Prime Cordial Labeling for Some Graphs

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Abstract
We present here prime cordial labeling for the graphs obtained by some graph operations on given graphs.

Keywords: Prime cordial labeling, Total graph, Vertex switching

1. Introduction
We begin with simple, finite, connected and undirected graph \( G = (V(G), E(G)) \). For all standard terminology and notations we follow (Harary F., 1972). We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

For a dynamic survey on graph labeling we refer to (Gallian J., 2009). A detailed study on variety of applications of graph labeling is reported in (Bloom G. S., 1977, p. 562-570).

Definition 1.2 Let \( G \) be a graph. A mapping \( f: V(G) \rightarrow \{0, 1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

For an edge \( e = uv \), the induced edge labeling \( f^*: E(G) \rightarrow \{0, 1\} \) is given by \( f^*(e) = |f(u) - f(v)| \). Let \( v_f(0), v_f(1) \) be the number of vertices of \( G \) having labels 0 and 1 respectively under \( f \) while \( e_f(0), e_f(1) \) be the number of edges having labels 0 and 1 respectively under \( f^* \).

Definition 1.3 A binary vertex labeling of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by (Cahit I., 1987, p. 201-207). After this many researchers have investigated graph families or graphs which admit cordial labeling. Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are product cordial labeling, total product cordial labeling and prime cordial labeling. The present work is focused on prime cordial labeling.

Definition 1.4 A prime cordial labeling of a graph \( G \) with vertex set \( V(G) \) is a bijection \( f: V(G) \rightarrow \{1, 2, 3, \ldots, |V(G)|\} \) defined by \( f(e = uv) = 1 \); if \( \gcd(f(u), f(v)) = 1 \), and \( f(v) \); otherwise

\( |e_f(0) - e_f(1)| \leq 1 \). A graph which admits prime cordial labeling is called a prime cordial graph.

Definition 1.5 Let \( G \) be a graph with two or more vertices then the total graph \( T(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and two vertices are adjacent whenever they are either adjacent or incident in \( G \).

Definition 1.6 The composition of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1[G_2] \) has vertex set \( V(G_1[G_2]) = V(G_1) \times V(G_2) \) and edge set \( E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2) \} \).

Definition 1.7 A vertex switching \( G_v \) of a graph \( G \) is the graph obtain by taking a vertex \( v \) of \( G \), removing all the edges incident to \( v \) and adding edges joining \( v \) to every other vertex which are not adjacent to \( v \) in \( G \).

2. Main Results
Theorem 2.1 \( T(P_n) \) is prime cordial graph, \( n \geq 5 \).
Proof: If \( v_1, v_2, v_3, \ldots, v_n \) and \( e_1, e_2, e_3, \ldots, e_n \) be the vertices and edges of \( P_n \) then \( v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \ldots, e_n \) are vertices of \( T(P_n) \).

We define vertex labeling \( f: V(T(P_n)) \to \{1, 2, 3, \ldots, |V(G)|\} \) as follows. We consider following four cases.

Case 1: \( n = 3, 5 \)

For the graph \( T(P_d) \) the possible pairs of labels of adjacent vertices are \((1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\). Then obviously \( e_1(0) = 1, e_1(1) = 6 \). That is, \( |e_1(0) - e_1(1)| = 5 \) and in all other possible arrangement of vertex labels \( |e_1(0) - e_1(1)| > 5 \). Therefore \( T(P_d) \) is not a prime cordial graph.

The case when \( n = 5 \) is to be dealt separately. The graph \( T(P_d) \) and its prime cordial labeling is shown in Fig 1.

Case 2: \( n \) odd, \( n \geq 7 \)

\[
f(v_1) = 2, f(v_2) = 4,
\]

\[
f(v_{1+2}) = 2(i + 3), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2
\]

\[
f(v_{n+2}\_j+1) = 3, f(v_{n+2}\_j+2) = 1, f(v_{n+2}\_j+3) = 7,
\]

\[
f(v_{n+2}\_j+2+i) = 4i + 9, 1 \leq i \leq \lfloor n/2 \rfloor - 2
\]

\[
f(e) = f(v_{n+2}\_j+i) + 2i, 1 \leq i \leq \lfloor n/2 \rfloor - 1,
\]

\[
f(e_{n+2}2) = 6, f(e_{n+2}\_j+1) = 9, f(e_{n+2}\_j+2) = 5,
\]

\[
f(e_{n+2}\_j+1+i) = 4i + 7, 1 \leq i \leq \lfloor n/2 \rfloor - 2
\]

