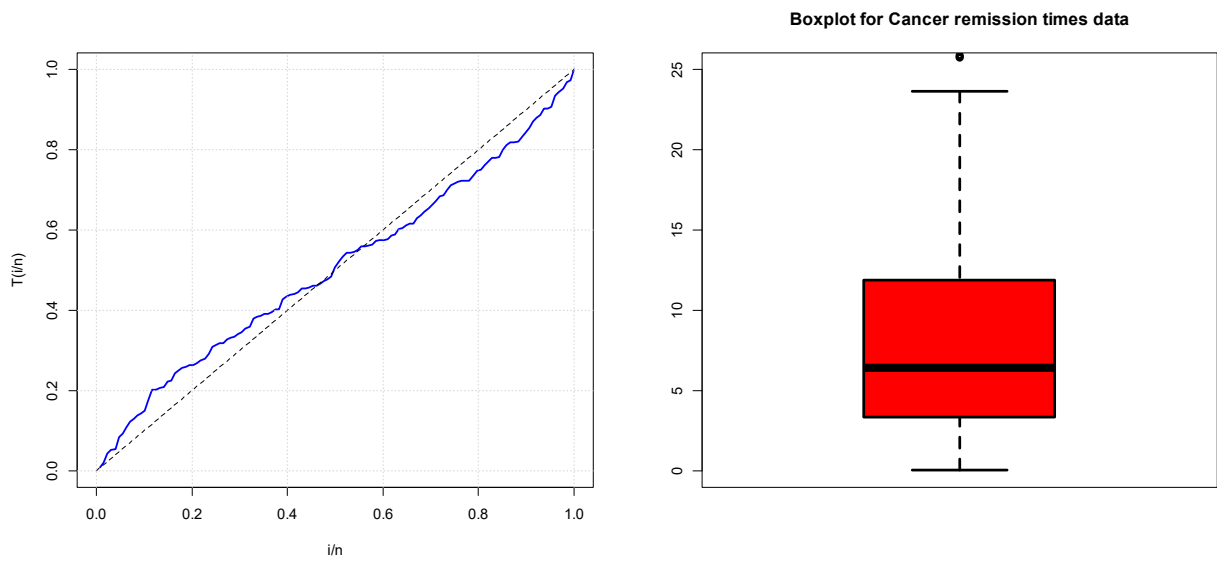|	Data 1	Data 2	Data 3
Minimum	0.08	0.010	1.00
First quartile	3.348	0.240	20.750
meadian	6.395	0.800	54.00
Mean	9.366	1.025	92.07
Third quartile	11.840	1.450	118.00
Maximum	79.050	7.890	603.0
Variance	110.425	1.253	11645.933
Skewness	3.326	3.047	2.157
Kurtosis	16.154	14.475	5.192

Diagram I Diagram II

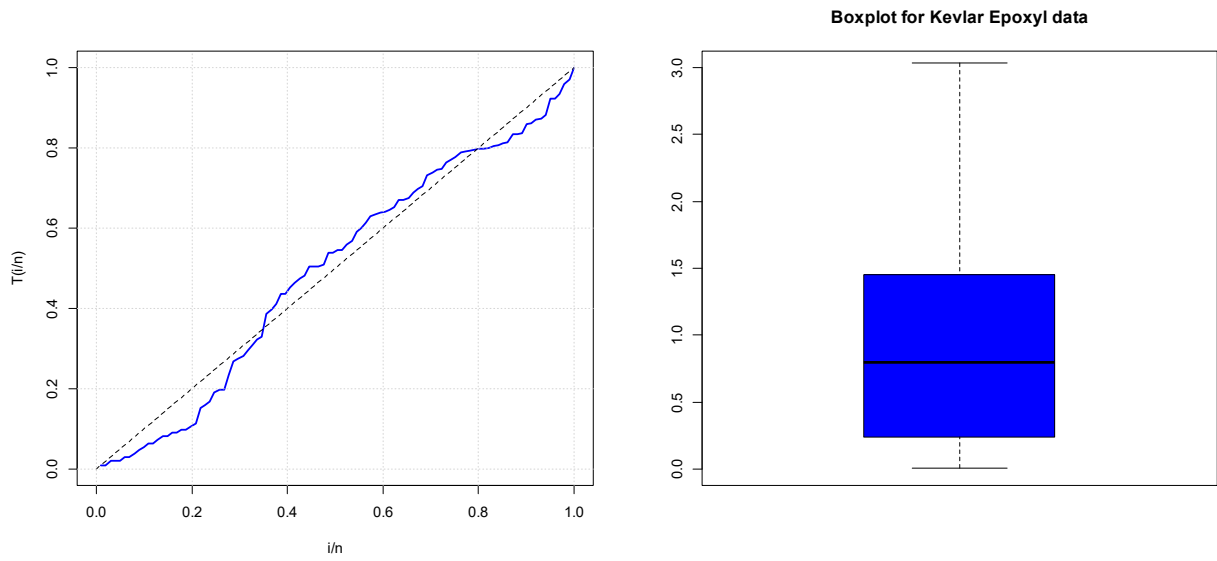Figure 4. TTT plot and the Boxplot for Cancer data

