A Bioeconomic Model of Two Equally Dominated Prey and one Predator System

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Abstract

The model is based on Lotka-Volterra dynamics with two competing fish species which are affected not only by harvesting but also by the presence of a predator, the third species. The prey populations are taken as equally dominating populations so that the coefficients of their interspecific competition are taken equal. The conditions of local and global stability of the model and the possibility of bioeconomic equilibrium are derived. Some numerical simulations are also done at the end of the paper. The asymptotic stability and global stability corresponding to the numerical examples are graphically shown.

Keywords: Harvesting, Global stability, Bioeconomic equilibrium

1. Introduction

The fishery management system consists of the interaction between the fish species and humans, and not just the fish population dynamics. The bioeconomic approach in fishery models combines fish population dynamics and the economic components of the fishery system. The bioeconomic models are developed to express and illustrate these through harvest activities. The response of fish stock to human activities (i.e. fishing effort, gear selection) and the economic consequences of specific harvest strategies can be examined by including the management objectives in the models on which management decisions are based. Hanneson (1993) applied the fishery bioeconomic model approach to identify economically optimal harvest strategies for the fishery for Arcto-Norwegian Cod, a species that exhibits considerable fluctuation in stock size. In recent past numerous works on bioeconomic modeling and optimal harvesting of biological resources have been published in leading journals. Some fabulous works are done by Clark [1990] on such articles. Later on some works on multispecies are done by Chaudhuri (1998), Dai and Tang (1998), Kar and Chaudhuri (2004), Kar (2006). Kar and Chattopadhyay (2010) discussed a dynamic reaction model on a multispecies model with stage-structure; Kar et al (2009) discussed some special features on multispecies bioeconomic model. Steinshamn (1998) has studied the application of bioeconomic approach in fluctuating fish stock. Bioeconomic models mainly aim towards the biologically persistence and economic consequences, which have critical importance on ecological system and our society. It is quite impossible to consider all natural, social and ecological constraints to formulate a general prey-predator model of multispecies and then to analyze the model. The prey and predator are all independent and the survival of any one is critically dependent on others. The ecological balance is the basic factor of non-extinction of any species from the nature. All commercially valuable fish species fluctuate in abundance, which leads to unavoidable changes over time in catches. Variability in catches is one of the most serious problems in fishery management. Because the market demand is price sensitive, fluctuating harvests will cause fluctuations. This may stabilize or destabilize fishing revenue, depending upon whether price elasticity of demand exceeds in one absolute value. On the other hand, from biological point of view, stability in fish abundance is desirable as it reduces the danger of extinction.

In this paper two prey species are considered whose dominance on each other is equal. The presence of a predator is also considered. The prey species are harvested only to meet a fixed demand in such a way that the some fraction of total biomass needed is received from one species and other part of total biomass needed is received from other prey species. The harvesting of predator species is not considered. We will consider the bioeconomic analysis of the model together with some other features. For references, bioeconomic exploitation of both the species in a Lotka-Volterra prey predator system was discussed by Chaudhuri and Saha Ray (1991), Krishna et al (1998) discussed the conservation of an exploited ecosystem with optimal taxation on harvesting, and Pradhan and Chaudhuri (1999) developed a two species model with taxation as a control instrument. The organization of the paper is as follows.

Formulation of the problem is given in section-2, existence of different steady state solutions is discussed in section-3, local stability analysis is analyzed in section-4 and the global stability is discussed in section-5. The bioeconomic equilibrium is found in section-6, some numerical examples are given in section-8. The paper is concluded with a brief conclusion in section-9.

2. Formulation of the model

The prey-predator fishery model is as follows. There are two fish populations which compete with each other for the use of a common resources and their dominance on each other is equal i.e. the interspecific competition between them are of equal strength. Both of these prey species are subjected to a constant harvesting in such a way that the total harvested mass is constant to meet a constant demand. There is predator (as for example a whale) feeding on both of the competing species. It is considered that the predator species is prohibited to harvest. In this model the logistic growth function for both the prey species (that is, the population density of each prey is resources limited) and the feeding rate of the predator species is assumed to increase linearly with each prey density.

The equations that govern the model are,

$$\frac{dx}{dt} = x[\lambda_1(1 - x/k_1) - \alpha y - \beta z - q_1(E/2 - \mu) \tag{2.1}$$

$$\frac{dy}{dt} = y[\lambda_2(1 - y/k_2) - \alpha x - \gamma z - q_2(E/2 + \mu) \tag{2.2}$$

$$\frac{dz}{dt} = z[\theta_1 \beta x + \theta_2 \gamma y - z] \tag{2.3}$$

Where $\lambda_1, \lambda_2, k_1, k_2, \alpha, \beta, \gamma, q_1, q_2, E, \theta_1, \theta_2$ are all positive constant. Particularly λ_1, λ_2 are biotic potentials; k_1, k_2 are environmental carrying capacities of the two prey species; α is the coefficient of interspecific competition; β, γ are the predation rate constants; θ_1, θ_2 are digesting factors; q_1, q_2 are catchability coefficients of *x* and *y*- species respectively; *E* is the total required biomass and μ is a positive constant whose variation indicates the variation in the harvested biomass of each species separately but the total harvested biomass remains *E.*

