Mechanical Engineering Research; Vol. 5, No. 2; 2015
ISSN 1927-0607 E-ISSN 1927-0615
Published by Canadian Center of Science and Education

Solutions of the Problems of a Viscoelastic Dynamic Contact between Smooth Curvilinear Surfaces of two Solid Bodies by the Application of the "Method of the Specific Forces"

Nikolay Goloshchapov[1]

[1] 71 Grosvenor Road, London, N3 1EY, UK

Correspondence: Nikolay Goloshchapov, 71 Grosvenor Road, London, N3 1EY, UK. E-mail: glshchpv@yahoo.co.uk

Received: October 20, 2015 Accepted: October 31, 2015 Online Published: November 4, 2015
doi:10.5539/mer.v5n2p59 URL: http://dx.doi.org/10.5539/mer.v5n2p59

Abstract

Solutions of the problems of a viscoelastic dynamic contact between smooth curvilinear surfaces of two solid bodies by the application of the "Method of the specific forces" have been given in the article, and the new conception for the definition of the elastic and the viscous forces in the common case of dynamics of a viscoelastic contact is proposed here by the further development of this method. Essence of this method is that, the forces of viscosity and the forces of elasticity can be found by integration of the specific forces acting inside an elementary volume of the contact zone. It is shown here, that this method allows finding the viscoelastic forces for any theoretical or experimental dependencies between the distance of mutual approach of two solid bodies and the diameter of the contact area. Also, the derivation of the integral equations of the viscoelastic forces, the equations for pressure in the contact is presented. Viscoelastic dynamic contacts between two spherical bodies, and between a spherical solid body and a semi-space at impact have been examined. Work and Energy in the phases of compression and restitution, and at the rolling shear have been derived. Approximate solutions for the differential equations of movement (displacement) by using the method of equivalent work have been derived. Equations for the normal contact stresses have been obtained. Also, equations for kinematic and dynamic parameters of the viscoelastic collision have been derived in this article. Examples of the comparison of theoretical results and conclusions have been given in the paper.

Keywords: viscoelastic forces, method specific forces, elementary distributed axial loads, geometry contact area, dynamic modules, dissipative energy, viscoelastic parameters, method equivalent work

1. Introduction

The objective of this paper is the further development of the application of the "Method of the specific forces (MSF)" for solutions to the problems of a viscoelastic dynamic contact between smooth curvilinear surfaces of two solid bodies. The new conception is proposed here, how to find the elastic and the viscous forces by an integration of the specific forces in the boundaries of the contact area, which can be found by considering of the geometry of the contact. This method has been already used by author for case of the collision between a spherical solid body and a semi-space (Goloshchapov, 2003, 2015).

It is assumed here that the surfaces of contact are smooth and in this case we are not considering the influence of roughness on the contact forces, and the initial velocities of contact V_x and $V\tau$ is less than the effective sound speed in the volume of deformation (Figure 1). Also the effect of the adhesive forces and a plastic deformations have not been considered in this paper.

As we know, the mechanics of an elastic contact problem between two smooth surfaces have been studied yet in the 19-th century by Herts (1882, 1896) and Boussinesq (1885), and then later, for example, it was examined by many others researchers, such as: Bowden and Tabor (1939); Landau and Lifshits (1944); Timoshenko and Goodier (1951); Archard (1957); Galin (1961); Sneddon(1965); Greenwood and Williamson (1966); Johnson, Kendall and Roberts (1971); Derjaguin, Muller and Toporov (1975); Bush, Gibson and Thomas (1975); Tabor (1977); Johnson (1985); Webster and Sayles, 1986; Stronge (2000); Persson, Bucher and Chiaia (2002); Wriggers (2006); Hyun and Robbins (2007). Also a viscoelastic contact between smooth and rough curvilinear surfaces of two solids already have been researched very widely and their results was published in many different manuscripts (Mindlin,1949; Radok. 1957; Hunter,1960; Goldsmith, 1960; Galin, 1961; Lee, 1962; Graham, 1965; Ting, 1966; Greenwood & Williamson, 1966; Simon, 1967; Jonas, 1982; Padovan, & Paramadilok, 1984; Johnson, 1985; Brilliantov, 1996; Brilliantov, Spahn, Hertzsch, & Poeschel, 1996; Ramírez, Poeschel, Brilliantov, & Schwager, 1999; Stronge, 2000; Barber & Ciavarella, 2000; Goloshchapov, 2001, 2003; Laursen, 2002; Dintwa, 2006; Carbone, Lorenz, Persson, & Wohlers, 2009; Harrass, Friedrich, & Almajid, 2010; Persson, 2010; Cummins, Thornton, & Cleary, 2012; Carbone & Putignano, 2013; Popov, 2015). In all these researches for a finding of the viscoelastic forces and stresses, the traditional theories and methods usually have been applied. But, in this paper, the novel theoretical and practical principals have been used for finding these forces and stresses.

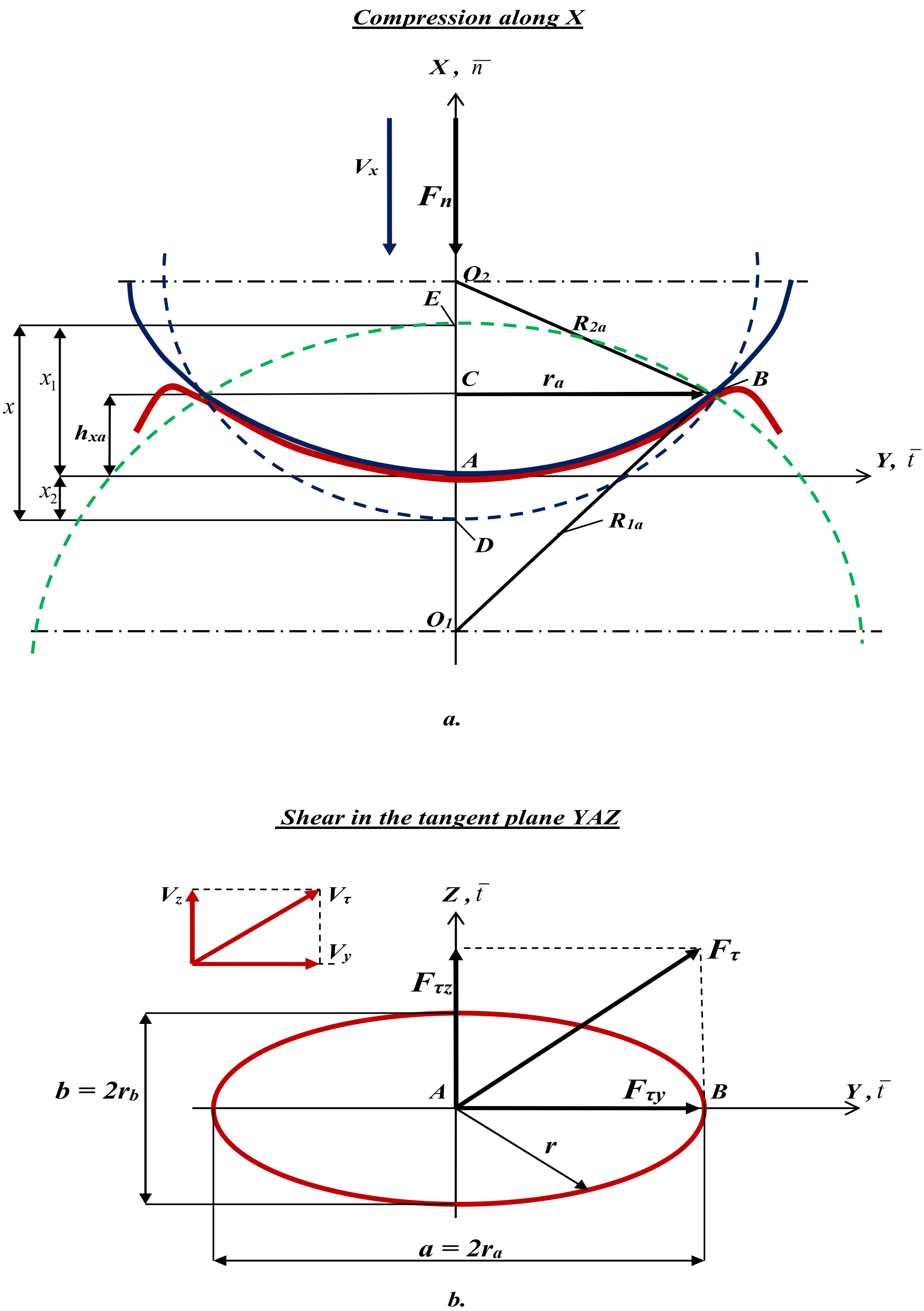


Figure 1. Illustration of the mutual approach between two curvilinear surfaces of two solid bodies along the axis ***X*** relative to the initial point of contact ***A***

The illustration of the mutual approach between two curvilinear surfaces of two solid bodies along the axis *X* relative to the initial point of contact *A* is depicted in the Figure 1. Here is not shown the displacement of the bodies along the axes *Z* and *Y*.

It is obviously that during the time of a dynamic contact, in the initial point *A*, the curvilinear surfaces are moving relative to each other with the normal relative velocity V_x and the tangential relative velocity V_τ between them under action of the normal force F_n and the tangential shear force F_τ. Also let: $F_{\tau y}$ and $F_{\tau z}$ are the tangential forces along axes *Y* and *Z*; *x* is the size of the mutual approach of the bodies along the axis *X*; *r* is a current radius of the contact area; h_{xa} is the depth of the contact surface in the plane *XAY* , or other words it is the depth of indentation of the more hard body into the surface of more soft body; h_{xb} is the depth of the contact surface in the plane *XAZ (This size is not shown in the Figure 1)*; x_1 and x_2 are normal compression deformations of the contacting surfaces relative to the initial point of the contact *A*; R_{1a} and R_{2a} are the radiuses of curvature of the contacting surfaces on the border of contact area in the plane *XAY*; $V_{\tau y}$ and $V_{\tau z}$ are the projections of the tangential relative velocity V_τ by axes *Y* and *Z*, or they are the tangential relative velocities of the displacement of surfaces of bodies along axes *Y* and *Z*; *a* is the big axis of an elliptic (oval) contact area; *b* is the small axis of an elliptic (oval) contact area; O_1 and O_2 are the centres of curvature of the contacting surfaces in the initial point of contact *A*; R_{1b} and R_{2b} are the radiuses of curvature of the contacting surfaces on the border of contact area in the plane *XAZ*.

It is obvious, that in a general case the area of contact is taking the shape of an oval or an ellipse, and in the same time, the contact surface takes the identical shape with the harder surface, and the surface, which has less hardness slips by a surface of the harder body. It is a micro-slip and we do not take in account the energy of this dissipative process in this paper. You can say that in process of sliding deformation under action of the tangential force, the area of contact does not take the elliptical shape, but it is the oval or other asymmetric figure.

But indeed, if the tangential force equal zero the contact area takes the right elliptical shape, but in the case of sliding deformation under the action of the tangential force, the elliptic shape of the contact area is transformed in the oval. Therefore, we can suppose, since the normal force stays same independently of the action of the tangential force, the normal contact pressure (or the normal contact stress) is not changed and stays same, and therefore the size of square of the contact is not changed too. According to this statement, we approximately can consider the area of contact like having the elliptical shape during all the contact time. Also in the case of an impact between the contact surfaces, when the rolling shear between them has place, we can state that the area of contact approximately keeps the elliptical shape too. Moreover, as we will see further, it does not matter, the area of contact is an ellipse or it is an oval, in the both cases, the derivation of the sizes of the contact area can be done in the same manner.

The viscoelastic forces can be found as the sums of the elastic forces and the viscous forces:

$$\begin{cases} F_n = F_{bn} + F_{cn} \\ F_{\tau y} = F_{b\,\tau y} + F_{c\,\tau y} \\ F_{\tau z} = F_{b\,\tau z} + F_{c\,\tau z} \end{cases} \tag{1}$$

Where: F_{cn} is the normal elastic force; F_{bn} is the normal viscous force; $F_{b\,\tau y}$ is the tangential viscous force by axis ***Y***; $F_{c\,\tau y}$ is the tangential elastic force by axis ***Y***; $F_{b\,\tau z}$ is the tangential viscous force by axis ***Z***; $F_{c\,\tau c}$ is the tangential elastic force by axis ***Z***.

The general tangential force can be found as sum

$$F_\tau = \sqrt{F_{\tau y}^2 + F_{\tau z}^2} \tag{2}$$

As it is known (Brilliantov et al., 1996; Goloshchapov, 2003, 2015; Jonson, 1985; Mindlin, 1949; Ramirez et al., 1999; Schafer et al., 1996; Schwager & Poschel, 2007; Stronge, 2000; Thornton, 2009 we can write all these viscoelastic forces, simply as:

$$\begin{cases} F_{bn} = b_x\dot{x} \qquad F_{cn} = c_x x \\ F_{b\tau y} = b_y\dot{y} \qquad F_{c\tau y} = c_y y \\ F_{b\tau z} = b_z\dot{z} \qquad F_{c\tau z} = c_z z \end{cases} \tag{3}$$

Where c_y, c_z, b_x, b_y, b_z are variable parameters depending of the displacement ***x, y*** and ***z***, and where: b_x is the effective parameter of viscosity (damping parameter), c_x is the effective parameter of elasticity (stiffness); b_y, b_z are the effective parameters viscosity at a shift; c_y, c_z are the effective parameters of elasticity (stiffness) at a shift. x, y, z are the relative displacements (deformations) between contacting bodies or between contacting surfaces in the initial point ***A***; $\dot{y}, \dot{x}, \dot{z}$ are relative velocities of displacements (deformations) between contacting bodies or between contacting surfaces in the initial point ***A***.