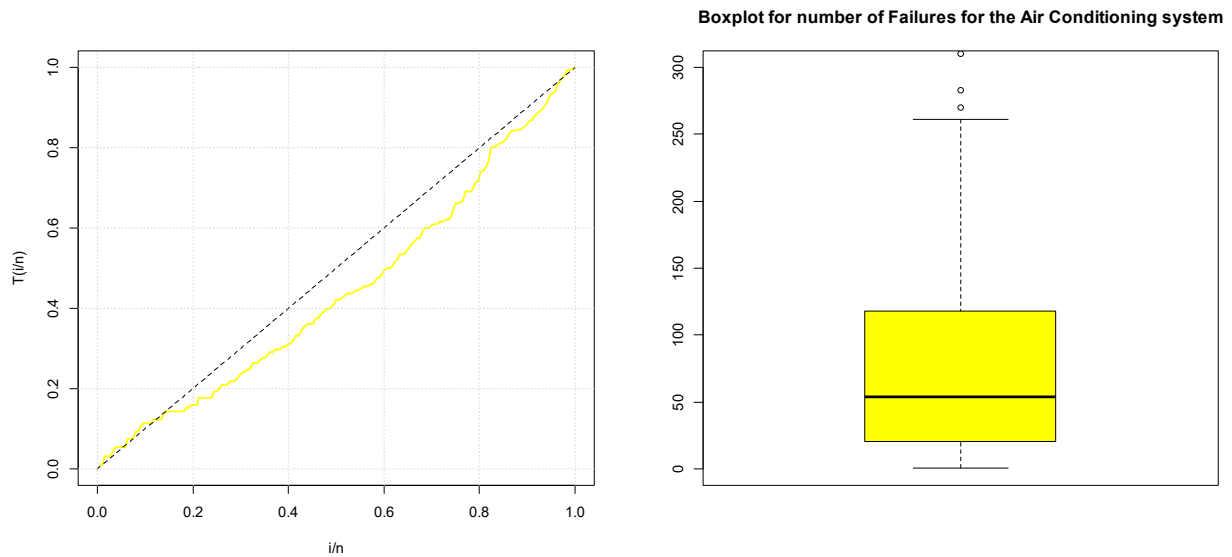Figure 5. TTT plot and the Boxplot for Kevlar Epoxy data

Figure 6. TTT plot and the Boxplot for Air condition data

Table 7. MLE's, Standard error (in parenthesis) and 95% Confidence interval (in curly bracket) for cancer remission data

Model	α	ν	γ	ω
LFP	7.0604(0.5734)	0.2434(0.0310)	7.0686(3.2671)	9.1513(4.1383)
	{5.9365,8.1843}	{0.1826,0.3042}	{0.6651,13.4721}	{1.0402,17.2624}
LIEP	0.8570(0.1865)	−6.7737(1.1381)	1.5282(0.230)	−
	{0.4915,1.2225}	{4.543,9.0044}	{1.0774,1.979}	−
FP	−	0.6419(0.1435)	1.0052(0.0580)	−6.4163(1.2857)
	−	{0.3606,0.9232}	{0.8915,1.1189}	{−8.9363, −3.8963}
IWP	−	0.9992(0.0556)	−	−4.4354(0.5281)
	−	{0.890,1.108}	−	{4.4354,0.5281}
IEP	−	0.6363(0.1451)	-6.4676(1.3156)	−
	−	{0.3519,0.9207}	{−9.0462, −3.889}	−
F	−	−	2.4304(0.2193)	0.7523(0.0424)
	−	−	{2.0005,2.8602}	{0.6692,0.8354}