3. Equilibrium analysis

The steady state solutions are the solutions of the equations,

$$x[\lambda_1(1 - x/k_1) - \alpha y - \beta z - q_1(E/2 - \mu)] = 0 \tag{3.1}$$

$$y[\lambda_2(1 - y/k_2) - \alpha x - \gamma z - q_2(E/2 + \mu)] = 0 \tag{3 2}$$

$$z[\theta_1 \beta x + \theta_2 \gamma y - z] = 0 \tag{3.3}$$

Solving these equations of biological equilibrium we get seven points of equilibria $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ and the conditions of their existence. They are all listed in the following table.

< Table 1>

We now write the values of *a, b,* $\widetilde{x}, \widetilde{y}, x^*, y^*, z^*$.

Here $a = \dfrac{\alpha q_2(\frac{\lambda_2}{q_2} - \frac{E}{2}) + \frac{\lambda_2}{k_2 q_1}(\frac{\lambda_1}{q_1} - \frac{E}{2})}{\frac{q_1\lambda_2}{k_2} + \alpha q_2}$ and $b = \dfrac{\frac{\lambda_1 q_2}{k_1}(\frac{\lambda_2}{q_2} - \frac{E}{2}) + \frac{\alpha}{q_1}(\frac{\lambda_1}{q_1} - \frac{E}{2})}{\frac{q_2\lambda_1}{k_1} + \alpha q_1}$,

$$\widetilde{x} = \frac{\frac{\lambda_2}{k_2}[\lambda_1 - q_1(\frac{E}{2} - \mu)] - \alpha[\lambda_2 - q_2(\frac{E}{2} + \mu)]}{\frac{\lambda_1\lambda_2}{k_1k_2} - \alpha^2},$$

$$\widetilde{y} = \frac{\frac{\lambda_1}{k_1}[\lambda_2 - q_2(\frac{E}{2} + \mu)] - \alpha[\lambda_1 - q_1(\frac{E}{2} - \mu)]}{\frac{\lambda_1\lambda_2}{k_1k_2} - \alpha^2},$$

$$x^* = \frac{(\alpha + \theta_2\beta\gamma)(\lambda_2 - q_2(\frac{E}{2} + \mu)) - (\frac{\lambda_2}{k_2} + \theta_2\gamma^2)(\lambda_1 - q_1(\frac{E}{2} - \mu))}{\theta_1(\alpha\beta\gamma - \frac{\lambda_2}{k_2}\beta^2) + \theta_2(\alpha\beta\gamma - \frac{\lambda_1}{k_1}\gamma^2) + (\alpha^2 - \frac{\lambda_1\lambda_2}{k_1k_2})},$$

$$y^* = \frac{(\alpha + \theta_1\beta\gamma)(\lambda_1 - q_1(\frac{E}{2} - \mu)) - (\frac{\lambda_1}{k_1} + \theta_1\beta^2)(\lambda_2 - q_2(\frac{E}{2} + \mu))}{\theta_1(\alpha\beta\gamma - \frac{\lambda_2}{k_2}\beta^2) + \theta_2(\alpha\beta\gamma - \frac{\lambda_1}{k_1}\gamma^2) + (\alpha^2 - \frac{\lambda_1\lambda_2}{k_1k_2})},$$

and $z^* = \dfrac{(\alpha\gamma\theta_2 - \frac{\lambda_2}{k_2}\theta_1\beta)(\lambda_1 - q_1(\frac{E}{2} - \mu)) - (\frac{\lambda_1}{k_1}\theta_2\gamma - \theta_1\alpha\beta)(\lambda_2 - q_2(\frac{E}{2} + \mu))}{\theta_1(\alpha\beta\gamma - \frac{\lambda_2}{k_2}\beta^2) + \theta_2(\alpha\beta\gamma - \frac{\lambda_1}{k_1}\gamma^2) + (\alpha^2 - \frac{\lambda_1\lambda_2}{k_1k_2})}$.

We see that if $(\frac{E}{2} + \mu) < \frac{\lambda_2}{q_2}$ then P_1, P_2 exist and if $(\frac{E}{2} - \mu) < \frac{\lambda_1}{q_1}$ then P_3, P_4 exist. That is if the harvesting effort to the species is less than its biotechnical productivity (BTP) then P_1, P_2, P_3, P_4 exist.