In this case we have \( e_f(0) = e_f(1) + 1 = 2(n-1) \)

Case 3: \( n = 2, 4, 6 \)

For the graph \( T(P_d) \) the possible pairs of labels of adjacent vertices are \((1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\). Then obviously \( e_1(0) = 4, e_1(1) = 7 \). That is, \( |e_1(0) - e_1(1)| = 3 \) and in all other possible arrangement of vertex labels \( |e_1(0) - e_1(1)| > 3 \). Thus \( T(P_d) \) is not a prime cordial graph.

The case when \( n = 6 \) is to be dealt separately. The graph \( T(P_d) \) and its prime cordial labeling is shown in Fig 2.

Case 4: \( n \) even, \( n \geq 8 \)

\[
f(v_1) = 2, f(v_2) = 4,
\]

\[
f(v_{1+2}) = 2(i + 3), \quad 1 \leq i \leq n/2 - 3
\]

\[
f(v_{n+2}) = 6, f(v_{n+2\_1}) = 9, f(v_{n+2\_2}) = 5,
\]

\[
f(v_{n+2\_2+i}) = 4i + 7, 1 \leq i \leq n/2 - 2
\]

\[
f(e_1) = f(v_{n+2\_1+i}) + 2i, 1 \leq i \leq n/2 - 1,
\]

\[
f(e_{n+2}2) = 3, f(e_{n+2\_1+i}) = 1, f(e_{n+2\_2+i}) = 7,
\]

\[
f(e_{n+2\_2+i}) = 4i + 9, 1 \leq i \leq n/2 - 3
\]

In this case we have \( e_f(0) = e_f(1) + 1 = 2(n-1) \)

That is, \( T(P_d) \) is a prime cordial graph, \( \forall n \geq 5 \).

Illustration 2.2 Consider the graph \( T(P_3) \). The labeling is as shown in Fig 3.

Theorem 2.3 \( T(C_n) \) is prime cordial graph, \( \forall n \geq 5 \).

Proof: If \( v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \ldots, e_n \) be the vertices and edges of \( C_n \) then \( v_1, v_2, v_3, \ldots, v_n, e_1, e_2, e_3, \ldots, e_n \) are vertices of \( T(C_n) \).

We define vertex labeling \( f: V(T(C_n)) \to \{1, 2, 3, \ldots, |V(G)|\} \) as follows. We consider following four cases.

Case 1: \( n = 4 \)

For the graph \( T(C_4) \) the possible pair of labels of adjacent vertices are \((1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8), (5, 6), (5, 7), (5, 8), (6, 7), (6, 8), (7, 8) \). Then obviously \( e_1(0) = 6, e_1(1) = 10 \). That is, \( |e_1(0) - e_1(1)| = 4 \) and all other possible arrangement of vertex labels will yield \( |e_1(0) - e_1(1)| > 4 \). Thus \( T(C_4) \) is not a prime cordial graph.
**Case 2:** \( n \) even, \( n \geq 6 \\
f(v_{1}) = 2, f(v_{2}) = 8, \\
f(v_{i+2}) = 4i + 10, \quad 1 \leq i \leq n/2 - 3 \\
f(v_{n/2}) = 2, f(v_{n/2 + 1}) = 8, \\
f(v_{n/2 + 2 + i}) = 4i + 9, \quad 1 \leq i \leq n/2 - 3 \\
f(e_{1}) = 4, f(e_{2}) = 10, \\
f(e_{n/2 + i + 1}) = 4i + 7, \quad 1 \leq i \leq n/2 - 2 \\

In view of the labeling pattern defined above we have \\
e_{f}(0) = e_{f}(1) = 2n.

**Case 3:** \( n = 3 \\
For the graph \( T(C_{3}) \) the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously \( e_{f}(0) = 4, e_{f}(1) = 8 \). That is, \( |e_{f}(0) - e_{f}(1)| = 4 \) and all other possible arrangement of vertex labels will yield \( |e_{f}(0) - e_{f}(1)| > 4 \). Thus \( T(C_{3}) \) is not a prime cordial graph.