In the past many old papers and others published recently (Mindlin, 1949; Simon, 1967; Johnson, 1985 ; Goloshchapov, 2003, 2015; Cundall & Strack,1979; Hertzsch, Spahn , Schafer, Dippel, & Wolf, 1996; Ramírez, Poeschel, Brilliantov, & Schwager, 1999; Stronge, 2000; Roylance, 2001; Brilliantov & Poeschel, 2004; Makse, Gland, Jnohnson, & Schwartz, 2004; Schwager & Poschel, Van Zeebroeck, 2005; Dintwa, 2006; Schwager & Poschel, 2007, 2008; Cheng, Subic, & Monir Takla, 2008; Becker, Schwagerand, & Pöschel, 2008; Schwager & Poschel, 2008; Thornton, 2009; Cummins, Thornton, & Cleary, 2012) have been used traditional theoretical rheological models, such as the "Linear Spring Dashpot Model"- (LS+D), the "Hertz Mindlin Spring Dashpot Model"- (HM+D), and the "Discrete Elements Method" - (DEM) and others . In all of these methods and models, for the definition of the effective parameter of elasticity c_x (Some authors name it like a stiffness, or spring parameter), the Hertz's theory of elastic contact between two surfaces (Landau and Lifshitz, 1944, 1965) has been used. Also, for the purpose of finding the tangential forces, the coefficient of friction was taken as a constant value. The more comprehensive analysis and review of these already known models and methods can be found, for example, in the monographs of the authors, such as: Stronge (2000); Van Zeebroeck (2005); Dintwa (2006); Li (2006). But, the most basic problem in the finding of the mechanical dynamic parameters of viscoelasticity c_y, c_z, b_x, b_y in the equations (3) is that, they are not the constant values. They all are variable magnitudes, because all dynamic mechanical and physical properties of the materials depend on dynamic conditions of loading (displacements, a velocity and a frequency) and temperature. But, the Hertz theory allows only the finding the normal elastic force. The existing methods still cannot give the complete answer, how these nonlinear parameters of viscoelasticity can be found for the practical application by using the dynamic modules of elasticity and viscosity, which usually can be found by using the known methods (Ferry, 1948, 1963; Moore, 1975; Van Krevelen 1972; Nilsen, 1978, 1994) For example as we know according the Hertz theory (Landau & Lifshtz, 1944, 1965) for the contact of two spherical surfaces $c_x = \frac{4}{3} E R^{1/2} x^{1/2}$. Where *E* is effective elasticity modulus, *R* is the effective radius of contact curvature. As we can see, the stiffness c_x is the nonlinear function of displacement ***x***, but we still have a problem in definition of others parameters c_y, c_z, b_x, b_y, b_z, which are the variable nonlinear functions too.

And also in already existing researches, the coefficient of friction usually is taken like a constant value for the purpose of finding the tangential forces, but according to Equations (1), (2) and (3) it can be defined as follows:

$$f = \frac{F_\tau}{F_n} = \frac{\sqrt{(F_{b\tau y} + F_{c\tau y})^2 + (F_{b\tau z} + F_{c\tau z})^2}}{F_{cn} + F_{bn}} = \frac{\sqrt{(b_y \dot{y} + c_y y)^2 + (b_z \dot{z} + c_z z)^2}}{c_x x + b_x \dot{x}} \tag{4}$$

As we can see from this equation, the coefficient of friction is not a constant value, but it changes during the time of contact, because the dynamic contact between two bodies is a non-equilibrium process, and all dynamic mechanical and physical properties of the materials depend on dynamic conditions of loading and temperature. For example, the dynamic elasticity modules are very yieldable to a changing of a velocity and temperature of the matter in the area of deformations (Ferry, 1948, 1963; Lee, 1962; Van Krevelen, 1972; Moor 1978; Nilsen, 1978, 1994; Lakes, 1998; Meyers, 1994; Menard, 1999; Roylance, 2001; Goloshchapov, 2001, 2003, 2015; Hosford, 2005; Popov, 2010, Popov & Hess, 2015).

Also, it is necessary to mention that, the researches in the field of the collision of viscoelastic particles (granules) with identical mechanical properties have been made by Brilliantov, Spahn, Hertzsch and Pöschel (1996). They have obtained the equation for the normal viscous force with variable viscosity parameter

$$F_{bn} = F_{dis}^{N} = 4 \frac{Y}{(1-\nu^2)} \sqrt{R^{eff}} A \sqrt{\xi} \xi^{1/2}, \tag{4*}$$

where $\xi = x$, $R^{eff} = R$, *Y* is the Young modulus or the elasticity modul, ν is the Poisson ratio,

$A = \frac{1}{3}\frac{(3\eta_2 - \eta_1)^2}{(3\eta_2 + 2\eta_1)}\left[\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right]$ is the damping viscous parameter, and where η_1 and η_2 are the viscous constants. But this theoretical result can only be used for the contact of the bodies with the same physical-mechanical properties, and in this case we have the problem of finding the viscous constants "η_1" and "η_2". If the contacting surfaces have different physical-mechanical properties this conception does not give the answer, because this is a yet more difficult problem.

Also the interesting method - "Method of Dimensionality Reduction (MDR)" has been presented by Popov (2015), but it can be used only in the case of contact between three-dimensional, axial-symmetric bodies and a foundation (a semi-space). In all these papers, to find the equations for tangential forces, the coefficient of friction again was taken as a constant value.

Thus, as we can see, the many problems still exist now in these research areas. Therefore, especially for the solving of these problems, such as the definition of the normal viscous force and the all tangential viscoelastic forces, and for the finding of the kinematic and the dynamic mechanical parameters between two contacting surfaces, such as the elasticity modulus and the viscosity modulus, the theoretical and experimental ways have been developed and represented in this article below.

2. Derivation of the Equations for the Viscoelastic Forces by the "Method of the Specific Forces (MSF)"

Let us assume that in the infinitesimal period of the time dt, when the mutual approach between a body and a semi-space is the infinitesimal magnitude dx (Figure 2), inside the elementary infinitesimal volume dV, which is arising around the initial point of the contact A (Figure1 and Figure2), the infinitesimal viscoelastic forces dF_n, $dF_{\tau y}$ and $dF_{\tau z}$ are beginning to act.

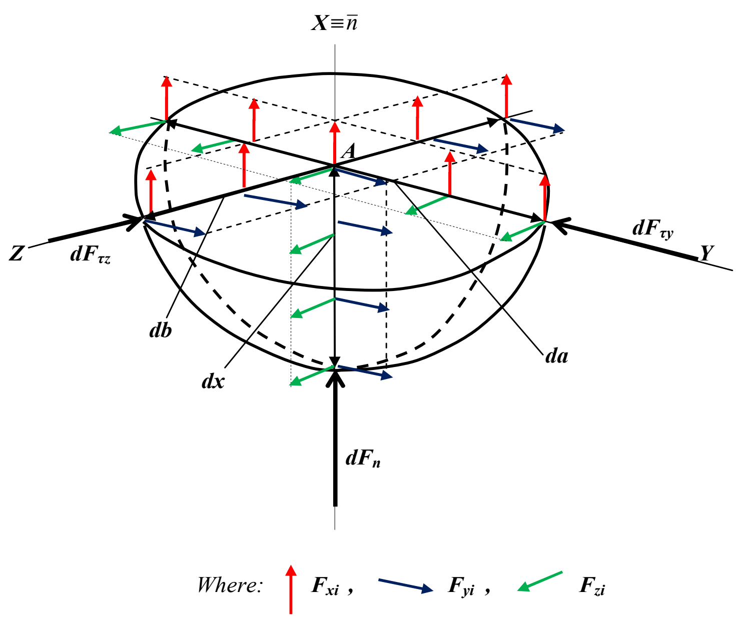


Figure 2. Illustration of the action of the specific elementary viscoelastic forces inside the infinitesimal volume ***dV*** in the vicinity of point ***A***

These forces can be found by the differentiation of the normal F_{xi}, and the tangential F_{yi} and F_{zi} specific forces by sizes da, db and dx:

$$dF_n = F_{xi}da + F_{xi}db\ ,\ dF_{\tau y} = F_{yi}db + F_{yi}dx\ ,\ dF_{\tau z} = F_{zi}da + F_{zi}dx \qquad (5)$$

Where: da and db the big and the small axes of the infinitesimal contact area; F_{xi} is the normal effective specific viscoelastic force; F_{zi} is the tangential effective specific viscoelastic force by axis ***Z***; F_{yi} is the tangential effective specific viscoelastic force by axis ***Y***.

Also it is very important to understand indeed that, these specific forces *Fxi*, *Fyi* and *Fzi* are the elementary axial loads distributed on the infinitesimal sizes, *da*, *db* and *dx* parallel to axes *X*, *Y* and *Z* (Figure2). The name “specific force" already was applied by author of article (Goloshchapov, 2015), and this name is the short name, which usually is used to denote the elementary distributed axial loads.

2.1 The Effective Dynamic Modules and the Effective Dynamic Viscosities

According to the “Newton’s Third Law” the effective specific forces and the specific forces between bodies have to be equal: $F_{xi} = F_{x1} = F_{x2}$, $F_{yi} = F_{y1} = F_{y2}$, $F_{zi} = F_{z1} = F_{z2}$. Where: F_{x1} and F_{x2} are the normal specific viscoelastic forces of the bodies by axis *X*; F_{y2} and F_{y1} is the tangential viscoelastic specific forces of the bodies by axis *Y*; F_{z2} and F_{z1} is the tangential viscoelastic specific forces by axis *Z*. (Here and further in this paper the subscript *1* is used for more soft body, and *2 is* used for more solid body). On the other hand, the specific viscoelastic forces can be found as the sum of the specific elastic forces and the specific viscous forces:

$$\begin{cases} F_{xi} = F_{xb} + F_{xc}, \; F_{x1} = F_{x1b} + F_{x1c}, \quad F_{x2} = F_{x2b} + F_{x2c} \\ F_{yi} = F_{yb} + F_{yc}, \; F_{y1} = F_{y1b} + F_{y1c}, \quad F_{y2} = F_{y2b} + F_{y2c} \\ F_{zi} = F_{zb} + F_{zc}, \; F_{z1} = F_{z1b} + F_{z1c}, \quad F_{z2} = F_{z2b} + F_{z2c} \end{cases} \tag{6}$$

Where: F_{xb} is the normal effective specific viscous force; F_{xc} is the normal effective specific elastic force; F_{x1b} and F_{x2b} are the normal specific viscous forces; F_{x1c} and F_{x2c} are the normal specific elastic forces; F_{yb} and F_{zb} are the tangential effective specific viscous forces; F_{yc} and F_{zc} is the tangential effective specific elastic forces; F_{y1b}, F_{y2b} and F_{z1b}, F_{z2b} are the tangential viscous specific forces; F_{y1c}, F_{y2c} and F_{z1c}, F_{z2c} are the tangential elastic specific forces.

Also let us suppose that the volume of deformation is the system of an infinitely large number of elementary discrete elements (Figure 3.) connected among themselves definitely. And also, in this case let us assume, that for the infinitesimal period of the contact time *dt* all deformations inside of each elementary discrete element are changing linearly and therefore all specific forces are changing linearly too. Based on this, the equations for all specific forces can be written as the linear functions:

$$\begin{cases} F_{xb} = \eta'_E \dot{x}, \; F_{xc} = E'x, \; F_{yb} = \eta'_G \dot{y}, \; F_{yc} = G'y, \; F_{zb} = \eta'_G \dot{z}, \; F_{zc} = G'z \\ F_{x1b} = \eta'_{1E} \dot{x}_1, \; F_{x1c} = E'_1 x_1, \; F_{y1b} = \eta'_{1G} \dot{y}_1, \; F_{y1c} = G'_1 y_1, \; F_{z1b} = \eta'_{1G} \dot{z}_1, \; F_{z1c} = G'_1 z_1, \\ F_{x2b} = \eta'_{2E} \dot{x}_2, \; F_{x2c} = E'_2 x_2, \; F_{y2b} = \eta'_{2G} \dot{y}_2, \; F_{y2c} = G'_2 y_2, \; F_{z2b} = \eta'_{2G} \dot{z}_2, \; F_{z2c} = G'_2 z_2 \end{cases} \tag{7}$$

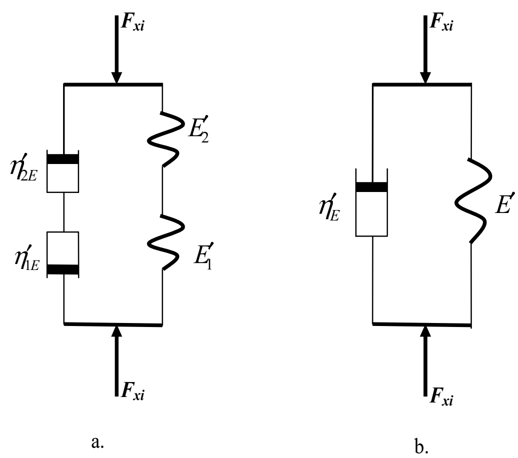


Figure 3. Illustration of the “Elementary discrete elements model (EDEM)”: a. the elementary discrete element of the normal contact between two bodies; b. the effective elementary discrete element of the normal contact

Where: E' is the effective dynamic elasticity modulus; η'_E is the effective dynamic viscosity; G' is effective dynamic elasticity modulus at the shear; η'_G is the effective dynamic viscosity at the shear; E'_1 and E'_2 are the dynamic elasticity modules. η'_{1E} and η'_{2E} are the dynamic viscosities; G'_1 and G'_2 is the dynamic elasticity modules at the shear. η'_{1G} and η'_{2G} is the dynamic viscosities at the shear.

In the proposed model, each elementary deformation between two bodies develops analogically like the deformation of the elementary discrete element, which is depicted in Figure 3.a. It is a simple case of the linear model of deformations of elementary discrete elements, and instead this model with four elements we can use its analogy - the model with two effective elements depicted in Figure 3.b. Also the "Elementary discrete elements model" for the normal forces can be used for the tangential forces in the same manner.