4. Local stability analysis

Now we consider the local stability analysis by variational principle. The variational matrix of the system at any point *P* is written as

$$V(P) = \begin{bmatrix} v_{11} & -\alpha x & -\beta x \\ -\alpha y & v_{22} & -\gamma y \\ \theta_1\beta z & \theta_2\gamma z & \theta_1\beta x + \theta_2\gamma y - 2z \end{bmatrix} \tag{4.1}$$

where

$$v_{11} = \lambda_1(1 - \frac{2x}{k_1}) - \alpha y - \beta z - q_1(\frac{E}{2} - \mu) \tag{4.2}$$

and

$$v_{22} = \lambda_2(1-\frac{2y}{k_2}) - \alpha x - \gamma z - q_2(\frac{E}{2}+\mu). \tag{4.3}$$

Now we analyze the local stability at the points of equilibria in terms of the community matrix evaluated at these points.
Here,

$$V(P_0) = \begin{bmatrix} \lambda_1 - q_1(\frac{E}{2}-\mu) & 0 & 0 \\ 0 & \lambda_2 - q_2(\frac{E}{2}+\mu) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

whose eigenvalues are $\xi_0^1 = 0$, $\xi_0^2 = \lambda_1 - q_1(\frac{E}{2}-\mu)$, $\xi_0^3 = \lambda_2 - q_2(\frac{E}{2}+\mu)$. Therefore by Routh-Hurwitz rule we ascertain that P_0 is an unstable node.

Now at P_1 the variational matrix is

$$V(P_1) = \begin{bmatrix} -\alpha y_1 - q_1(\frac{E}{2}-\mu) & 0 & 0 \\ -\alpha y_1 & \lambda_2(1-\frac{2y_1}{k_2}) - q_2(\frac{E}{2}+\mu) & -\gamma y_1 \\ 0 & 0 & \theta_2\gamma y_1 \end{bmatrix} \tag{4.5}$$

where $(0, y_1, 0)$ are the co-ordinates of P_1 as given in Table1. The eigenvalues of this matrix are

$$\xi_1^1 = \theta_2\gamma\frac{k_2}{\lambda_2}[\lambda_2 - q_2(\frac{E}{2}+\mu)] \ ,$$

$$\xi_1^2 = -\alpha\frac{k_2}{\lambda_2}[\lambda_2 - q_2(\frac{E}{2}+\mu)] - q_1(\frac{E}{2}-\mu),$$

and

$$\xi_1^3 = \lambda_2[1 - 2(\lambda_2 - q_2(\frac{E}{2}+\mu))] - q_2(\frac{E}{2}+\mu).$$

If P_1 exists then $\xi_1^1 > 0$, $\xi_1^2 < 0$ and $\xi_1^3 < 0$. Thus P_1 is also an unstable equilibrium.

Now

$$V(P_2) = \begin{bmatrix} \lambda_1 - \alpha y_2 - \beta z_2 - q_1(\frac{E}{2}-\mu) & 0 & 0 \\ -\alpha y_2 & -\lambda_2\frac{y_2}{k_2} & -\gamma y_2 \\ \theta_1\beta z_2 & \theta_2\gamma z_2 & -z_2 \end{bmatrix} \tag{4.6}$$

where (0, y_2, z_2) are the co-ordinates of P_2 . The characteristic equation of $V(P_2)$ is

$$[\lambda_1 - \alpha y_2 - \beta z_2 - q_1(\frac{E}{2}-\mu) - \xi][\xi^2 + (\frac{\lambda_2 y_2}{k_2} + z_2)\xi + (\gamma^2\theta_2 y_2 z_2 + \frac{\lambda_2 y_2 z_2}{k_2})] = 0 \tag{4.7}$$

Therefore one of the eigenvalues is

$$\xi_2^1 = \lambda_1 - \alpha y_2 - \beta z_2 - q_1(\frac{E}{2} - \mu) \tag{4.8}$$

which is negative or positive according as

$$\mu < or > \frac{E}{2} - \frac{\lambda_1 - \alpha y_2 - \beta z_2}{q_1} \tag{4.9}$$

The other two eigenvalues are the roots of the quadratic

$$\xi^2 + (\frac{\lambda_2 y_2}{k_2} + z_2)\xi + (\gamma^2 \theta_2 y_2 z_2 + \frac{\lambda_2 y_2 z_2}{k_2}) = 0 \tag{4.10}$$

The sum of whose roots = $-(\frac{\lambda_2 y_2}{k_2} + z_2) < 0$ and product $= (\gamma^2 \theta_2 + \frac{\lambda_2}{k_2}) y_2 z_2 > 0$, the signs of sum and product of roots negative and positive respectively. Therefore the roots of (4.10) are real and negative or complex conjugates with negative real parts. Thus P_2 is asymptotically stable if $\mu < \frac{E}{2} - \frac{\lambda_1 - \alpha y_2 - \beta z_2}{q_1}$.