**Case 4:** \( n \) odd, \( n \geq 5 \\
f(v_{1}) = 2, \\
f(v_{1 + i}) = 4(i + 1), \quad 1 \leq i \leq \left\lfloor n/2 \right\rfloor - 1 \\
f(v_{n/2 + 1}) = 6, f(v_{n/2 + 2}) = 9, f(v_{n/2 + 3}) = 5, \\
f(e_{1}) = 4, \\
f(e_{n/2 + 1}) = 4i + 9, \quad 1 \leq i \leq n - \left\lfloor n/2 \right\rfloor - 3 \\

In view of the labeling pattern defined above we have \\
e_{f}(0) = e_{f}(1) = 2n. \\
Thus \( f \) is a prime cordial labeling of \( T(C_{n}) \).

**Illustration 2.4** Consider the graph \( T(C_{6}) \). The labeling is as shown in Fig 4.

**Theorem 2.5** \( P_{2} \square P_{m} \) is prime cordial graph \( \forall m \geq 5. \)

**Proof:** Let \( u_{1}, u_{2}, u_{3}, \ldots, u_{m} \) be the vertices of \( P_{m} \) and \( v_{1}, v_{2} \) be the vertices of \( P_{2} \). We define vertex labeling \( f: V(P_{2} \square P_{m}) \to \{1, 2, 3, \ldots, |V(G)|\} \) as follows. We consider following four cases.

**Case 1:** \( m = 2, 4 \)

For the graph \( P_{2} \square P_{2} \) the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Then obviously \( e_{f}(0) = 1, e_{f}(1) = 5 \). That is, \( |e_{f}(0) - e_{f}(1)| = 4 \) and in all other possible arrangement of vertex labels we have \( |e_{f}(0) - e_{f}(1)| > 4 \). Therefore \( P_{2} \square P_{2} \) is not a prime cordial graph.

For the graph \( P_{3} \square P_{2} \) the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8) (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8) \). Then obviously \( e_{f}(0) = 7, e_{f}(1) = 9 \). i.e. \( |e_{f}(0) - e_{f}(1)| = 2 \) and in all other possible arrangement of vertex labels we have \( |e_{f}(0) - e_{f}(1)| > 2 \). Thus \( P_{2} \square P_{2} \) is not a prime cordial graph.

**Case 2:** \( m \) even, \( m \geq 6 \)

\[ f(u_{1}, v_{1}) = 2, f(u_{2}, v_{1}) = 8, \]
\[ f(u_{2 + i}, v_{1}) = 4i + 10, \quad 1 \leq i \leq m/2 - 3 \]
\[ f(u_{m/2}, v_{1}) = 12, \]
\[ f(u_{m/2 + 1}, v_{1}) = 4i - 3, \quad 1 \leq i \leq m/2 \]
\(f(u_1, v_2) = 4, f(u_2, v_2) = 10,\)
\(f(u_{i+1}, v_2) = 4i + 12, \quad 1 \leq i \leq \frac{m}{2} - 3\)
\(f(u_{m/2}, v_2) = 6, f(u_{m/2 + 1}, v_2) = 3,\)
\(f(u_{m/2 + 1 + i}, v_2) = 4i + 12, \quad 1 \leq i \leq \frac{m}{2} - 3\)

Using above pattern we have
\(e_f(0) = e_f(1) = 54\)

Case 3: \(m = 3\)

For the graph \(P_2[P_3]\) the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously \(e_f(0) = 4, e_f(1) = 7\). That is, \(|e_f(0) - e_f(1)| = 3\) and in all other possible arrangement of vertex labels we have \(|e_f(0) - e_f(1)| > 3\). Thus \(P_2[P_3]\) is not a prime cordial graph.

Case 4: \(m \text{ odd, } m \geq 5\)

\(f(u_i, v_1) = 4(1+i), \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\)
\(f(u_{\left\lfloor \frac{n}{2} \right\rfloor}, v_1) = 2,\)
\(f(u_{\left\lfloor \frac{n}{2} \right\rfloor + 1}, v_1) = 3,\)
\(f(u_{\left\lfloor \frac{n}{2} \right\rfloor + 2}, v_1) = 4i + 9, \quad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\)

In view of the above defined labeling pattern we have
\(e_f(0) = e_f(1) = 2n + \left\lfloor \frac{m}{2} \right\rfloor - 1\).