It is obvious that the according to the initial conditions, when $t = 0$, $x = 0$, the specific elastic forces acting by X are equal at the initial instant of the contact (in this point they are equal zero); and the according to the boundary conditions, when $x = x_m$, they are equal at the instant of the maximum compression between two surfaces (in this point they reach the maximum value). But at the same time, the specific viscous forces are equal at the initial instant of the contact, when $t = 0$, $x = 0$ (in this point they are equal to the maximum value, because the velocity $\dot{x}$ has the maximum value), and they are equal at the instant of the maximum compression between two surfaces, when $x = x_m$ (in this point they are equal zero, because the velocity $\dot{x}=0$). All of these forces in the Equation 7 are linear continuous functions and if they are equal for these two values of the argument x, they have to be equal for any other values as well, or by other words, they are equal in any instant of the time of the contact. Analogically this conclusion is valid for other specific forces acting by axes Y and Z. Thus consequently we can write that

$$F_{xc}=F_{x1c}=F_{x2c}\,,\ \ F_{xb}=F_{x1b}=F_{x2b}\,,\ \ F_{yc}=F_{y1c}=F_{y2c}\,,\ \ F_{yb}=F_{y1b}=F_{y2b}\ \text{ and }\ F_{zb}=F_{z1b}=F_{z2b}\,, \tag{7*}$$

and hence we get respectively

$$\begin{cases} E'x = E'_1x_1=E'_2x_2\,, & \eta'_E\dot{x}=\eta'_{1E}\dot{x}_1=\eta'_{2E}\dot{x}_2 \\ G'y=G'_1y_1=G'_2y_2\,; & \eta'_G\dot{y}=\eta'_{1G}\dot{y}_1=\eta'_{2G}\dot{y}_2 \\ G'z=G'_1z_1=G'_2y_2\,; & \eta'_G\dot{z}=\eta'_{1G}\dot{z}_1=\eta'_{2G}\dot{z}_2 \end{cases} \tag{8}$$

Since as $x = x_1 + x_2$, $y = y_1 + y_2$, $z = z_1 + z_2$ according to (8) we can write the equations for the effective dynamic modules and the effective dynamic viscosities as

$$E' = \frac{E'_1E'_2}{E'_1+E'_2}\,,\ \ \eta'_E = \frac{\eta'_{1E}\eta'_{2E}}{\eta'_{1E}+\eta'_{2E}}\,,\ \ G' = \frac{G'_1G'_2}{G'_1+G'_2}\,,\ \ \eta'_G = \frac{\eta'_{1G}\eta'_{2G}}{\eta'_{1G}+\eta'_{2G}} \tag{9}$$

And also we can write that

$$x_1 = D_1x \ \text{ and } \ x_2 = D_2x \tag{10}$$

Where: $D_1 = \dfrac{E'_2}{E'_1+E'_2}$ and $D_2 = \dfrac{E'_1}{E'_1+E'_2}$ are the coefficients of deformation.

2.2 Integral Equations for the Viscoelastic Forces

Now, since according to (5) and (6) we can write that

$$\begin{cases} dF_n = (F_{xb} + F_{xc})da + (F_{xb} + F_{xc})db \ , \\ dF_{\tau y} = (F_{yb} + F_{yc})db + (F_{yb} + F_{yc})dx \ , \\ dF_{\tau z} = (F_{zb} + F_{zc})da + (F_{zb} + F_{zc})dx \ , \end{cases} \qquad (10^*)$$

and according to (7), since $F_{xb} = \eta'_E \dot{x}$, $F_{xc} = E'x$, $F_{yb} = \eta'_G \dot{y}$, $F_{yc} = G'y$, $F_{zb} = \eta'_G \dot{z}$, $F_{zc} = G'z$, the twelve expressions for the infinitesimal viscoelastic forces can be written separately as:

$$\begin{cases} dF_{abn} = \eta'_E \dot{x} da \, , \, dF_{acn} = E' x da \, , \, dF_{bbn} = \eta'_E \dot{x} db \, , \, dF_{bcn} = E' x db \, , \\ dF_{bby} = \eta'_G \dot{y} db , \, dF_{bcy} = G' y db , \, dF_{hby} = \eta'_G \dot{y} dx \, , \, dF_{hcy} = G' y dx , \\ dF_{abz} = \eta'_G \dot{z} da , \, dF_{acz} = G' z da , \, dF_{hbz} = \eta'_G \dot{z} dx \, , \, dF_{hcz} = G' z dx \end{cases} \qquad (11)$$

And since the limits of integration h_{xa} and h_{xb} are known, the integral equations for all viscoelastic forces can be written respectively as

$$\begin{cases} F_{abn} = \eta'_E \dot{x} \int da \ , \ F_{acn} = E' \int x da \ , \ F_{bbn} = \eta'_E \dot{x} \int db \ , \ F_{bcn} = E' \int x db \ , \\ F_{bby} = \eta'_G \dot{y} \int db \ , \ F_{bcy} = G' y \int db \ , \ F_{hby} = \eta'_G \dot{y} \int_0^{h_{xa}} dx \ , \ F_{hcy} = G' y \int_0^{h_{xa}} dx \ , \\ F_{abz} = \eta'_G \dot{z} \int da \ , \ F_{acz} = G' z \int da \ , \ F_{hbz} = \eta'_G \dot{z} \int_0^{h_{xb}} dx \ , \ F_{hcz} = G' z \int_0^{h_{xb}} dx \end{cases} \qquad (12)$$

2.3 The Geometry of the Area of the Contact and the Pressure Distribution

And now, the important moment, it can be seen that for a finding of the solutions for all Equations (12) we have to know only the equations or the formulas for $a = f(x)$, $b = f(x)$, and for $h_{xa} = f(x)$ and $h_{xb} = f(x)$. For example, we can use that $r = (Rx)^{1/2}$ according to the Hertz theory, but according to this theory, the area of contact is a flat surface and the depth of indentation (the depth of the contact surface) $h_x = 0$. But in reality the area of contact usually is not a flat, it is a curvilinear surface. In Hertz's theoretical models has been taken that the contacting surfaces deform together without of the micro-sliding, but in reality each surface deforms independently. Therefore, to find the radiuses of the contact area r_a and r_b in reality, first of all, let us consider the geometry of contact between two curvilinear surfaces in the normal section in the plane *XAY*, like it is depicted in the illustration in Figure1. It is obviously that, in the time of indentation of more hard surface into a soft surface, the contact surface takes a curvilinear shape, where the point *B* (see Figure1) is a special point where the deformations always equal zero, and the border of the area of contact always pass through this point *B*. According to this statement, the distance O_2B between this point and the centre of curvature O_2 of the surface of more hard body will not be changed in the period of time of contact. This distance always equals to the radius of curvature R_2. Also the distance O_1B between this point and the centre of curvature O_1 of the surface of less hard body will not be changed in the period of time of contact too. This distance always equals to the radius of curvature R_1. Hence, obviously that $O_2B = O_2D = R_{2a}$ and $O_1B = O_1E = R_{1a}$, and also we can write that $O_1C + O_2C = (R_{1a} + R_{2a}) - x$, and since as $O_1C = (R_{1a}^2 - r_a^2)^{1/2}$ and $O_2C = (R_{2a}^2 - r_a^2)^{1/2}$, after a simple calculation, if to neglect by members of smallest order, we get the next equation for the radius of contact area $r_a = f(x)$ equals *a/2*, and then analogically in same way we can find the radius of contact area $r_b = f(x)$ equals *b/2* as follows:

$$r_a^2 = 2R_a x - x^2 \, , \ r_b^2 = 2R_b x - x^2 \qquad (13)$$

Where $R_a = \dfrac{R_{1a} R_{2a}}{R_{1a} + R_{2a}}$ is the effective initial radius of contact curvature in the plane *XAY* and $R_b = \dfrac{R_{1b} R_{2ab}}{R_{1b} + R_{2b}}$ is the effective initial radius of contact curvature in the plane *XAZ*. The Equation (13) is not convenient for using and therefore, let us rewrite them as

$$r_a^2 = k_{pa}^2 R_a x\,,\ \ r_b^2 = k_{pb}^2 R_b x \tag{14}$$

where

$$k_{pa} = \sqrt{2 - \frac{x}{R_a}}\,,\ \ k_{pb} = \sqrt{2 - \frac{x}{R_b}} \tag{15}$$

are the correlation coefficients. Practically for the solution of the contact problems, the correlation coefficient can be found by the method of iterations and a consecutive approximation.

Obvious from (13) that the surface of the contact takes the elliptical shapes in the planes *XAY* and *XAZ*, but for a simplification, its shape can be approximated by the parabolic functions (14) respectively as

$$x = \frac{1}{k_{pa}^2 R_a} r_a^2\,,\ \ x = \frac{1}{k_{pb}^2 R_b} r_b^2 \tag{16}$$

Since the surface of the contact has approximately the parabolic shape, let us to take that the radial distribution of the pressure inside of this area changes analogically according to the parabolic function as

$$P = P_c\left(1 - \frac{r_y^2}{r_a^2}\right),\ \ P = P_c\left(1 - \frac{r_z^2}{r_b^2}\right) \tag{17}$$

Where: r_y is a current radius of the contact area by axis *Y*; r_z is a current radius of the contact area by axis *Z*; P_c is the maximum magnitude of the pressure in the centre of the contact area.

Further since the square under these functions in the Equation17 and the square under the linear function of the mean pressure P_m in the contact area are equal, we can write that

$$P_c\int_0^{r_a}\left(1 - \frac{r_y^2}{r_a^2}\right)dr_y = P_m r\,,\ \ P_c\int_0^{r_b}\left(1 - \frac{r_z^2}{r_b^2}\right)dr_z = P_m r \tag{18}$$

Then after the integration follows

$$P_c\left(r - \frac{1}{3}r\right) = P_m r\,, \tag{19}$$

and finally the ratio between maximum and the mean pressure in the contact zone can be found as

$$P_c = \frac{3}{2}P_m \tag{20}$$

Now let us to define the depths of the contact surface (see Figure 1) in the planes *XAY* and *XAZ*. The expressions for the radiuses of contact area can be found also as follows

$$r_a^2 = R_{2a}^2 - \left(R_{2a}^2 - (x_2 + h_{xa})\right)^2,\ \ r_b^2 = R_{2b}^2 - \left(R_{2b}^2 - (x_2 + h_{xb})\right)^2 \tag{21}$$

After a simple geometric calculation, if to neglect by members the smallest order, we obtain the next equations for the radiuses of contact area:

$$r_a^2 = 2R_{2a}(x_2 + h_{xa}) - (x_2 + h_{xa})^2\,,\ \ r_b^2 = 2R_{2b}(x_2 + h_{xb}) - (x_2 + h_{xb})^2 \tag{22}$$

Then after the comparison equations (22) and (13) we can write that

$$2R_{2a}(x_2 + h_{xa}) \approx 2R_a x\ \ ,\ \ 2R_{2b}(x_2 + h_{xb}) \approx 2R_b x \tag{23}$$

Finally since $x_2 = D_2 x$, the formulas for h_{xa} and h_{xb} can be written as follows

$$h_{xa} = \left(\frac{R_a - D_2 R_{2a}}{R_{2a}} \right) x = k_{ha} x \,, \quad h_{xb} = \left(\frac{R_b - D_2 R_{2b}}{R_{2b}} \right) x = k_{hb} x \tag{24}$$

Where $k_{ha} = \left(\frac{R_a - D_2 R_{2a}}{R_{2a}} \right)$ and $k_{hb} = \left(\frac{R_b - D_2 R_{2b}}{R_{2b}} \right)$ are the coefficients of depth of the contact surface.

Also let us remark that, in the case a very small deformations, when $R_a >> x$ and $R_b >> x$ according to Equations (15) follows $k_{pa} \approx k_b \approx \sqrt{2}$, and we can write that

$$r_a = (2R_a x)^{1/2}, \quad r_b = (2R_b x)^{1/2} \tag{25}$$

But, according to the Hertz theory (Landau and Lifshitz 1944, 1965) $r = (Rx)^{1/2}$, but it is possible only in the one case when $k_{pa} = k_{pb} = 1$.

2.4 The Equations for all Viscoelastic Forces

Since as $a = 2r_a$ and $b = 2r_b$, according to the Equation (14) we can write that

$$a = 2k_{pa} R_a^{1/2} x^{1/2} \quad , \quad b = 2k_{pb} R_b^{1/2} x^{1/2} \tag{26}$$

The taking of the derivatives *da/dx* and *db/dx* gives us

$$da = \frac{k_{pa} R_a^{1/2}}{x^{1/2}} dx \quad , \quad db = \frac{k_{pb} R_b^{1/2}}{x^{1/2}} dx \,, \tag{27}$$

and then after an integration of the integral equations for the normal forces from (12), we get

$$F_{acn} = k_{pa} E' R_a^{1/2} \int x^{1/2} dx = \frac{2}{3} k_{pa} E' R_a^{1/2} x^{3/2}, \quad F_{bcn} = k_{pb} E' R_b^{1/2} \int x^{1/2} dx = \frac{2}{3} k_{pb} E' R_b^{1/2} x^{3/2} \tag{28}$$

Since as $F_{cn} = F_{acn} + F_{bcn}$ we get

$$F_{cn} = \frac{2}{3} E' x^{3/2} (k_{pa} R_a^{1/2} + k_{pb} R_b^{1/2}) = \frac{2}{3} E' \psi x^{3/2} \,, \tag{29}$$

where $\psi = (k_{pa} R_{pa}^{1/2} + k_{pb} R_{pb}^{1/2})$, [$m^{1/2}$] is the parameter of curvature.