Using the values of y_2, z_2 we get that P_2 is asymptotically stable if,

$$\mu < \frac{(\frac{E}{2} - \frac{\lambda_1}{q_1}) q_1 (\frac{\lambda_2}{k_2} + \theta\gamma^2) + (\alpha + \theta_2 \beta\gamma)(\lambda_2 - \frac{q_2 E}{2})}{q_1(\frac{\lambda_2}{k_2} + \theta\gamma^2) + q_2(\alpha + \theta_2\beta\gamma)}.$$

Moreover P_2 exists if $\mu < (\frac{\lambda_2}{q_2} - \frac{E}{2})$, therefore P_2 is an asymptotically stable equilibrium if,

$$\mu < \text{Min}\left\{ \frac{(\frac{E}{2} - \frac{\lambda_1}{q_1}) q_1 (\frac{\lambda_2}{k_2} + \theta\gamma^2) + (\alpha + \theta_2 \beta\gamma)(\lambda_2 - \frac{q_2 E}{2})}{q_1(\frac{\lambda_2}{k_2} + \theta\gamma^2) + q_2(\alpha + \theta_2\beta\gamma)}, (\frac{\lambda_2}{q_2} - \frac{E}{2}) \right\} \tag{4.11}$$

At $P_3(x_3, 0, z_3)$, where x_3, z_3 are given by $x_3 = \dfrac{\lambda_1 - q_1(\frac{E}{2} - \mu)}{\frac{\lambda_1}{k_1} + \theta_1\beta^2}$, $z_3 = \theta_1\beta \dfrac{\lambda_1 - q_1(\frac{E}{2} - \mu)}{\frac{\lambda_1}{k_1} + \theta_2\beta^2}$, the variational matrix is

$$V(P_3) = \begin{bmatrix} -\lambda_1 x_3 / k_1 & -\alpha x_3 & -\beta x_3 \\ 0 & \lambda_2 - \alpha x_3 - \gamma z_3 - q_2(\frac{E}{2} + \mu) & 0 \\ \theta_1 \beta z_3 & \theta_1 \gamma z_3 & -z_3 \end{bmatrix} \tag{4.12}$$

one of whose eigenvalues is $\xi_3^1 = \lambda_2 - \alpha x_3 - \gamma z_3 - q_2(\frac{E}{2} + \mu)$ (4.13)

and other two eigenvalues are the roots of the following quadratic in ξ,

$$\xi^2 + (\frac{\lambda_1 x_3}{k_1} + z_3)\xi + (\frac{\lambda_1}{k_1} + \theta_1{}^2\beta) x_3 z_3 = 0 \tag{4.14}$$

Under the assumption that P_3 exists in R_+^3,

Sum of the roots of (4.14) = $-(\frac{\lambda_1 x_3}{k_1} + z_3) < 0$

Product of the roots of (4.14) = $(\frac{\lambda_1}{k_1} + \theta_1{}^2 \beta) > 0$

Therefore, other eigenvalues ξ_3^2, ξ_3^3 of (4.12) are real negative or complex conjugates with negative real parts. Hence P_3 is asymptotically stable if, $\xi_3^1 < 0$ that is if $\lambda_2 - \alpha x_3 - \gamma z_3 - q_2(\frac{E}{2} + \mu) < 0$. Using the values of x_3, z_3 this condition can be simplified after little calculation to the form

$$\mu > \frac{(\lambda_2 - \frac{q_2 E}{2})(\frac{\lambda_1}{k_1} + \theta_1 \beta^2) - (\alpha + \theta_1 \beta)(\lambda_1 - \frac{q_1 E}{2})}{(\alpha + \theta_1 \beta \gamma) q_1 + q_2 (\frac{\lambda_1}{k_1} + \theta_1 \beta^2)}.$$

Again P_3 exists if $\mu > (\frac{E}{2} - \frac{\lambda_1}{q_1})$.

Thus P_3 is asymptotically stable if

$$\mu > \text{Max}\left\{ \frac{(\lambda_2 - \frac{q_2 E}{2})(\frac{\lambda_1}{k_1} + \theta_1 \beta^2) - (\alpha + \theta_1 \beta)(\lambda_1 - \frac{q_1 E}{2})}{(\alpha + \theta_1 \beta \gamma) q_1 + q_2 (\frac{\lambda_1}{k_1} + \theta_1 \beta^2)}, (\frac{E}{2} - \frac{\lambda_1}{q_1}) \right\} \tag{4.15}$$

At $P_4(x_4, 0, 0)$, where $x_4 = k_1[1 - \frac{q_1}{\lambda_1}(\frac{E}{2} - \mu)]$, the community matrix is given by

$$V(P_4) = \begin{bmatrix} -\lambda_1 x_4 / k_1 & -\alpha x_4 & -\beta x_4 \\ 0 & \lambda_2 - \alpha x_4 - q_2(\frac{E}{2} + \mu) & 0 \\ 0 & 0 & \theta_1 \beta x_4 \end{bmatrix} \tag{4.16}$$

whose eigenvalues are $\xi_4^1 = \theta_1 \beta x_4$, $\xi_4^2 = -\frac{\lambda_1}{k_1} x_4$, and $\xi_4^3 = \lambda_2 - \alpha x_4 - q_2(\frac{E}{2} + \mu)$.