Table 8. Measures of goodness-of-fit of Cancer remission data

MLE	$-ll$	AIC	CAIC	BIC	HQIC	K	A^*	W^*	PV
LFP	412.127	832.253	832.578	843.662	836.889	0.0363	0.0459	0.4314	0.9505
LIEP	423.207	852.413	852.607	860.970	855.889	0.1623	1.3297	0.2044	0.0023
FP	427.135	860.270	860.463	868.826	863.746	0.0944	2.5375	0.4031	0.2038
IWP	429.306	862.612	862.708	868.316	864.929	0.1077	2.5378	0.4028	0.1022
IEP	427.132	858.265	858.360	863.968	860.582	0.0965	2.5244	0.4007	0.1839
F	444.001	892.002	892.098	897.706	894.319	0.410	11.1780	1.9946	2.2e-16

Table 9. MLE's, Standard error (in parenthesis) and 95% Confidence interval (in curly bracket) for49/Epoxy data

$Model$	α	ν	γ	ω
LFP	0.5899(0.1914)	0.4416(0.0406)	−5.0361(1.1548)	9.8611(3.3049)
	{0.2148,0.9650}	{0.362,0.5212}	{−7.2995, −2.7727}	{3.3835,16.3387}
$LIEP$	0.0563(0.0131)	−4.1031(0.8534)	0.7924(0.1229)	−
	{0.0306,0.0820}	{−5.7758, −2.4304}	{0.5515,1.0333}	−
FP	−	2.6433(0.3657)	0.2594(0.0296)	15.9277(5.6611)
	−	{1.9265,3.3601}	{0.2014,0.3174}	{4.8319,27.0235}
IWP	−	0.4468(0.0245)	−	2.7522(0.4095)
	−	{0.3988,0.4948}	−	{1.9496,3.5548}
IEP	−	0.0670(0.0122)	−4.5136(0.7695)	−
	−	{0.0431,0.0909}	{−6.0218, −3.0054}	−
F	−	−	0.4206(0.0585)	0.6141(0.0424)
	−	−	{0.3059,0.5353}	{0.5310,0.6972}

Table 10. Measures of goodness-of-fit for 49/Epoxy data

MLE	$-ll$	AIC	$CAIC$	BIC	$HQIC$	K	A^*	W^*	PV
LFP	107.494	222.988	223.404	233.448	227.222	0.1133	2.3507	0.4410	0.1495
$LIEP$	130.812	267.623	267.871	275.469	270.799	0.9411	3.4488	0.6431	2.2e-16
FP	114.193	234.386	234.633	242.231	237.562	0.1649	3.5739	0.6649	0.0082
IWP	131.263	266.525	266.648	271.756	268.642	0.2040	6.1356	1.1365	0.0005
IEP	132.207	268.414	268.537	273.645	270.532	0.2399	5.7642	1.0745	1.78e-05
F	132.441	268.882	269.004	274.112	270.999	0.4176	10.5943	1.9962	9.992 e-16

Table 11. MLE's, Standard error (in parenthesis) and 95% Confidence interval (curly bracket) for air condition failure data

$Model$	α	ν	γ	ω
LFP	6.2447(2.9967)	0.4016(0.0459)	−4.7004(1.6543)	26.5895(16.9552)
	{0.3712,12.1182}	{0.3116,0.4916}	{−7.9428, −1.4580}	{−6.6427,59.8217}
$LIEP$	6.6206(1.5056)	−5.1296(0.9439)	1.0924(0.1248)	−
	{3.6696,9.5716}	{−6.9796, −3.2796}	{0.8478,1.3370}	−
FP	−	5.1506(1.4678)	0.9394(0.0488)	−5.0641(1.1124)
	−	{2.2737,8.0275}	{0.8437,1.0351}	{−7.2444, −2.8838}
IWP	−	0.8072(0.0378)	−	−14.6451(1.7261)
	−	{0.7331,0.8813}	−	{−18.0283, −11.2619}
IEP	−	6.1265(1.3240)	−5.1192(0.9989)	−
	−	{3.5315,8.7215}	{−7.077, −3.1614}	−
F	−	−	11.2205(1.3427)	0.7466(0.0374)
	−	−	{8.5888,13.8522}	{0.6733,0.8199}