If $k_{pa} = k_{pb} = 1$ and if the contact area is a circle $R_a = R_b = R$ we have the same solution that have been obtained for the contact between spherical surfaces by using the Hertz theory (Landau & Lifshitz, 1944, 1965):

$$F_{cn} = \frac{4}{3} E' R^{1/2} x^{3/2} \tag{30}$$

Thus, it is obvious that the proposed method of the finding of the normal elastic forces definitely is valid and correct. It can be seen that, if we know a functional dependency between ***r*** and ***x***, we can always find the elastic force. But, if this method is correct for the definition this force, hence it should be valid for the definition of all viscoelastic forces in the equations (12). The equation for the normal viscous forces can be found in the same way by integration:

$$\begin{cases} F_{abn} = k_{pa} \eta'_E \dot{x}(t) \int \frac{R_a^{1/2}}{x^{1/2}} dx = 2k_{pa} \eta'_E R_a^{1/2} \dot{x}(t) x^{1/2} \\ F_{bbn} = k_{pb} \eta'_E \dot{x}(t) \int \frac{R_b^{1/2}}{x^{1/2}} dx = 2k_{pb} \eta'_E R_b^{1/2} \dot{x}(t) x^{1/2} \end{cases} \tag{31}$$

Here $\dot{x}(t)$ are the function linear independent from *x*, and it cannot be integrated by *x*, and it stays outside of integrals. And since as $F_{bn} = F_{abn} + F_{bbn}$ follows

$$F_{bn} = 2\eta'_E \dot{x} x^{1/2} (k_{pa} R_a^{1/2} + k_{pb} R_b^{1/2}) = 2\eta'_E \psi \dot{x} x^{1/2} \tag{32}$$

In the case when the contact area is a circle $k_{pa} = k_{pb} = k_p$ and $R_a = R_b = R$ we get

$$F_{bn} = 4k_p \eta'_E R^{1/2} \dot{x} x^{1/2} \tag{33}$$

Since as *x* and *y* are linearly independent and $da = \frac{k_{pa} R_a^{1/2}}{x^{1/2}} dx$, $db = \frac{k_{pb} R_b^{1/2}}{x^{1/2}} dx$, and $h_{ax} = k_{ha} x$, $h_{xb} = k_{hb} x$, after an integration of the equations for the tangential forces from Equation (12) their solutions can be written as follows:

$$\begin{cases} F_{bby} = \eta'_G \dot{y} \int db = 2k_{pb} \eta'_G R_b^{1/2} x^{1/2} \dot{y}, \quad F_{bcy} = G' y \int db = 2k_{pb} G' R_b^{1/2} x^{1/2} y \\ F_{hby} = \eta'_G \dot{y} \int_0^{h_{xa}} dx = k_{ha} \eta'_G x \dot{y}, \quad F_{hcy} = G' y \int_0^{h_{xa}} dx = k_{ha} x y, \\ F_{abz} = \eta'_G \dot{z} \int da = 2k_{pa} \eta'_G R_a^{1/2} x^{1/2} \dot{z} \quad F_{acz} = G' z \int da = 2k_{pa} G' R_a^{1/2} x^{1/2} z \\ F_{hbz} = \eta'_G \dot{z} \int_0^{h_{xb}} dx = k_{hb} x \dot{z}, \quad F_{hcz} = G' z \int_0^{h_{xb}} dx = k_{hb} x z \end{cases} \tag{34}$$

Here $\dot{y}, y, \dot{z}, z$ are the functions linear independent from *x*, and they cannot be integrated by *x*, and they stay outside of integrals.

The equations for the tangential elastic and viscous forces can be written now, as the sum of the elastic and the viscous tangential forces from Equation (34):

$$\begin{cases} F_{b\tau y} = F_{hby} + F_{bby} = \eta'_G P_{bx} \dot{y}, \quad F_{c\tau y} = F_{hcy} + F_{bcy} = G' P_{bx} y \\ F_{b\tau z} = F_{hbz} + F_{bbz} = \eta'_G P_{ax} \dot{z}, \quad F_{c\tau z} = F_{hcz} + F_{acz} = G' P_{ax} z \end{cases} \tag{35}$$

Where:

$$P_{bx} = k_{ha} x + 2k_{pb} R_b^{1/2} x^{1/2}, \quad P_{ax} = k_{hb} x + 2k_{pa} R_a^{1/2} x^{1/2} \tag{36}$$

Thus finaly, according to Equations (1), (31), (34) and (35) the next system of equations for general viscoelastic forces can be written respectively:

$$\begin{cases} F_n = F_{bn} + F_{cn} = 2\eta'_E \psi \dot{x} x^{1/2} + \dfrac{2}{3} E' \psi x^{3/2} \\ F_{\tau y} = F_{b\tau y} + F_{c\tau y} = \eta'_G P_{bx} \dot{y} + G' P_{bx} y \\ F_{\tau z} = F_{b\tau z} + F_{c\tau z} = \eta'_G P_{ax} \dot{z} + G' P_{ax} z \end{cases} \tag{37}$$

And thus, we can write the expressions for the variable viscoelasticity parameters as follows:

$$b_x = 2\eta'_E \psi x^{1/2},\ c_x = \frac{2}{3} E' \psi x^{1/2},\ b_y = \eta'_G P_{bx},\ c_y = G' P_{bx},\ b_z = \eta'_G P_{ax},\ c_z = G' P_{ax} \tag{38}$$

It is not possible the finding of the viscoelastic forces separately for each contacting body if to use the Hertz theory and others already existing theories, but it is possible by using the "Method of the specific forces (MSF)". The integral Equation (12) separately for each contacting body can be written as follows:

$$\begin{cases} F_{abni} = \eta'_{iE} \dot{x}_i \int da,\ F_{acni} = E'_i \int x_i da,\ F_{bbni} = \eta'_{iE} \dot{x}_i \int db,\ F_{bcni} = E'_i \int x_i db, \\ F_{hbyi} = \eta'_{iG} \dot{y}_i \int_0^{h_{xa}} dx,\ F_{hcyi} = G'_i y_i \int_0^{h_{xa}} dx,\ F_{hbzi} = \eta'_{iG} \dot{z}_i \int_0^{h_{xb}} dx,\ F_{hczi} = G'_i z_i \int_0^{h_{xb}} dx, \\ F_{bbyi} = \eta'_{iG} \dot{y}_i \int db,\ F_{bcyi} = G'_i y_i \int db,\ F_{abzi} = \eta'_{iG} \dot{z}_i \int da,\ F_{aczi} = G'_i z_i \int da \end{cases} \tag{39}$$

Where usually the index $\boldsymbol{i} = 1$ use for a soft body and $\boldsymbol{i} = 2$ use for a hard body.

Since as $x = \dfrac{x_1}{D_1} = \dfrac{x_2}{D_2}$, we can write that $a = 2k_{pa} R_a^{1/2} \dfrac{x_1^{1/2}}{D_1^{1/2}}$ and as well $a = 2k_{pa} R_a^{1/2} \dfrac{x_2^{1/2}}{D_2^{1/2}}$, and $b = 2k_{pb} R_b^{1/2} \dfrac{x_1^{1/2}}{D_1^{1/2}}$ and as well $b = 2k_{pb} R_b^{1/2} \dfrac{x_2^{1/2}}{D_2^{1/2}}$, then after differentiation we get:

$$da = \frac{k_{pa} R_a^{1/2}}{D_1^{1/2} x_1^{1/2}} dx_1,\ da = \frac{k_{pa} R_a^{1/2}}{D_2^{1/2} x_2^{1/2}} dx_2,\ db = \frac{k_{pb} R_b^{1/2}}{D_1^{1/2} x_1^{1/2}} dx_1,\ db = \frac{k_{pb} R_b^{1/2}}{D_2^{1/2} x_2^{1/2}} dx_2 \tag{40}$$

After a substituting the expressions (40) into (39) and then after their integration we can get the equations for the viscoelastic forces separately for each body. If we can find the viscoelastic forces separately for each body hence we can find separately the viscoelastic stresses for each body too. It is not possible to do by using the Hertz theory or by using others already existing theories and methods.

The normal viscoelastic stresses, which equal to the mean pressure P_m in the contact area, can be found as

$$P_m = \frac{F_n}{S_x}, \tag{41}$$

where $S_x = \dfrac{\pi}{4} ab$ is the contact area, and according to Equation (26) follows $S_x = \pi k_{pa} k_{pb} R_a^{1/2} R_b^{1/2} x$. In correspondence with Equations (20), (37) and (41) the expression for maximum value of the normal contact pressure (stresses) can be written as follows:

$$P_c = \frac{\psi(3\eta'_E \dot{x} + E'x)}{\pi k_{pa} k_{pb} R_a^{1/2} R_b^{1/2} x^{1/2}} \tag{42}$$

3. Dynamic Contact between two Spherical Bodies at Impact

In the case of the contact between two spheres, when the contact area is a circle $R_a = R_b = R$, $r_a = r_b = r$, $a = b = 2r$, $F_{\tau z} = 0$, $F_\tau = F_{\tau y}$, $F_{b\tau} = F_{b\tau y}$, $F_{c\tau} = F_{c\tau y}$ (see Figure1) follows that $k_p = k_{pa} = k_{pb}$, $h_x = h_{xa} = h_{xb}$ and $k_h = k_{ha} = k_{hb}$, $\psi = 2k_p R^{1/2}$, and then according to Equations (1) and (37) we get the next system of equations for the main viscoelastic forces:

$$\begin{cases} F_n = F_{bn} + F_{cn} = b_x \dot{x} + c_x x = 4k_p \eta'_E R^{1/2} \dot{x} x^{1/2} + \dfrac{4}{3} k_p E' R^{1/2} x^{3/2} \\ \\ F_\tau = F_{b\tau} + F_{c\tau} = b_y \dot{y} + c_y y = \eta'_G P_x \dot{y} + G' P_x y \end{cases} \tag{43}$$

Where $P_x = k_h x + 2k_p R^{1/2} x^{1/2}$, and where the expressions for the variable viscoelasticity parameters in the Equation (3) can be written as follows

$$b_x = 4\eta'_E R^{1/2} x^{1/2}; \quad c_x = \frac{2}{3} E' R^{1/2} x^{1/2}, \quad b_y = \eta'_G P_{bx}; \quad c_y = G' P_{bx} \tag{44}$$

Also, for this case, the expression for the depth of the contact surface can be written as follows

$$h_x = \left(\frac{R - D_2 R_2}{R_2} \right) x = k_h x \tag{44*}$$

4. Contact between a Spherical Solid Body and a Semi-Space at Impact

This example is given here, because the results, which can be obtained in this case, can be used also for others cases of a dynamic contacts, such as for the dynamic contact at impact between two spherical bodies or between two cylindrical bodies.

Let a spherical body, having the average statistical mass m_2, the radius R and the initial velocity V_0, comes into viscoelastic contact under an arbitrary angle of attack α to the surface of semi-space at the initial instant of the time $t = 0$, at the initial point of contact 0 (Figure 4). And let the vectors of velocities are applied to the centre of mass of the body (the point C_0). Also in Figure4 are designated: $V_{0x} = V_0 \sin\alpha$, $V_{0y} = V_0 \cos\alpha$ are the initial normal and tangential velocities of a body; $M = F_\tau l_x$ is the reactive moment; l_x is the shoulder of tangential force; ω is the angular velocity and ε is the angular acceleration around of the centre of mass of a body; V_d is the volume of deformations, which is forming in the course of contact. Also the geometry of surface of the contact zone is characterised by the geometrical parameters, such as (Figure 4): x_1 is the normal deformation of the surface of semi-space in the middle of the contact area, which as well, is the approach of a semi-space relative to a body; x_2 is the normal deformation of the surface of a body in the middle of the contact area, which as well, is the approach of a semi-space relative to a semi-space; r - the radius of the contact area; h_x is the depth of the contact surface, or other words, h_x is the depth of indentation of a body into the surface of semi-space.

Also, it is seen here (Figure4) that at the initial instant of the time, the body with the centre of mass in the point C_0 comes into contact with the surface of semi-space at the initial point of the contact 0 with coordinates $x = 0$ and $y = 0$, but at the instant of the time t, the centre of mass of a body (the point C_t) takes the position with coordinates x and y.

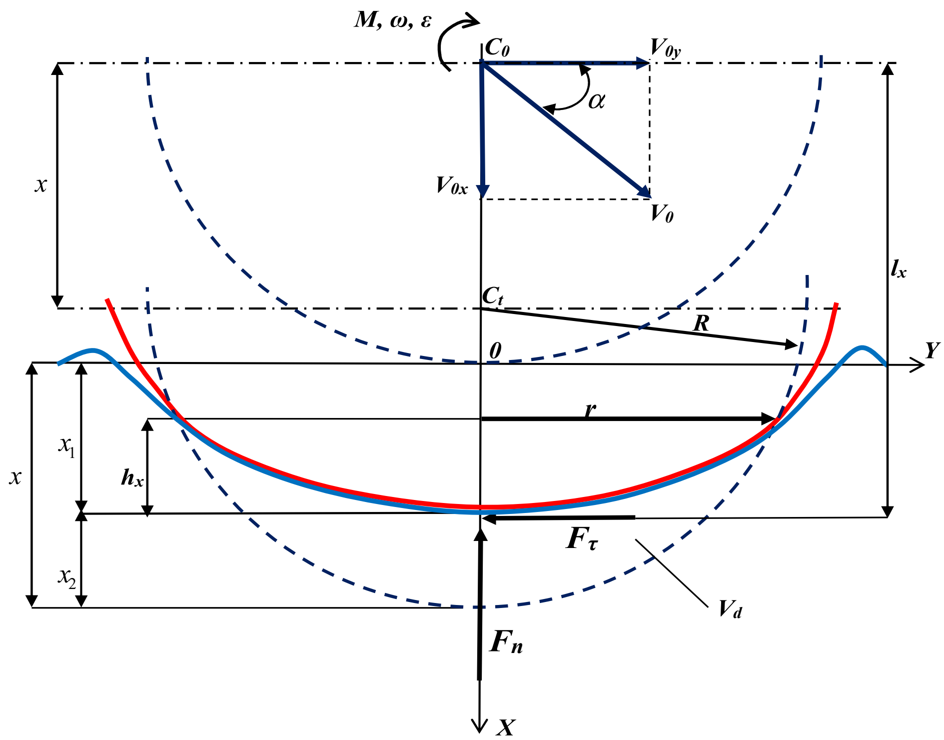


Figure 4. Schematic illustration of the indentation of a spherical body into a semi-space

It is obviously that the viscoelastic forces F_n and F_τ are acting in the contact area between the surfaces of the contact and according to Newton's Second Law we can write:

$$\begin{cases} F_n = -m\ddot{x} \\ F_\tau = -m\ddot{y} \\ M \;\; = -J_z\,\ddot{\varphi} \end{cases} \tag{45}$$

Where: m is the effective or reduced mass of the contacting bodies; $\ddot{y}, \ddot{x}$ - the accelerations of the centre of mass of the body; J_z is the moment of inertia of a body; φ the angle of rotation of the body around the centre of mass; $\ddot{\varphi}$ is the angular acceleration of a body around the centre of mass.