Under the condition of existence of P_4 we see that $\xi_4^1 > 0$, $\xi_4^2 < 0$ and so P_4 is an unstable equilibrium.

The variational matrix at $P_5(\tilde{x}, \tilde{y}, 0)$ is given by

$$V(P_5) = \begin{bmatrix} -\lambda_1 \tilde{x} / k_1 & -\alpha \tilde{x} & -\beta \tilde{x} \\ -\alpha \tilde{y} & -\frac{\lambda_2 \tilde{y}}{k_2} & -\gamma \tilde{y} \\ 0 & 0 & \theta_1 \beta \tilde{x} + \theta_2 \gamma \tilde{y} \end{bmatrix} \tag{4.16}$$

Whose one eigenvalues is $\xi_5^1 = \theta_1 \beta \tilde{x} + \theta_2 \gamma \tilde{y}$ and other two eigenvalues are the roots of the quadratic $\xi^2 + (\frac{\lambda_1}{k_1}\tilde{x} + \frac{\lambda_2}{k_2}\tilde{y})\xi + (\frac{\lambda_1 \lambda_2}{k_1 k_2} - \alpha^2)\tilde{x}\tilde{y} = 0$. For existence of $P_5(\tilde{x}, \tilde{y}, 0)$ we assume $\tilde{x}, \tilde{y}$ to be positive and hence, if P_5 exists then $\xi_5^1 > 0$. This in turn implies that the critical point P_5 is an unstable one.

At $P_6(x^*, y^*, z^*)$ the variational matrix is given by

$$V(P_6) = \begin{bmatrix} -\lambda_1 x^*/k_1 & -\alpha x^* & -\beta x^* \\ -\alpha y^* & -\frac{\lambda_2 y^*}{k_2} & -\gamma y^* \\ \theta_1 \beta z^* & \theta_2 \gamma z^* & -z^* \end{bmatrix} \quad (4.17)$$

The characteristic equation is

$$\xi^3 + b_1 \xi^2 + b_2 \xi + b_3 = 0 \quad (4.18)$$

where,

$$b_1 = \frac{\lambda_1}{k_1}x^* + \frac{\lambda_2}{k_2}y^* + z^* ,$$

$$b_2 = \frac{\lambda_2}{k_2}y^* z^* + \theta_2 \gamma^2 y^* z^* + \frac{\lambda_1 \lambda_2}{k_1 k_2}x^* y^* + \frac{\lambda_1}{k_1}x^* z^* + \theta_1 \beta^2 x^* z^* - \alpha^2 x^* y^* ,$$

$$b_3 = [\frac{\lambda_1 \lambda_2}{k_1 k_2} + \frac{\lambda_1}{k_1}\theta_2 \gamma^2 - \alpha^2 - \theta_1 \alpha\beta\gamma - \theta_2 \alpha\beta\lambda - \frac{\lambda_2}{k_2}\theta_1 \beta^2]x^* y^* z^* .$$

Using Routh-Hurwitz criteria it can be shown that $P_6(x^*, y^*, z^*)$ is asymptotically stable if, (i) $\frac{\lambda_1}{k_1} > \frac{\alpha\theta_1\beta}{\theta_2\gamma}$ and (ii) $\frac{\lambda_2}{k_2} > \frac{\alpha\theta_2\beta}{\theta_1\gamma}$. (4.18)

We observe that among the equilibria only three points P_2, P_3, P_6 may be asymptotically stable with some restrictions. Of which, the stability at P_2, P_3 depends on the value of the demand related harvesting variation (μ) of prey species, but interestingly the stability of the interior equilibrium P_6 does not depend on μ. Thus the persistence of prey species is not affected by the variation of respective harvesting coefficients provided the total biomass harvested remains fixed.

5. Global stability

For examination of global stability of the interior equilibrium we consider a suitable Lyapunov function

$$v(x, y, z) = (x - x^*) - x^* \ln(x/x^*) + (y - y^*) - y^* \ln(y/y^*) + (z - z^*) - z^* \ln(z/z^*).$$

So the time derivative of v is given by,

$$\frac{dv}{dt} = (x - x^*)\frac{\dot{x}}{x} + (y - y^*)\frac{\dot{y}}{y} + (z - z^*)\frac{\dot{z}}{z}$$

i.e. $$\frac{dv}{dt} = (x - x^*) [\lambda_1(1 - x/k_1) - \alpha y - \beta z - q_1(E/2 - \mu)] +$$

$$(y - y^*)[\lambda_2(1 - y/k_2) - \alpha x - \gamma z - q_2(E/2 + \mu) + (z - z^*)[\theta_1 \beta x + \theta_2 \gamma y - z]$$