Table 12.Measures of goodness-of-fit for air condition failure data

MLE	$-ll$	AIC	$CAIC$	BIC	$HQIC$	KS	A^*	W^*	PV
LFP	1033.17	2074.35	2074.57	2087.29	2079.59	0.0397	0.2675	0.0341	0.9288
$LIEP$	1048.30	2102.61	2102.74	2112.32	2106.54	0.9872	3.0343	0.4337	$2.2e-16$
FP	1047.84	2101.68	2101.81	2111.38	2105.61	0.0676	1.9501	0.2955	0.3572
IWP	1055.17	2114.34	2114.40	2120.81	2116.96	0.0866	2.6848	0.4048	0.1189
IEP	1048.601	2101.20	2101.27	2107.67	2103.82	0.0803	2.0998	0.3205	0.1772
F	1061.42	2126.84	2126.90	2133.31	2129.46	0.4025	11.1658	1.9067	$2.2e-16$

Based on the values obtained which were recorded in Tables 8, 10 and 12, it is clear that the Lehman Type II Frechet Poisson distribution provide the best fit for the three real life data considered having possessed the smallest AIC, CAIC,

BIC, HQIC, A^*, W^* and the largest PV among all others competing models. Figures 7, 8 and 9 also illustrate the flexibility of Lehmann Type II Frechet Poisson distribution in modeling real life data.

6. Conclusion

This work examined the flexibility, tractability and applicability of Lehmann Type II Frechet Poisson distribution. Some structural properties of the newly developed distribution are derived and population parameters are obtained using maximum likelihood estimation method. Simulation study and three real life data were used to illustrate the model usefulness in modeling life data. Among other competing models considered the Lehmann Type II Frechet Poisson distribution provides the best fit. We recommend that further studies should be carried by using different estimation methods such as moment method, least square method etc. and compare the performance of the estimation techniques.

Conflicts of interest

The authors want to declare that there is no conflict of interest during and after the preparation of the manuscript.
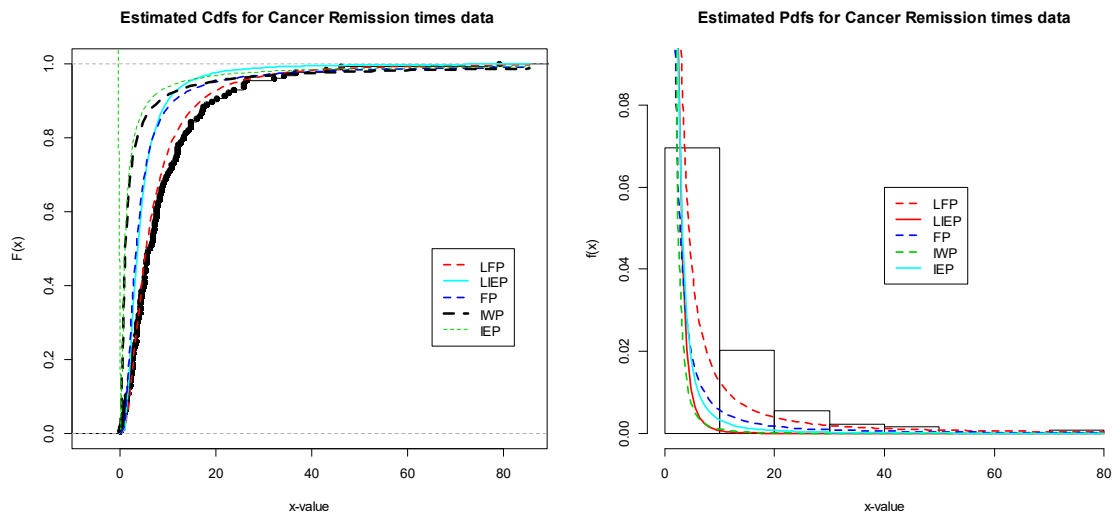
Acknowledgement