Remark: The term "effective mass" already have been used by Stronge (2000), Dintwa, (2006), Bordbar, Hyppänen (2007), Antypov, Elliott, Hancock (2011), and by many others authors. Also the mass m was called like the reduced mass by Landau (1944, 1965), Brilliantov (1996).

We can see here that $x = x_1 + x_2$ is the distance of the mutual approach (the total deformation) between a body and a semi-space, and as well, in the same instant of the time, it is the displacement of the centre of mass of a body relative to the initial point of contact *0* by axis *X*. At impact of two bodies, the effective mass m enters usually like the mass of the third body, and the movement (the displacement) x of the centre of mass of this third body takes equal to the distance of the mutual approach (or a compression, an overlapping) of the colliding bodies. At impact of two bodies, according to the second law of Newton, we can write that

$$F_n = -m\frac{dV_{0x}}{dt} = -m_1\frac{dV_{1x}}{dt} = -m_2\frac{dV_{2x}}{dt},$$

where $V_{0x} = V_{1x} + V_{2x}$. These equations are valid only for the movement of the centres of mass of the bodies. All authors, who use these equations, for example, Stronge (2000), Dintwa, (2006), Bordbar, Hyppänen (2007), Antypov, Elliott, Hancock (2011), Landau (1944, 1965), Brilliantov (1996) take that x is the displacement of centre mass of this third body and x is the mutual approach (a compression, an overlapping) between the bodies too. From these two expressions follows that $\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$ and since as in the case of the collision of a body and a semi-space $m_1 >> m_2$, and hence we can take that $m = m_2$. As well, if the semi-space is immovable, when the velocity of a semi-space $V_{1x} = 0$, follows that $m = m_2$. Hence it is proved that in the case of collision between a body and a semi-space, the mass of a body m_2 is equal to the effective mass *m*. Further in this paper, the mass of body is designated by the symbol *m*. Consequently, the distance of the mutual approach between a body and a semi-space is equal to the displacement *x* of the centre of mass of a body. Analogically as for *x* we can write that $y = y_1 + y_2$ is the displacement of the centre of mass of a body relative to the initial point of contact *0* by axis *Y*, where y_1 - is the tangential deformation of the surface of a semi-space y_2 is the tangential deformation of the surface of a body.

It is obvious that, in the case of the contact between a spherical body and a semi-space, the effective radius *R* and the radius of a body are equal, and if take in account (25) and that $M = F_\tau l_x$, then if the dynamic viscosities replace by the dynamic viscosity modulus according to the known expressions (Ferry, 1948, 1963; Lee, 1962; Van Krevelen, 1972; Moor 1978; Nilsen, 1978, 1994; Lakes 1998; Meyers 1994; Menard 1999; Goloshchapov, 2001, 2003, 2015; Hosford 2005; Popov 2010, Popov and Hess 2015)

$$\frac{E''}{\omega_x} = \eta'_E \text{ and } \frac{G''}{\omega_y} = \eta'_G, \tag{46}$$

the equations for the normal and the tangential viscoelastic forces and for the reactive moment can be written as follows:

$$\begin{cases} F_n = F_{bn} + F_{cn} = \dfrac{4k_p E''}{\omega_x} R^{1/2}\dot{x}x^{1/2} + \dfrac{4}{3}k_p E' R^{1/2} x^{3/2} \\ F_\tau = F_{b\tau} + F_{c\tau} = \dfrac{G''}{\omega_y} P_x \dot{y} + G' P_x y \\ M = \left(\dfrac{G''}{\omega_y}\dot{y} + G' y \right) P_x l \end{cases} \tag{47}$$

Where E'' is the effective viscosity modulus under, G'' is the effective viscosity modulus at shear, ω_x is the normal angular frequency of damped oscillations of the volume of deformation $\boldsymbol{V_d}$ by the axis $\boldsymbol{X}$, ω_y is the tangential angular frequency of damped oscillations of the volume of deformation $\boldsymbol{V_d}$ by the axis $\boldsymbol{Y}$.

We have to mark that in the case of contact between a spherical body and a semi-space, when $R_2 = R$ (see Equation 44*) follows that $k_h = (1 - D_2) = D_1$, and hence

$$h_x = x_1 = D_1 x \tag{47*}$$

Viscosity modules can be found by using the known (Ferry,1948; Moore,1975; Van Krevelen, 1972; Nilsen, 1978; Landel, 1994) formula

$$\frac{E''}{E'} = \frac{G''}{G'} = tg\beta, \tag{48}$$

where β is the angle of mechanical losses.

Let us notice that often the dynamic modulus of elasticity is named yet as the accumulation (storage) modulus, and the dynamic modulus of viscosity is named yet as the loss modulus.

And now, according to (45) and (47), the differential equations of the movement (displacement) of the centre of mass of a body can be written as follows:

$$\begin{cases} m\ddot{x} + 4k_p \dfrac{E''R^{1/2}}{\omega_x} \dot{x}x^{1/2} + \dfrac{4}{3} k_p E'R^{1/2} x^{3/2} = 0 \\ m\ddot{y} + \dfrac{G''}{\omega_y} P_x \dot{y} + G'P_x y = 0 \\ J_z \ddot{\varphi} + \left(\dfrac{G''}{\omega_y} P_x \dot{y} + G'P_x y \right) l_x = 0 \end{cases} \tag{49}$$

Or it can be also written in the canonical form as

$$\begin{cases} m\ddot{x} + b_x \dot{x} + c_x x = 0 \\ m\ddot{y} + b_y \dot{y} + c_y y = 0 \\ J_0 \ddot{\varphi} + l_x \cdot \left(b_y \dot{y} + c_y y \right) = 0 \end{cases} \tag{50}$$

Where, the formulas for the variable viscoelasticity parameters in the system of equation (50) can be written as

$$b_x = \frac{4E''R^{1/2}}{\omega_x} x^{1/2}, \quad c_x = \frac{4}{3} E'R^{1/2} x^{1/2}, \quad b_y = \frac{G''}{\omega_y} P_x, \quad c_y = G'P_x \tag{51}$$

4.1 Work and Energy

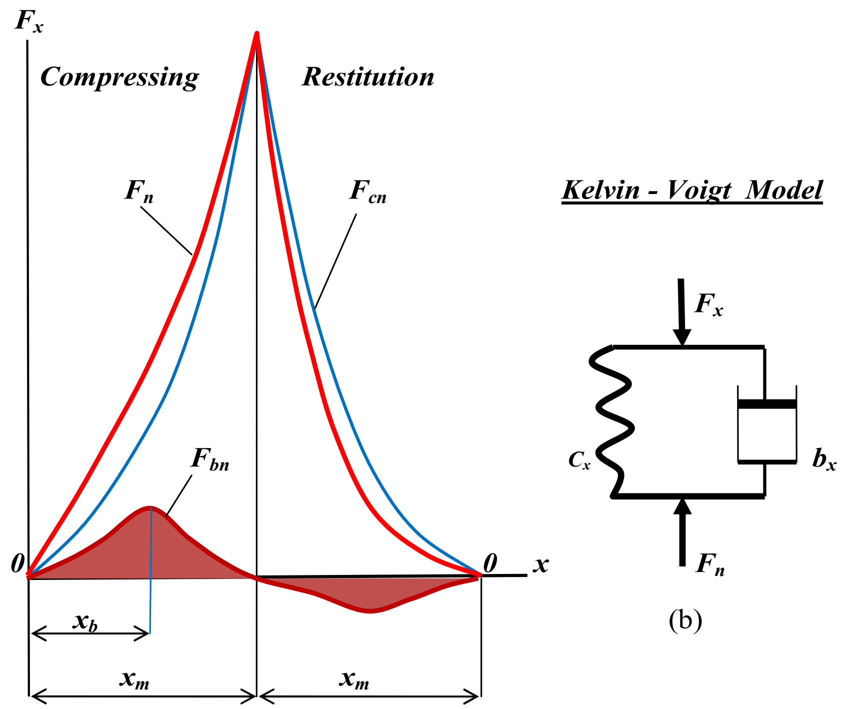
As we know the period of time at impact includes two principally different phases such as, the phase of the compression and the phase of the restitution. Also in the duration of a collision, the full initial kinetic energy of a body $W_0 = \frac{mV_0^2}{2}$ is divided into the two independent parts such as, the normal initial kinetic energy of a body $W_{0x} = \frac{mV_{0x}^2}{2}$ and the tangential initial kinetic energy of a body $W_{0y} = \frac{mV_{0y}^2}{2}$. On the other hand, the full kinetic energy of a body at the instant of rebound $W_t = \frac{mV_t^2}{2}$ (where V_t is the velocity of the centre of mass of a body in the instant of rebound) is included two independent parts such as, the normal kinetic energy of a body at the instant of rebound $W_{tx} = \frac{mV_{tx}^2}{2}$ (where V_{tx} is the normal velocity of the centre of mass of a body in the instant of the rebound) and the tangential kinetic energy of a body at the instant of rebound $W_{ty} = \frac{mV_{ty}^2}{2}$ (where V_{ty} is the tangential velocity of the centre of mass of a body in the instant of rebound). Therefore, the description of the processes of the compression and the restitution along the axis X, and the shear along the axis Y are given independently in this part of the paper. The basic problems here are the finding the equations for the work of viscoelastic forces, the dissipative energy in the phases of the compression, the restitution and at the shear. Also it is necessary to determine the coefficients of restitution, the maximum size of the compression between a body and a semi-space, the dynamic modules of elasticity and viscosity.

4.1.1 Work and Energy in the Phases of Compression and Restitution

The graphical illustration of the functional dependences between the normal viscoelastic forces and the displacement of the centre of mass of a body is depicted in Figure 5: (a). Also the “Rheological model of Kelvin-Vogt”, which usually is used for the viscoelastic contact, is represented in Figure 5: (b). As we can see, the viscosity force has the extremum in some point of the time equal τ_b.

It is obvious that the normal initial kinetic energy of the body W_{0x} is spent for the work A_{xm} of the normal viscoelastic force F_n in the compression phase. But on other hand, A_{xm} can be found as the sum of the works A_{xcm} and A_{xbm}, where A_{xcm} is the work of the normal elastic force F_{cn} and A_{xbm} is the work of the normal viscous force F_{bn} in the compression phase. Also we can say that the part of the kinetic energy W_{0x} is transformed into the potential energy of the nonlinear elastic element (spring) (Figure5: (b)) and the other part of this kinetic energy is dissipated during the time of deformation at the compression of the nonlinear viscous element (dashpot). However, on the other hand, the work A_{xt} of the normal viscoelastic force F_n in the restitution phase is equal to the normal energy of a body W_{tx} at the instant of rebound, and also A_{xt} can be found as the difference between A_{xct} and A_{xbt}, where A_{xct} is the work of the normal elastic force F_{cn} and A_{xbt} is the work of the normal viscous force F_{bn} in the restitution phase. Consequently, we can write that

$$\begin{cases} A_{xm} = A_{xcm} + A_{xbm} = W_{0x} = \dfrac{mV_{0x}^2}{2} \\ \\ A_{xt} = A_{xct} - A_{xbt} = W_{tx} = \dfrac{mV_{tx}^2}{2} \end{cases} \tag{52}$$



(a)

Figure 5: (a) - The graphical illustration of the functional dependences between the normal viscoelastic forces and the displacement x of the centre of mass of a body; (b) - The "Nonlinear Rheological Model of Kelvin-Vogt", where c_x and b_x are not the constant magnitudes

It is obvious that $A_{xcm} = A_{xct}$ and hence the potential energy which has been accumulated inside of the elastic element (spring) fully returns back to the body in the instant of rebound. The works A_{xcm} and A_{xbm} at the compression can be found by integration:

$$A_{xcm} = \int_0^{x_m} F_{cn}\,dx = \frac{4}{3}k_p E' R^{1/2} \int_0^{x_m} x^{3/2}\,dx = \frac{8}{15}k_p E' R^{1/2} x_m^{5/2} \tag{53}$$

and

$$A_{xbm} = \int_0^{x_m} F_{bn} dx = \int_0^{x_m} \frac{4k_p E'' R^{1/2}}{\omega_x} \dot{x} x^{1/2} dx = \left(\frac{4k_p E'' R^{1/2}}{\omega_x} \right) \frac{\int_0^{x_m} \int dx x^{1/2} dx}{\int_0^{\tau_1} dt} = \frac{8k_p E'' R^{1/2} x_m^{5/2}}{5\omega_x \tau_1} \quad (54)$$

Analogically the works A_{xct} and A_{xbt} in the restitution phase can be found as follows:

$$A_{xct} = -\int_{x_m}^{0} F_{cn} dx = -\int_{x_m}^{0} \frac{4}{3} k_p E' R^{1/2} x^{3/2} dx = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \quad (55)$$

and

$$A_{xbt} = -\int_{x_m}^{0} F_{bn} dx = -\int_{x_m}^{0} \frac{4k_p E'' R^{1/2}}{\omega_x} \dot{x} x^{1/2} dx = -\left(\frac{4k_p E'' R^{1/2}}{\omega_x} \right) \frac{\int_{x_m}^{0} x^{1/2} dx \int dx}{\int_{\tau_1}^{\tau_x} dt} = \frac{8k_p E'' R^{1/2} x_m^{5/2}}{5\omega_x \tau_2} \quad (56)$$

Where: $\tau_x = \tau_1 + \tau_2$ is the period time of the contact; τ_1 is the period time of the compression; τ_2 is the period time of the restitution; x_m is the maximum magnitude of the compression between a body and a semi-space (also it is the maximum displacement of the centre of mass of a body, which is equal to the maximum of mutual approach between a body and a semi-space).