Along the steady state solutions of the model, after some little mathematical calculation we get,

$$\frac{dv}{dt} = -[\frac{\lambda_1}{k_1}(x-x^*)^2 + 2\alpha(x-x^*)(y-y^*) + \beta(1-\theta_1)(x-x^*)(z-z^*) + \gamma(1-\theta_2)(y-y^*)(z-z^*)$$

$$+\frac{\lambda_2}{k_2}(y-y^*)^2 + (z-z^*)^2]$$

$$= -X^T AX \tag{5.1}$$

where $$A = \begin{bmatrix} \frac{\lambda_1}{k_1} & \alpha & \beta(1-\theta_1)/2 \\ \alpha & \frac{\lambda_2}{k_2} & \gamma(1-\theta_2)/2 \\ \beta(1-\theta_1)/2 & \gamma(1-\theta_2)/2 & 1 \end{bmatrix} \tag{5.2}$$

$$X = \begin{bmatrix} x-x^* \\ y-y^* \\ z-z^* \end{bmatrix} \tag{5.3}$$

Therefore $\frac{dv}{dt} < 0$ if A is positive definite. Now the principal minors of A are $\frac{\lambda_1}{k_1}$, $\frac{\lambda_1\lambda_2}{k_1k_2} - \alpha^2$, $\frac{\lambda_1\lambda_2}{k_1k_2} - \alpha^2 - \frac{\lambda_1}{k_1}\gamma^2(1-\theta_2)^2 - \frac{\lambda_2\beta^2}{2k_2}(1-\theta_1)(1-\theta_2) + \frac{1}{2}\alpha\beta\gamma(1-\theta_1)(1-\theta_2)$ which are all positive i.e. A is positive definite when $\frac{\lambda_1\lambda_2}{k_1k_2} > \alpha^2$ and $\theta_1 = \theta_2 = 1$. Thus we arrive at the following theorem.

Theorem 5.1 The sufficient conditions that the interior equilibrium $P_6(x^*, y^*, z^*)$ is globally asymptotically are that $\frac{\lambda_1\lambda_2}{k_1k_2} > \alpha^2$ and $\theta_1 = \theta_2 = 1$.

6. Bioeconomic Equilibrium

The concept of bioeconomic equilibrium is a combined concept of biological equilibrium as well as economic equilibrium. The equations of biological equilibrium are $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$. The economic equilibrium happens when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvest).

Let, $c =$ constant fishing cost per unit effort,

$p_1 =$ constant price per unit biomass of first species,

$p_2 =$ constant price per unit biomass of second species.

The net revenue at any time is given by,

$$\pi(x, y, z, \mu) = \text{TR-TC} = p_1 q_1 (\frac{E}{2} - \mu)x + p_2 q_2 (\frac{E}{2} + \mu) - cE . \tag{6.1}$$

Now the equations of biological equilibrium are

$$\dot{x}=0 \Rightarrow x\text{=0 or } \frac{E}{2}=\frac{\lambda_1}{q_1}-\frac{\lambda_1 x}{k_1 q_1}-\frac{\alpha y}{q_1}-\frac{\beta z}{q_1}+\mu$$

$$\dot{y}=0 \Rightarrow y\text{=0 or } \frac{E}{2}=\frac{\lambda_2}{q_2}-\frac{\lambda_2 x}{k_2 q_2}-\frac{\alpha x}{q_2}-\frac{\gamma z}{q_2}-\mu$$

$$\dot{z}=0 \Rightarrow z\text{=0 or } \theta_1\beta x+\theta_2\gamma y-z=0.$$

Hence the equations of non-trivial biological equilibrium are,

$$\left(\frac{\lambda_1}{k_1 q_1}-\frac{\alpha}{q_2}\right)x+\left(\frac{\alpha}{q_1}-\frac{\lambda_2}{k_2 q_2}\right)y+\left(\frac{\beta}{q_1}-\frac{\gamma}{q_2}\right)z+\left(\frac{\lambda_2}{q_2}-\frac{\lambda_1}{q_1}-2\mu\right)=0$$

and
$$\theta_1\beta x+\theta_2\gamma y-z=0. \tag{6.2}$$

The bioeconomic equilibrium $R(x_\infty, y_\infty, z_\infty)$ is the point at which the line in (6.2) meets the plane $\pi(x,y,z,\mu)=0$, in the first octant. Therefore unique interior bioeconomic equilibrium exists if all four minors of order thereof the matrix

$$\begin{bmatrix} \frac{\lambda_1}{k_1 q_1}-\frac{\alpha}{q_2} & \frac{\alpha}{q_1}-\frac{\lambda_2}{k_2 q_2} & \frac{\beta}{q_1}-\frac{\gamma}{q_2} & -\left(\frac{\lambda_2}{q_2}-\frac{\lambda_1}{q_1}-2\mu\right) \\ \theta_1\beta & \theta_2\gamma & -1 & 0 \\ p_1 q_1\left(\frac{E}{2}-\mu\right) & p_2 q_2\left(\frac{E}{2}+\mu\right) & 0 & cE \end{bmatrix}$$

are all positive or all negative.