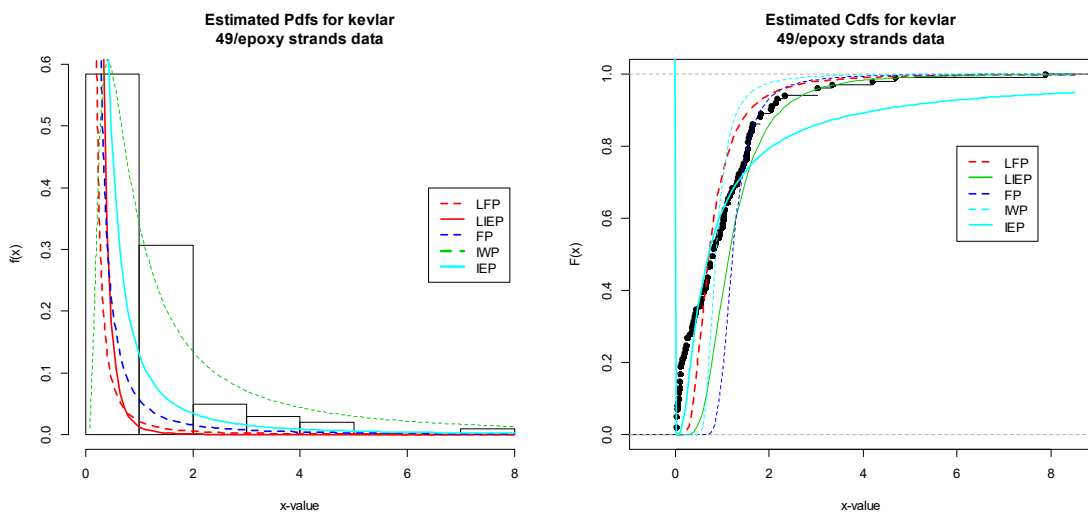Figure 7. Estimated *pdf* and *cdf* function and other competing models for cancer remission data

Figure 8. Estimated *pdf* and *cdf* function and other competing models for Kevlar 49/epoxy data
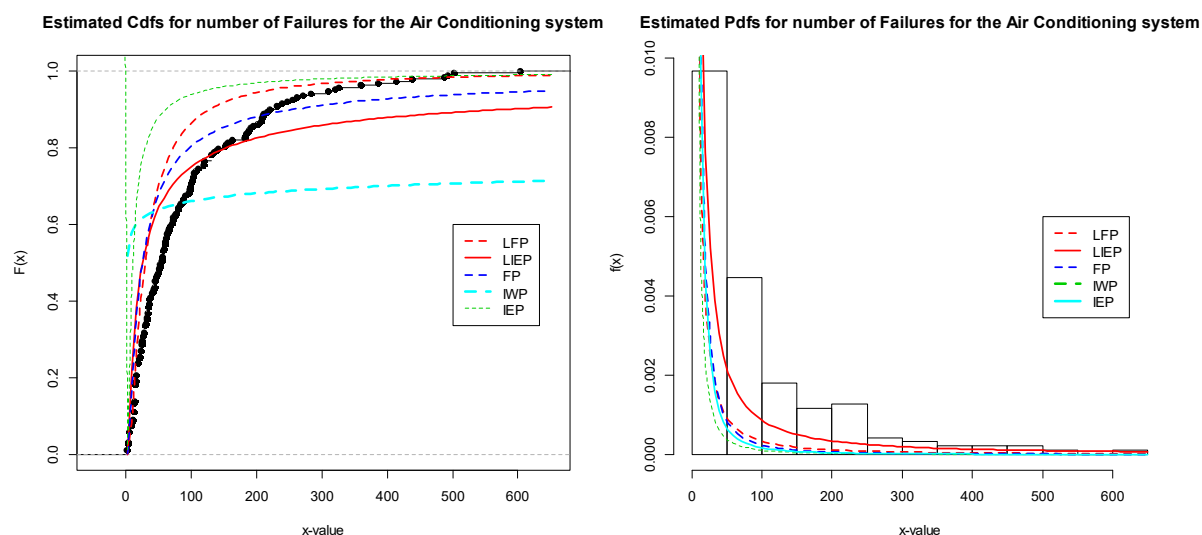
Figure 9. Estimated *cdf* and *pdf* function and other competing models for air condition failure data

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