According (52), (53), (54), (55) and (56) the equations for the work of the compression and the restitution can be written as follows:

$$\begin{cases} A_{xm} = A_{xcm} + A_{xbm} = \dfrac{8}{15} k_p R^{1/2} x_m^{5/2} (E' + \dfrac{3E''}{\omega_x \tau_1}) \\ \\ A_{xt} = A_{xct} - A_{xbt} = \dfrac{8}{15} k_p R^{1/2} x_m^{5/2} (E' - \dfrac{3E''}{\omega_x \tau_2}) \end{cases} \quad (57)$$

and according (48) and (57) we can write

$$\begin{cases} A_{xm} = \dfrac{8}{15} k_p E' R^{1/2} x_m^{5/2} \left(1 + \dfrac{3tg\beta}{\omega_x \tau_1} \right) \\ \\ A_{xt} = \dfrac{8}{15} k_p E' R^{1/2} x_m^{5/2} \left(1 - \dfrac{3tg\beta}{\omega_x \tau_2} \right) \end{cases} \quad (58)$$

Since as $A_{xm} = W_{0x} = \dfrac{mV_{0x}^2}{2}$, $A_{xt} = W_{tx} = \dfrac{mV_{tx}^2}{2}$, and by using the first of the equations (58), we get the formula for x_m respectively

$$x_m = \left[\frac{15 m \omega_x \tau_1 V_{0x}^2}{16 (3tg\beta + \omega_x \tau_1) k_p E' R^{1/2}} \right]^{2/5} \quad (59)$$

Also, we can define the energetic coefficient of restitution e_x, which equals to the square of the kinematic coefficient of restitution k_x (further it will be named simply the coefficient of restitution), like the ratio between W_{tx} and W_{0x}:

$$e_x = k_x^2 = \frac{V_{tx}^2}{V_{0x}^2} = \left(\frac{\omega_x \tau_2 - 3tg\beta}{\omega_x \tau_1 + 3tg\beta}\right)\frac{\tau_1}{\tau_2} \tag{60}$$

Since as we can take that

$$x_m \approx \frac{/V_{0x}/}{2}\tau_1 = \frac{/V_{tx}/}{2}\tau_2 , \tag{61}$$

we get that

$$k_x = \frac{\tau_1}{\tau_2} , \tag{62}$$

and using (60) and (62) we get that

$$tg\beta = \frac{\omega_x \tau_1}{3} \times \frac{1 - k_x}{k_x} \tag{63}$$

Thus, we have got the equation, which binds the coefficient of restitution and the tangent of the angle of mechanical losses. So, if $k_x = 1$, $tg\beta \rightarrow 0$ we get the totally elastic impact, but if $k_x = 0$, $tg\beta \rightarrow \infty$ then we get the totally viscous impact. Using (63) we can write the formula for the restitution coefficient as

$$k_x = \left[\frac{\omega_x \tau_1}{(3tg\beta + \omega_x \tau_1)}\right] \tag{64}$$

If to compare the Equations (59) and (64) we can finally get the expression for the maximum magnitude of the compression between a body and a semi-space respectively as

$$x_m = \left[\frac{15\, mV_{0x}^2}{16\, k_p E' R^{1/2}} k_x\right]^{2/5} \tag{65}$$

In the case of a totally elastic impact, when $k_x = 1$ and $k_p = 1$ we get the same result, as it has been obtained by L. Landau (1944, 1965) according to the Hertz theory for the absolutely elastic contact.

4.1.2 Work and Energy at the Rolling Shear

It is obvious that, in the during time of the displacement and the rolling shear along axis Y, the tangential initial kinetic energy of a body $\boldsymbol{W_{0y}}$ is spent for the work A_y of the tangential viscoelastic force F_τ . The work A_y can be found as the sum of the works A_{yb} and A_{yc}, where A_{yb} is the work of the tangential viscous force $F_{b\tau}$ and A_{yc} is the work of the tangential elastic force. But on other hand, it is obvious as well, that the work A_{yb} is transformed into the dissipative energy Q_ω and the work A_{yc} is transformed into the work A_ω of the rotation of the body around the centre of mass of a body. Thus, according to the "Law of preservation of energy for a non-conservative (dissipative) mechanical systems", we can write the equations for the displacement of the centre of mass of a body and for the rotation of a body relative to the centre of mass of a body, as follows below:

$$\begin{cases} \dfrac{m}{2}\left(\dfrac{dy}{dt}\right)^2 + A_y = \dfrac{mV_{oy}^2}{2} \\ \dfrac{J_z}{2}\left(\dfrac{d\varphi}{dt}\right)^2 + A_\omega + Q_\omega = \dfrac{J_z \omega_0^2}{2} \end{cases} \tag{66}$$

Where: $A_y = \int F_\tau dy$; $A_\omega = -\int M\, d\varphi$; $Q_\omega = \int F_{b\tau} dy$, and where $M = F_\tau l_x$.

Since $F_\tau = F_{c\tau} + F_{b\tau}$ and if $R >> x$, we can take that $M = F_\tau R = (F_{c\tau} + F_{b\tau})R$, and since as $d\varphi = dy / R$, hence

$$A_{\omega} = -\int M\,d\varphi = -\int (F_{c\tau} + F_{b\tau})dy \tag{67}$$

Also since, if the initial angular velocity ω_0 equals zero we can write the Equation (66) for the boundary conditions in the instant of the time $t = \tau_1$ of the maximum compression $x = x_m$ and $y = y_1$ as follows:

$$\begin{cases} \dfrac{mV_{my}^2}{2} + \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy = \dfrac{mV_{oy}^2}{2} \\ \dfrac{J_z\omega_m^2}{2} - \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy + \int_0^{y_1} F_{b\tau}dy = 0 \end{cases} \tag{68}$$

Where: V_{my} is the velocity at the instant of the time $t = \tau_1$; ω_m is the angular velocity relative to the centre of mass of a body at the instant of the time $t = \tau_1$; y_1 is displacement of the centre of mass of a body along axis Y at the instant of the time $t = \tau_1$. The Equation (68) can be rewritten as

$$\begin{cases} \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy = \dfrac{mV_{oy}^2}{2} - \dfrac{mV_{my}^2}{2} \\ \dfrac{J_z\omega_m^2}{2} = \displaystyle\int_0^{y_1} F_{c\tau}dy \end{cases} \tag{69}$$

Also at the point of the rebound, when $t = \tau_x$ we get

$$\begin{cases} \dfrac{mV_{ty}^2}{2} + \displaystyle\int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy = \dfrac{mV_{my}^2}{2} \\ \dfrac{J_z\omega_t^2}{2} - \displaystyle\int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy + \int_{y_1}^{y_t} F_{b\tau}dy = \dfrac{J_z\omega_m^2}{2} \end{cases} \tag{70}$$

Then we can write that

$$\begin{cases} \dfrac{mV_{ty}^2}{2} + \displaystyle\int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy = \dfrac{mV_{my}^2}{2} \\ \dfrac{J_z\omega_t^2}{2} - \dfrac{J_z\omega_m^2}{2} = \displaystyle\int_{y_1}^{y_t} F_{c\tau}dy \end{cases} \tag{71}$$

The summation of the systems (69) and (71) together yields the following result

$$\begin{cases} \dfrac{mV_{ty}^2}{2} = \dfrac{mV_{0y}^2}{2} - \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy - \int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy \\ \dfrac{J_z\omega_t^2}{2} = \displaystyle\int_0^{y_1} F_{c\tau}dy + \int_{y_1}^{y_t} F_{c\tau}dy \end{cases} \tag{72}$$

We can rewrite Equation (72) in the next order

$$\begin{cases} A_y = \dfrac{mV_{0y}^2}{2} - \dfrac{mV_{ty}^2}{2} = \int\limits_0^{y_1} (F_{c\tau} + F_{b\tau})dy + \int\limits_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy \\ \dfrac{J_z \omega_t^2}{2} = \int\limits_0^{y_1} F_{c\tau} dy + \int\limits_{y_1}^{y_t} F_{c\tau} dy \end{cases} \tag{73}$$

Finally we get

$$\begin{cases} A_y = \dfrac{mV_{0y}^2}{2} - \dfrac{mV_{ty}^2}{2} = A_{ybm} + A_{ycm} + A_{ybt} + A_{yct} \\ A_{yc} = \dfrac{J_z \omega_t^2}{2} = A_{ycm} + A_{yct} \end{cases} \tag{74}$$

Where: $A_{ybm} = \int\limits_0^{y_1} F_{b\tau} dy$ is the work of the tangential viscous force $\overline{F_{b\tau}}$ in the compression period τ_1 ;

$A_{ycm} = \int\limits_0^{y_1} F_{c\tau} dy$ is the work of the tangential elastic force $\overline{F_{c\tau}}$ in the compression period τ_1 ;

$A_{ybt} = -\int\limits_{y_1}^{y_t} F_{b\tau} dy$ is the work of the tangential viscous force $\overline{F_{b\tau}}$ in the restitution period τ_2 ;

$A_{yct} = -\int\limits_{y_1}^{y_t} F_{c\tau} dy$ is the work of the tangential elastic force $\overline{F_{c\tau}}$ in the restitution period τ_2 .

Since as from Equation 47 follows that $F_{b\tau} = \dfrac{G''}{\omega_y} P_x \dot{y}$ and $F_{c\tau} = G' P_x y$, all these works in (74) can be found by integration, as follows:

$$\begin{cases} A_{ybm} = \dfrac{G''}{\omega_y} \int\limits_0^{P_m} dP_x \int\limits_0^{y_1} \dot{y} dy = \dfrac{G''}{\omega_y} P_m \dfrac{\int_0^{y_1} \int dydy}{\int_0^{\tau_1} dt} = \dfrac{G''}{2\omega_y} P_m \dfrac{y_1^2}{\tau_1} \\ A_{ycm} = G' \int\limits_0^{P_m} dP_x \int\limits_0^{y_1} y dy = \dfrac{G'}{2} P_m y_1^2 \\ A_{ybt} = -\dfrac{G''}{\omega_y} \int\limits_{P_m}^{0} dP_x \int\limits_{y_1}^{y_t} \dot{y} dy = -\dfrac{G''}{\omega_y} (0 - P_m) \dfrac{\int_{y_1}^{y_t} \int dydy}{\int_{\tau_1}^{\tau_x} dt} = \dfrac{G''}{2\omega_y} P_m \dfrac{y_t^2 - y_1^2}{\tau_x - \tau_1} \\ A_{yct} = -G' \int\limits_{P_m}^{0} dP_x \int\limits_{y_1}^{y_t} y dy = \dfrac{G'}{2} P_m (y_t^2 - y_1^2) \end{cases} \tag{75}$$

Where

$$P_m = D_1 x_m + 2k_p R^{1/2} x_m^{1/2} \tag{76}$$

The full changing of the energy of the dissipative system at the rolling shear can be found as the difference between A_y and A_{yc} from (74):

$$\Delta W_y = A_y - A_{cy} = \frac{mV_{0y}^2}{2} - \frac{mV_{ty}^2}{2} - \frac{J_z \omega_t^2}{2} = A_{ybm} + A_{ybt} = A_{yb} \tag{77}$$

According to the Equation (74) the conclusion can be drawn that the work A_{yc} is transformed into the kinetic energy of the rotation of the body relative its centre of mass, but on the other hand the work A_{yb} is transformed into dissipative energy Q_ω in the process of the internal friction. Accordingly, using (74) and (75) we have

$$A_{yc} = \frac{J_z \omega_t^2}{2} = \frac{G'}{2} P_m y_t^2 \tag{78}$$

Hence the equation for the angular velocity at the instant time of rebound can be written as follows

$$\omega_t = \left(\frac{G' P_m}{J_z} \right)^{\frac{1}{2}} y_t \tag{79}$$

Since the work A_{yb} of the viscous tangential force $F_{b\tau}$ is equal to the dissipative energy Q_ω , using Equation (75) we get

$$A_{by} = Q_\omega = A_{ybm} + A_{ybt} = \frac{G''}{2\omega_y} P_m \left(\frac{y_1^2}{\tau_1} - \frac{y_1^2}{\tau_2} + \frac{y_t^2}{\tau_2} \right) \tag{80}$$

Since as $\tau_1 = k_x \tau_2$, finally we get

$$A_{by} = Q_\omega = A_{ybm} + A_{ybt} = \frac{G''}{2\omega_y \tau_2 k_x} P_m \left[y_1^2 (1 - k_x) + k_x y_t^2 \right] \tag{81}$$

5. Approximate solution to the Differential Equations of the Displacement by using the Method of the Equivalent Works

For practical application of the differential Equation (50) with the variable viscoelasticity parameters, we can find their approximate solutions in the same manner as for the equations with the equivalent constant viscoelasticity parameters, if we choose the equivalent constant parameters B_x, C_x and B_y, C_y so that the work A_{xcm} and A_{xbm}, A_{ycm} and A_{ybm} with the variable viscoelasticity parameters c_x, b_x and c_y, b_y will be equal to the works with the constant viscoelasticity parameters. Thus, according to this statement, and since as the work A_{xcm} and A_{xbm} , A_{ycm} and A_{ybm} are known from Equations (53), (54) and (75), we can write next equations in the phase of the compression

$$\begin{cases} A_{xcm} = C_x \int_0^{x_m} x dx = \frac{1}{2} C_x x_m^2 = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \\ \\ A_{xbm} = B_x \int_0^{x_m} \dot{x} dx = B_x \frac{\int_0^{x_m} x dx}{\int_0^{\tau_1} dt} = B_x \frac{x_m^2}{2\tau_1} = \frac{8 k_p E'' R^{1/2}}{5 \Omega_x \tau_1} x_m^{5/2} \end{cases} \tag{82}$$

and also in the phase of the rolling shear for the period of the compression time

$$\begin{cases} A_{ycm} = C_y \int\limits_0^{y_1} y dy = \frac{1}{2} C_y y_1^2 = \frac{1}{2} G' P_m y_1^2 \\ \\ A_{ybm} = B_y \int\limits_0^{y_1} \dot{y} dy = B_y \frac{\int_0^{y_1} \int dy dy}{\int_0^{\tau_1} dt} = B_y \frac{y_1^2}{2\tau_1} = \frac{G''}{2\omega_y \tau_1} P_m y_1^2 \end{cases} \tag{83}$$

Hence, according to the results obtained in (82) and in (83), we can write the expressions for the equivalent constant viscoelasticity parameters, respectively as:

$$B_x = \frac{16 E'' k_p R^{1/2}}{5\omega_x} x_m^{1/2}, \quad C_x = \frac{16}{15} k_p E' R^{1/2} x_m^{1/2}, \quad B_y = \frac{G''}{\omega_y} P_m, \quad C_y = G' P_m \tag{84}$$

Thus, the Equation (50) with variable parameters can be rewritten as the equations with constant parameters as follows:

$$\begin{cases} m\ddot{x} + B_x \dot{x} + C_x x = 0 \\ m\ddot{y} + B_y \dot{y} + C_y y = 0 \\ J_z \ddot{\varphi} + l_x \cdot \left(B_y \dot{y} + C_y y \right) = 0 \end{cases} \tag{85}$$

The Equation (85) are the equations of the damped oscillations and the solutions to these equations are known:

$$\begin{cases} x = \frac{\upsilon_{0x}}{\omega_x} e^{-\delta_x t} \sin(\omega_x t) \\ \\ y = \frac{\upsilon_{0y}}{\omega_y} e^{-\delta_y t} \sin(\omega_y t) \end{cases} \tag{86}$$

Where: $\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$; $\delta_x = \frac{B_x}{2m}$ is the normal damping factor; $\omega_{0x} = \sqrt{\frac{C_x}{m}}$ is the angular frequency of the harmonic oscillations by axis X ; $\omega_y = \sqrt{\omega_{0y}^2 - \delta_y^2}$; $\delta_y = \frac{B_y}{2m}$ is the tangential damping factor; $\omega_{0y} = \sqrt{\frac{C_y}{m}}$ is the angular frequency of the harmonic oscillations by axis Y.