Thus we see that though the stability local / global of an interior equilibrium does depend on μ, interior bioeconomic equilibrium depends on μ.

7. Numerical simulation

For numerical analysis we take the following set of values of parameters.

λ_1=2.09, k_1=200, α=0.001, β=0.01 q_1=0.04, E=10, μ=0.5, λ_2=2.07, k_2=300,

γ=0.02, q_2=0.01, θ_1=1, θ_2=1, p_1=2, p_2=3, c=50.

In this example the steady state solutions are $P_0(0,0,0)$, $P_1(0,292.029,0)$, $P_2(0,276.027,5.520)$, $P_3(181.775,0,0)$, $P_4(182.775,0,0)$, $P_5(157.007,269.274,0)$, $P_6(152.4975,250.9593,6.54416)$.

The variation of the three species with time is given in the figure 1.

<Figure1>

For this set of values of parameters the system is stable at the interior equilibrium $P_6(152.4975,250.9593,6.54416)$. This is best described in figure 2.

< Figure 2>

< Figure 3>

We see that the model in terms of global stability at interior equilibrium is not sensible to μ while the existence of bioeconomic equilibrium depends on μ .

For the above example in which $\mu = 0.5$ the bioeconomic equilibrium is attained at the point *R* (1361.4190, 59.9343, 14.8128). If we take $\mu = 0.9$ instead of $\mu = 0.5$ then the bioeconomic equilibrium point is shifted to *R* (1478.4606, 85.1125, 16.4869).

Concluding remarks

In this paper, we aimed at the discussion of the effects of harvesting in a two prey species equally competitive system in presence of a predator species. We have studied the existence and local/global stability of the possible steady states. We then discussed the existence of bioeconomic equilibrium of the exploited system. We have considered the prey species as equally dominating each other in terms of interspecific competition.

Keeping total biomass to be harvested fixed and demand oriented harvesting variation of prey species we derive all the results. At last, some numerical examples are considered to examine our theoretical results. We used Matlab to get numerical results. We observed that global stability of the model does not depend on the variation coefficient μ , while the bioeconomic equilibrium does.

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Table 1.

Points of equilibrium	Co-ordinates	Conditions of existence
P_0	(0,0,0)	Trivial
P_1	$(0,\frac{k_2}{\lambda_2}[\lambda_2 - q_2(E/2+\mu)],0)$	$0<\mu<(\frac{\lambda_2}{q_2}-\frac{E}{2})$
P_2	$(0,\frac{\lambda_2 - q_2(\frac{E}{2}+\mu)}{\frac{\lambda_2}{k_2}+\theta_2\gamma^2},\theta_2\gamma\frac{\lambda_2 - q_2(\frac{E}{2}+\mu)}{\frac{\lambda_2}{k_2}+\theta_2\gamma^2})$	$0<\mu<(\frac{\lambda_2}{q_2}-\frac{E}{2})$
P_3	$(\frac{\lambda_1 - q_1(\frac{E}{2}-\mu)}{\frac{\lambda_1}{k_1}+\theta_1\beta^2},0,\theta_1\beta\frac{\lambda_1 - q_1(\frac{E}{2}-\mu)}{\frac{\lambda_1}{k_1}+\theta_2\beta^2})$	$0<\frac{E}{2}-\frac{\lambda_1}{q_1}<\mu$
P_4	$(\frac{k_1}{\lambda_1}[\lambda_1 - q_1(E/2-\mu)],0,0)$	$0<\frac{E}{2}-\frac{\lambda_1}{q_1}<\mu$
P_5	$(\tilde{x},\tilde{y},0)$ The values of $\tilde{x},\tilde{y}$ are given bellow.	$a<\mu<b$ or $b<\mu<a$, according as $\frac{\lambda_1\lambda_2}{k_1k_2}>$ or $<\alpha^2$ where *a, b* are given bellow.
P_6	(x^*,y^*,z^*) The values of x^*,y^*,z^* are given bellow.	The range of values of μ is such that x^*,y^*,z^* are all positive.