It is obviously that the period of time of the contact τ_x is equal to the semi-period of damped oscillations $T_x/2$ by axis X.

$$\tau_x = \frac{T_x}{2} = \frac{\pi}{\omega_x} \tag{87}$$

Since as $\tau_x = \tau_1 + \tau_2$ and also by using Equations (60), (62),(63) and (87) we get:

$$tg\beta = \frac{\pi}{3} \times \frac{(1 - k_x)}{(1 + k_x)} \tag{88}$$

The equation for the restitution coefficient we can write now as follows:

$$k_x = \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)} \tag{89}$$

If $tg\beta = 0$ hence $k_x = 1$, it is a totally elastic impact, but if $tg\beta = \pi/2$ hence $k_x = 0$ and $x = 0$, it is absolutely plastic impact. Both of these two cases are not possible in nature.

Finally, from (65) and (89) follows that

$$x_m = \left[\frac{15\,mV_{0x}^2}{16\,k_p E' R^{1/2}} \times \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)} \right]^{2/5} \tag{90}$$

Thus we have a very simple way to calculate x_m, if we know the value of $tg\beta$. According to the Equation (9), (46) and (48) $tg\beta$ can be calculated by formula

$$tg\beta = \frac{E''}{E'} = \frac{E_1'' E_2'' (E_1' + E_2')}{E_1' E_2' (E_1'' + E_2'')} \tag{91}$$

The equations for the velocities of the centre of mass of a body can be received by differentiation of (86):

$$\begin{cases} \dot{x} = \dfrac{V_{0x}}{\omega_x} e^{-\delta_x t} \left[\omega_x \cos(\omega_x t) - \delta_x \sin(\omega_x t)\right] \\ \\ \dot{y} = \dfrac{V_{0y}}{\omega_y} e^{-\delta_y t} \left[\omega_y \cos(\omega_y t) - \delta_y \sin(\omega_y t)\right] \end{cases} \tag{92}$$

Using (92) for the velocity, the duration of the time of the impact equals to the period of the time of the contact can be found now from the conditions $\dot{x} = V_{tx}$ and $t = \tau_x$ as

$$\tau_x = -\frac{\ln k_x}{\delta_x} \quad , \tag{93}$$

where

$$\delta_x = \frac{B_x}{2m} = \frac{8k_p E'' R^{1/2}}{5m\,\omega_x} x_m^{1/2} = \frac{8k_p E' tg\beta}{5\pi m} \tau_x R^{1/2} x_m^{1/2} \,, \tag{94}$$

and since $tg\beta$ is known from (88), by using (65),(93) and (94) we get

$$\tau_x^2 = -\frac{2(1+k_x)\ln k_x}{V_{0x}^{2/5}(1-k_x)k_x^{1/5}} \times \left(\frac{5m}{8k_p E' R^{1/2}} \right)^{4/5} \tag{95}$$

6. Determination of the Dynamics Modules by the Method of the "Temperature -Time Superposition"

The dynamic elasticity and viscosity modules for high velocities of the collision can be found, if to follow the principles of the "Time-temperature superposition" according to the equation of the "WLF" Williams - Landel - Ferry or Arrhenius (Ferry, 1963; Van Krevelen, 1972; Moore, 1975; Nilsen and Landel, 1994). First of all we have to define experimentally the effect of temperature for the period of the contact time τ_x, and for the coefficient of restitution k_x at the fixed initial velocity of impact. For example, if we define these parameters for velocity at 2 m/s, then using the principles of the "Time-temperature superposition" we can determine their values for any velocities interesting for us, for example for velocity 100 m/c and for temperature 100 0C. After this, when τ_x and k_x will be known, we can find the value of $tg\beta$ and the dynamic modules E'' and E'. If to use the equation (95), the expression for the calculation of the effective dynamic elasticity module can be written as follows

$$E' = \frac{5m}{8k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times \left(\frac{(-2\ln k_x)(1+k_x)}{k_x^{1/5}(1-k_x)} \right)^{5/4} \tag{96}$$

And, if to use (48), (88) and (96) we get the formula for the calculation of the effective dynamic viscosity module

$$E'' = \frac{15\pi m}{24k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times (-2\ln k_x)^{5/4} \left(\frac{(1+k_x)}{k_x(1-k_x)} \right)^{1/4} \tag{97}$$

Obviously, if $k_x = 0$, then $E'' = 0$ too. We can find G' and G'' in the analogical way.

8. Analyse and Conclusion

It is a very important now to confirm of the correctness of the offered theories and methods, obtained in this article, if to compare them with others already available. For example, the equation for the elastic force $F_{cn} = \frac{4}{3} k_p E' R^{1/2} x^{3/2}$ have been obtained by the using of the "Method of the specific forces". In the case, when $k_p = 1$ we have the same solution like in the Hertz theory. Hence the Hertz theory gives the partial results in comparisons with proposed "MSF" method, because MSF" let to find the viscoelastic forces for any curvilinear contact between two surfaces, but Hertz case can be using only for the flat contact. The obtained result proves us that, the "Method of the specific forces" is definitely valid for finding of the normal elastic force, because if we know a functional dependency between r and x, we can always find the elastic force. It is obviously that, if it gives the correct way for the definition elastic force, and also as it was represented, it is valid for the definition of the viscous force and the tangential viscoelastic forces. We cannot find viscous force and the tangential viscoelastic forces by using the Hertz's theoretical model, but we can do this by using the "Method of the Specific Forces". It is obviously that, for the finding the normal viscous and the tangential viscoelastic forces, we can take $k_p = 1$, like according the Hertz theory, but we should be aware that, in this case, the contact area is a flat surface according to 2D tensor of deformations and h_x – the depth of indentation has to be equal to zero. But in reality, the contact surface takes the curvilinear shape, therefore, alternatively in this paper, the way of the finding of the radius of the contact area, by considering the geometry of contact between two curvilinear surfaces, have been proposed. It was received that the radius of contact area $r = f(x)$can be find by the equation $r^2 = 2Rx - x^2$. Since this equation is not convenient in using, and therefore it was proposed the finding ***r*** as $r^2 = k_p^2 Rx$, where $k_p = \sqrt{2 - \frac{x}{R}}$ is the correlation coefficient, which can be found by the method of iterations and consecutive approximations. If a deformation is small, when $R >> x$, hence we can take $k_p = \sqrt{2}$. And, if contact area is a flat, when $h_x = 0$, follows from Equation 47* that $D_1 = 0$. Practically the area of contact can be considered as a flat surface only in the case, when the surface of a semi-space in many times harder than the surface of a body. For example, it is possible in the case of impact between a rubber ball and a steel plane. Hence, it is obviously, that the "MSF" is the universal method, which can be used for any functional dependencies between the radius or the diameter of the contact area and the distance of the mutual approach (the total deformation) between two curvilinear surfaces. But nevertheless, we still have the question: What kind of the equation is better to take for finding of the radius of contact area, by the Hertz theory or directly by the way of consideration the geometry of the contact, like it is proposed in this article? Objectively to answer this question, we have to analyse simply logically the way as these equations were received. It was taken according to the Hertz theory that the contact surface is a flat, and the deformations are very small, the contact pressure is distributed analogically as an electrical potential (Remark: an electrical potential is the scalar function, but a pressure is the vector function), and then, on the basis of this main statements, the equation between the radius of the contact area and the normal elastic force, and the equation between the distance of the mutual approach and the normal elastic force as the effect have been obtained. Then only after that, in result of the comparison of these two equations by excluding the normal force (Landau, 1944, 1965), the expression $r^2 = Rx$ have been received. But in this article the analogical functional dependence have been proposed as the cause, in the result of the direct consideration the geometry of the contact. It was shown that, in the time of indentation of more hard surface into a soft surface, the contact surface takes a curvilinear elliptical shape (the function $r^2 = 2Rx - x^2$ is the elliptical function, which can be approximated by the parabolic function $r^2 = k_p^2 Rx$), where the point *B* (see Figure1) is a special point where the deformations always equal zero, and the border of the area of contact always pass through this point *B*. Therefore, the proposed geometrical method is more exact, than according to the Hertz's theoretical model.

On the other hand, the equation for the normal viscous force $F_{bn} = 4k_p \frac{E''}{\omega_x} R^{1/2} \dot{x} x^{1/2}$ gives the similar result as it has place in the contact between two bodies with identical mechanical properties in the equation (4*), which have been obtained by Brilliantov, N. V., Spahn, F., Hertzsch, J.-M., and Poeschel, T. (1996). After the comparison of two these equations, since as $x = \xi$ and $R = R^{eff}$ we obtain the following

$$A = \frac{E'' k_p (1 - \nu^2)}{Y \omega_x} \tag{98}$$

But since as $\omega_x = \frac{\pi}{\tau_x}$ and in the quasi-static conditions $E' = Y$ we get

$$A = \frac{k_p tg\beta}{\pi} \times \left(1 - \nu^2\right) \times \tau_x \tag{99}$$

If $\beta = 0$ hence $A = 0$ too, it is a totally elastic impact. Thus we can find the parameter ***A*** by a very simple way using the “Method of the Specific Forces”.

Also the equation (65) to determine the maximum displacement x_m have been derived. It is obvious, that in the case of k_x = 1 and k_p =1 we have the same result, as was obtained by Landau (1944,1965) for a totally elastic impact by using the Hertz Theory. It proves the correctness of the way of finding the Equation 65. But we have to understand that, this equation has the borders of application which can be found if to solve the next equation

$\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$. First of all since as $\omega_{0x}^2 = \frac{C_x}{m}$ and $\delta_x = \frac{B_x}{2m}$, we can write that $\delta_x = \frac{3\omega_{0x}^2}{2\omega_x} tg\beta$ and we get the next algebraic equation $\omega_x^4 - \omega_{0x}^2 \omega_x^2 + \omega_{0x}^4 tg^2\beta = 0$. This equation has only the one valid solution $\omega_x^2 = \frac{\omega_{0x}^2}{2}\left(1 + \sqrt{1 - 9tg^2\beta}\right)$ and it has the valid root only when $1 - 9tg^2\beta \geq 0$, therefore $tg\beta = \frac{E''}{E'} \leq \frac{1}{3}$, and according to Equation 88 we get for a viscoelastic contact that

$$k_x \geq \frac{\pi - 1}{\pi + 1} \tag{100}$$

In the case when $k_x \lhd \frac{\pi - 1}{\pi + 1}$ the plastic deformations will be have place in the zone of the contact.

Also it is necessary to proof the correctness of the definition of the work for the normal viscous force in Equation (54) and in Equation (82), because two ways in the order of integrations are possible to apply here such as, that have been taken for finding the solution in Equation (54) and in Equation (82) and like it is shown below

$$A_{xbm} = \left(\frac{4k_p E'' R^{1/2}}{\omega_1}\right) \frac{\int_0^{x_m} \int x^{1/2} dx dx}{\int_0^{\tau_1} dt} = \frac{16 k_p E'' R^{1/2} x_m^{5/2}}{15 \omega_1 \tau_1} \tag{101}$$

It is simple to proof that the 1-st variant of the order of integration in Equation (54) and in Equation (82) is correct and the second variant in Equation (101) is not correct. It obviously that the attitude between the normal viscous force and the normal elastic force can be find as follows

$$\frac{b_x \dot{x}}{c_x x} = \frac{B_x \dot{x}}{C_x x} = \frac{3tg\beta}{\omega_1} \frac{\dot{x}}{x} \tag{102}$$

Or hence we can write respectively

$$\frac{b_x}{c_x} = \frac{B_x}{C_x} = \frac{3tg\beta}{\omega_1} \tag{103}$$

As we can see this attitude is a constant value and it should not change at dependence by the order of integration. If we take the second variant of integration according to Equation (101), the values of the constants of viscoelasticity C_x and B_x will be changed. Thus therefore, in the first variant of integration as it is taken in Equation (54) and in Equation (82) we have a valid solution.

In conclusion, first of all, let us to mark, that the method of specific viscoelastic forces allows to find the equations for all viscoelastic forces. The proposed method is a principally different with others in which are using the Hertz's theory, the classical theory of elasticity and the tensor algebra. In this method the new conception is proposed, how to find the elastic and viscous forces by an integration of the specific forces in the infinitesimal boundaries of the contact area. The radius of contact area can be taken according the Hertz theory or can be found by the considering the geometry of the contact. This method can be used in researches of the contact dynamics of any shape of contacting surfaces. Also in the article the method of the solution of the differential equations of a movement has been proposed and they have been solved. This method also can be used for determination of the dynamic mechanical properties of materials, and it can be used in the design of wear-resistant elements and coverings for components of machines and equipment, which are working in harsh conditions where they are subjected to the action of flow or jet abrasive particles. Also the theoretical and experimental statements which are presented here can be useful in the design of elements and details machines and mechanism which are being in the conditions of the dynamic contact. The results of the experimental and theoretical research and the method of the specific forces presented in this article can be used for the determination of the viscoelastic forces, contact stresses, durability and fatigue life for a wide spectrum of the tasks relevant to collisions between solid bodies under different loading conditions. Opportunities exist to use the obtained results practically in the design and development of new advanced materials, wear-resistant elastic coatings and elements for pneumatic and hydraulic systems, stop valves, fans, centrifugal pumps, injectors, valves, gate valves and in other installations. Also the using of this theory gives an opportunity for the development of analytical and experimental methods allowing optimising the basic dynamic and mechanical visco-elastic qualities already existing materials and in the development new advanced materials and elements of machines. Also this theory can be used not only for visco-elastic contact and also for any other kind of contacts, such as the elasto- plastic contact and for the elasto-visco-plastic contact too.

References

Antypov, D., Elliott, J. A., & Bruno, C., & Hancock, B. C. (2011). Effect of particle size on energy dissipation in viscoelastic granular collisions. *Phys. Rev. E, 84, 021303.* Retrieved from http://PhysRevE.84.021303

Archard, J. F. (1957). Elastic deformation and the laws of friction. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 243*(1233), 190--205.

Barber, J. R., & Ciavarella, M. (2000). Contact mechanics. *International Journal of Solids and Structures, 37*. http://dx.doi.org/10.1016/S0020-7683(99)00075-X

Becker, V., Schwager, T., & Pöschel, T. (2008). Coefficient of tangential restitution for the linear dashpot model. *Phys. Rev. E, 77, 011304.* http://dx.doi.org/10.1103/PhysRevE.77.011304

Bowden, F. P., & Tabor, D. (1939). The area of contact between stationary and between moving surfaces. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 169*(938), 391-413

Brilliantov, N. V., Spahn, F., Hertzsch, J.-M., & Pöschel, T. (1996). Model for collisions in granular gases. *Phys. Rev. E,* 53, 5382. Retrieved from arxiv.org

Bordbar, M. H., & Hyppänen, T. (2007). Modeling of Binary Collision between Multisize Viscoelastic Spheres. *Journal of Numerical Analysis in Sciences and Engineering (ESCMSE), 2*(3-4), 115-128.

Boussinesq J. (1885). Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques. Paris, 1885, Gauthier-Villars.

Bush, A. W., Gibson, R. D., & Thomas, T. R. (1975). The elastic contact of a rough surface. *Wear, 35*(1), 87-111.

Carbone, G., Lorenz, B., Persson, B. N. J., & Wohlers, A. (2009). Contact mechanics and rubber friction for randomly rough surfaces with anisotropic statistical properties. *The European Physiscal Journal E – Soft Matter, 29*(3), 275– 284. http://dx.doi.org/10.1140/epje/i2009-10484-8

Carbone, G., & Putignano, C. (2013). A novel methodology to predict sliding/rolling friction in viscoelastic materials: theory and experiments. *Journal of Mechanics and Physics of Solids.* http://dx.doi.org/10.1016/j.jmps.2013.03.005

Cheng, N., Subic, A., & Takla, M. (2008). Development of a fast-solving numerical model for the structural analysis of cricket balls. *Sports Technol., 1*(2–3), 132–144. Retrieved from http:// publication/227591545

Cummins, S. J., Thornton, C., & Cleary, P. W. (2012). CONTACT FORCE MODELS IN INELASTIC COLLISIONS, Ninth International Conference on CFD in the Minerals and Process Industries CSIRO, Melbourne, Australia. Retrieved from http://www.cfd.com.au/cfd_conf12/PDFs/175CUM.pdf

Cundall, P. A., & Strack, O. D. L. (1979). A discrete numerical model for granular assemblies. *Geotechnique, 29*, 47-65.

Derjaguin, B. V., Muller, V. M., & Toporov, Y. P. (1975). Effect of contact deformations on the adhesion of particles. *Journal of Colloid and Interface Science, 53*(2), 314-326.

Dintwa, E. (2006). Development of accurate contact force models for use with Discrete Element Method (DEM) modelling of bulk fruit handling processes. *DISSERTATIONES DE AGRICULTURA, Doctoraatsproefschrift nr. 726 aan de faculteit Bio-ingenieurswetenschappen van de K.U.Leuven, Katholieke Universiteit Leuven, Faculteit Bio-ingenieurswetenschappen*.

Ferry, J. D. (1948). Viscoelastic Properties of Polymer Solutions. *Journal of Research of the National Bureau of Standards, Research Paper, RP1903Volume, 41, US.*

Ferry, J. D. (1963). Viscoelasticpropertiesof polymers. New York: John Wiley & Sons, Inc.

Galin, L. A. (1961). Contact problems in the theory of elasticity. *Dept. of Mathematics, School of Physical Sciences and Applied Mathematics, North Carolina State College.*

Greenwood, J. A., & Williamson, J. B. P. (1966). Contact of nominally flat surfaces. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 295*, 300-319.

Goldsmith, W. (1960). *Impact. The Theory and Physical Behaviour of Colliding Solids*. Adward Arnold Publisher Ltd, 371 pages.

Goloshchapov, N. N. (2001). Determination methods of cycle-thermo-kinetics viscoelastic and relaxation characteristic of elastomers under impact. *Cycles, Materials of the 3-d International conference, North-Caucasian State Technical University, Stavropol, Russia.*

Goloshchapov, N. N. (2003). Tribo-cyclicity. Cycles. Materials of 5th International conference. North-Caucasian State Technical University, Russia.

Goloshchapov, N. N. (2003). Processes of abrasive wear elastomers and their connection with viscoelasticity qualities. *2-d International scientific conference "Tribology in Environmental Design", Collected articles*. London: Professional Engineering Publishing l.

Goloshchapov, N. (2015). Application of the method of the specific forces in the mechanics of a viscoelastic collision between a solid body and a semi-space. *International Journal of Damage Mechanics, 24*(6), 915-943. Retrieved from http://ijd.sagepub.com/content/24/6/915

Graham, G. A. C. (1965). The contact problem in the linear theory of viscoelasticity. *International Journal of Engineering Science, 3*, 27. http://www.sciencedirect.com/science/article/pii/0020722565900182

Harrass, M., Friedrich, K., & Almajid, A. A. (2010). Tribological behavior of selected engineering polymers under rolling contact, *Tribology International, 43*, 635-646.

Hertz H. (1882). Über die Berührung fester elastischer Körper (On the contact of elastic solids*). J. Reine und angewandte Mathematik,* 92, 156-171.

Hertz, H. (1896). Über die berührung fester elastischer Körper (On the contact of rigid elastic solids). *In: Miscellaneous Papers.* J. reine und angewandte Mathematik 92, Macmillan, London, 156 pages. English translation.

Hosford, W. F. (2005). Mechanical Behaviour of Materials. New York: Cambridge University Press. Retrieved from http://assets.cambridge.org/97805218/46707/frontmatter/9780521846707_frontmatter.pdf

Hunter, S. C. (1960). The Hertz problem for a rigid spherical indenter and a viscoelastic half space. *Journal of the Mechanics and Physics of Solids, 8*, 219-234.

Hyun, S., & Robbins, M. O. (2007). Elastic contact between rough surfaces. Effect of roughness at large and small wavelengths. *Trobology International, 40*, 1413-1422. K. L.

Johnson, K. L, Kendall, K., & Roberts, A. D. (1971). Surface energy and the contact of elastic solids. *Proc. R. Soc. London A, 324*, 301-313. Retrieved from http://rspa.royalsocietypublishing.org/content/324/1558/30

Johnson, K. L. (1985). Contact mechanics. *Cambridge University Press.*

Jonas, A. Z. (1982). Impact dynamics. *Wiley, - Technology & Engineering - 452 pages*.

Lakes, R. S. (1998). Viscoelastic Solids. *II Series: Mechanical Engineering*. CRC Press.

Landau, L. D., & Lifshits, E. M. (1944). *Theory of Elastisity. Theoretical Physics* (vol.4). Moscow.

Landau, L. D., & Lifshitz, E. M. (1965). *Theory of Elasticity*. Oxford: Oxford University Press,.

Laursen, T. A. (2002). Computational Contact and Impact Mechanics. *Fundamentals of Modeling Interfacial Phenomena in Nonlinear Finite Element Analysis.* New York: Springer Verlag.

Lee, E. H. (1962). Viscoelasticity. In W. Flugge (Ed.), Handbook of Engineering Mechanics. New York: McGraw-Hill.

Li. H. (2006). Impact of cohesion forces on practical mixing and segregation. Doctoral dissertation. University of Pittsburgh. Retrieved from http://d-scholarship.pitt.edu/9699/

Makse, H. A., Gland, N., Johnson, D. L., & Schwartz, L. (2004). Granular Packings: Nonlinear elasticity, sound propagation and collective relaxation Dynamics. *Dated: Phys. Rev. E, 70*, 061302.

Menard, K. P. (1999). Dynamic Mechanical Analysis. *A Practical Introduction.* CRC Press LLC.

Meyers, M. A. (1994). Dynamic Behavior of Materials. *J. Wiley.*

Mindlin, R. D. (1949). Compliance of elastic bodies in contact. *Trans. ASME J. Appl. Mech.,* 16, 259-268

Moore, D. F. (1975). *Principles and Application of Tribology.* New York: Pergamon Press, Technology & Engineering.

Moore, D. F. (1978). Principles and Application of Tribology. *Mir, Moscow, Russia.*

Nilsen, L. (1978). Mechanical properties of polymers and polymeric compositions. *Moscow.*

Nilsen, L. E., & Landel, R. F. (1994). Mechanical properties of polymers and polymeric compositions (2^{nd} ed.). Rev., Series: Mechanical Engineering, Marcel Dekker. Ink., New York.

Padovan, J., & Paramadilok, O. (1984). Transient and steady state viscoelastic rolling contact. *Comput Struct, 20*, 545-553.

Persson, B. N. J., Bucher, F., & Chiaia, B. (2002). Elastic contact between randomly rough surfaces. *Comparison of theory with numerical results, Physical Review B, 65*(18), 184106.

Persson, B. N. J. (2010). Rolling friction for hard cylinder and sphere on viscoelastic solid. *Eur. Phys. J. E, 33*, 327-333. Retrieved from http://link.springer.com/article/10.1140%2Fepje%2Fi2010-10678-y

Radok, J. R. M. (1957). Viscoelastic stress analysis. *Q. App. Math, 15*.

Popov. V. L. (2010). *Contact mechanics and friction: Physical principals and application.* Springer/

Popov. V. L., & Hess, M. (2015). *Method of dimensionality reduction in contact mechanics and friction*. Springer. Retrieved from http://www.twirpx.com/file/1756685/

Ramírez, R., Poeschel T., Brilliantov, N. V., & Schwager, T. (1999). Coefficient of restitution of colliding viscoelastic spheres. *Phys. Rev. E, 60*. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/11970301

Roylance, D. (2001). Engineering viscoelasticity. *Department of Materials Science and Engineering, Massachusetts Institute of Technology Cambridge, MA 02139.* Retrieved from http://ocw.mit.edu/courses/materials-science-and-engineering/3-11-mechanics-of-materials-fall-1999/modules/visco.pdf

Schafer, J., Dippel, S., & Wolf, D. E. (1996). Force schemes in simulations of granular materials. *Journal de Physique I, 6*, 5-20. Retrieved from https://hal.archives-ouvertes.fr/jpa-00247176/document

Schwager, T., & Poschel, T. (2007). Coefficient of restitution and linear-dashpot model revisited. *Granular Matter,* 9, 465-469. Retrieved from http://link.springer.com/article/10.1007%2Fs10035-007-0065-z

Schwager, T., & Poschel, T. (2008). Coefficient of restitution for viscoelastic spheres: The effect of delayed recovery. *Phys. Rev. E, 78*, 051304. Retrieved from http://journals.aps.org/pre/abstract/10.1103/PhysRevE.78.051304

Simon, R. (1967). *Development of a Mathematical Tool for Evaluating Golf Club Performance*. ASME Design Engineering Congress, New York.

Sneddon, I. N. (1965). The Relation between Load and Penetration in the Axisymmetric Boussinesq Problem for a Punch of Arbitrary Profile. *Int. J. Eng. Sci., 3*, 47–57.

Stronge, W. J. (2000). *Impact Mechanics*. Cambridge: Cambridge University Press.

Tabor, D. (1977). Surface forces and surface interactions. *Journal of Colloid and Interface Science, 58*(1), 2-13.

Tabor, D. (1977). The hardness of solids. *Journal Colloid Interface Sci., 58*, 145-179.

Thornton, C. (2009). Special Issue on the 4th. Int. Conf. on Discrete Element Methods. *Powder Technol., 193*, 216-336.

Thornton, C., Cummins, S. J., & Cleary, P. W. (2012). An investigation of the comparative behaviour of alternative contact force models during inelastic collisions. *Powder Technol., 233*, 30-46.

Timoshenko, S., & Goodier, J. N. (1951). Theory of Elasticity. N.Y.: MCGRAW – Hill.

Ting T. (1966). The contact stresses between a rigid indenter and a viscoelastic half space. *Journal of Applied Mechanics, 33*, 845.

Van Krevelen, D. W. (1972). *Properties of polymers correlations with chemical structure.* Amsterdam-London-New York: Esliier Publishing.

Van Zeebbroeck, M. (2005). The Discrete Element Method (DEM) to Simulate Fruit Impact Damage during Transport and Handling. DISSERTATIONES DE AGRICULTURA. *Proefschrift voorgedragen tot het behalen van de graad van Doctor in de Toegepaste Biologische Wetenschappen door.* Retrieved from https://lirias.kuleuven.be/bitstream/1979/46/2/doctoraatmichaelvanzeebroeck.pdf

Webster M. N., & Sayles, R. S. (1986). A numerical model for the elastic frictionless contact of real rough surfaces. *ASME, 108*, 314-320.