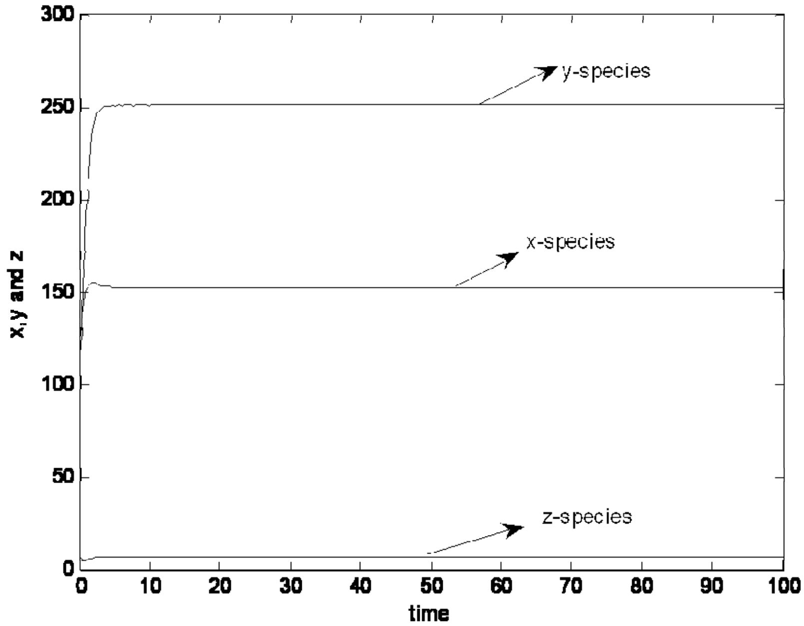

Figure 1. Solution curve

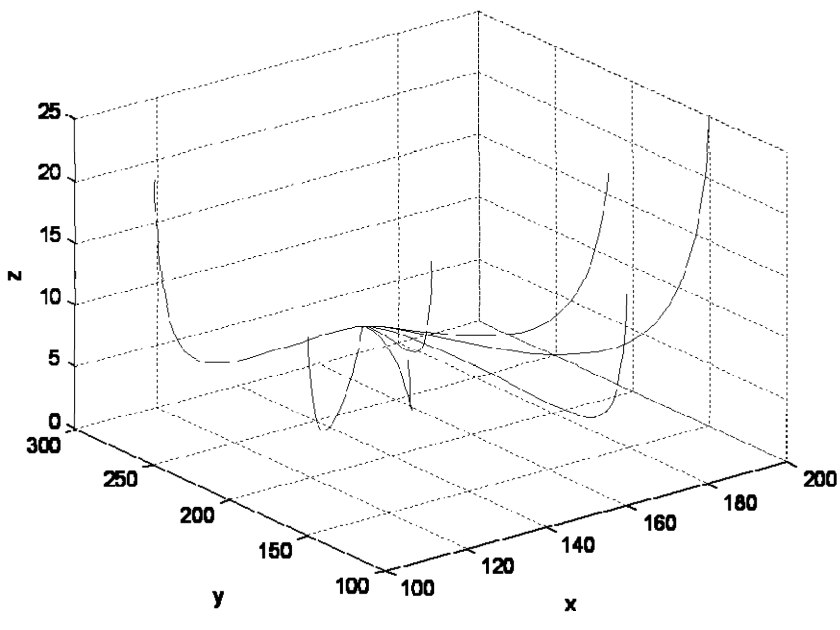

Figure 2. Phase diagram with $\mu = 0.5$

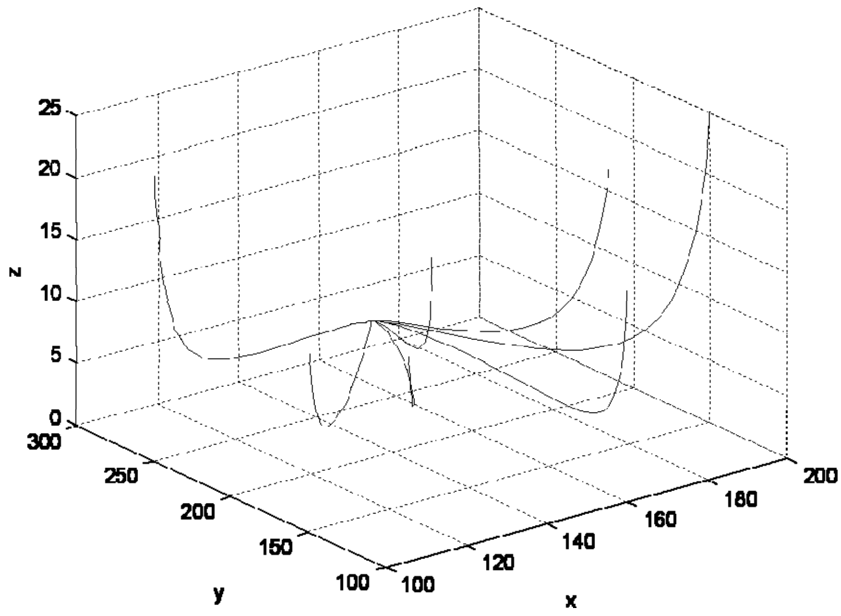

Figure 3. Phase diagram with $\mu = 0.9$.