Thus in case 2 and case 4 the graph \(P_2[P_m]\) satisfies the condition \(|e_f(0) - e_f(1)| \leq 1\).

That is, \(P_2[P_m]\) is a prime cordial graph \(\forall m \geq 5\).

Illustration 2.6 Consider the graph \(P_2[P_5]\). The prime cordial labeling is as shown in Fig 5.

Theorem 2.7 Two cycles joined by a path \(P_m\) is a prime cordial graph.

Proof: Let \(G\) be the graph obtained by joining two cycles \(C_n\) and \(C'_n\) by a path \(P_m\). Let \(v_1, v_2, v_3, \ldots, v_n, v'_1, v'_2, v'_3, \ldots, v'_n\) be the vertices of \(C_n, C'_n\) respectively. Here \(u_1, u_2, u_3, \ldots\) are the vertices of \(P_m\). We define vertex labeling \(f: V(G) \rightarrow \{1, 2, 3, \ldots\}\) as follows. We consider following four cases.

Case 1: \(m \text{ odd, } m \geq 5\)

\(f(u_i) = f(v_i) = 2, f(v_i) = 4,\)
\(f(v_{i+2}) = 2(i + 3), \quad 1 \leq i \leq n - 2\)
\(f(u_{i+1}) = f(v_i) + 2i, \quad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 2\)
\(f(u_{\left\lfloor \frac{m}{2} \right\rfloor}) = 6, f(u_{\left\lfloor \frac{m}{2} \right\rfloor + 1}) = 3, f(u_{\left\lfloor \frac{m}{2} \right\rfloor + 2}) = 1\)
\(f(u_{\left\lfloor \frac{m}{2} \right\rfloor + 2 + i}) = 2i + 3, \quad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\)

In view of the above defined labeling pattern we have
\(e_f(0) = e_f(1) = n + \left\lfloor \frac{m}{2} \right\rfloor\).

Case 2: \(m = 3\)

\(f(u_i) = f(v_i) = 6, f(v_i) = 2, f(v_i) = 4,\)
\(f(v_{i+2}) = 2(i + 3), \quad 1 \leq i \leq n - 3\)
\(f(u_{i+1}) = f(v_i) + 2i, \quad 1 \leq i \leq n - 3\)
\(f(u_{2}) = 3, f(v'_{1}) = f(u_1) = 1, f(v'_{2}) = 5\)
\(f(v'_{2+i}) = 2i + 5, 1 \leq i \leq n - 2\)

In view of the above defined labeling pattern we have
\( e_f(0) = e_f(1) = n + 1 \)

**Case 3:** \( m \) even, \( m \geq 4 \)

\[
\begin{align*}
  f(u_i) &= f(v_i) = 2, f(v_2) = 4, \\
  f(v_{i+2}) &= 2(i + 3), \quad 1 \leq i \leq n - 2 \\
  f(u_{m/2}) &= 6, \quad f(u_{m/2+i}) = 3, \quad f(u_{m/2+2+i}) = 1 \\
  f(u_{m/2+2+i}) &= 2i + 3, \quad 1 \leq i \leq m/2 - 2 \\
  f(v^{'i}) &= f(u_n), \quad f(v^{'+i}) = f(v^{'1}) + 2i, \quad 1 \leq i \leq n - 1
\end{align*}
\]

In view of the above defined labeling pattern we have

\[
e_f(0) = e_f(1) + 1 = n + m/2
\]

**Case 4:** \( m = 2 \)

\[
\begin{align*}
  f(u_1) &= f(v_1) = 2 \\
  f(v_{i+2}) &= 2(i + 1), \quad 1 \leq i \leq n - 1 \\
  f(v^{'i+1}) &= f(u_2) = 1 \\
  f(v^{'i+1}) &= 2i + 1, \quad 1 \leq i \leq n - 1
\end{align*}
\]

In view of the above defined labeling pattern we have

\[
e_f(0) + 1 = e_f(1) = n + 1.
\]

Thus in all cases graph \( G \) satisfies the condition

\[
|e_f(0) - e_f(1)| \leq 1.
\]

That is \( G \) is a prime cordial graph.

**Illustration 2.8** Consider the graph joining to copies of \( C_3 \) by the path \( P_7 \). The prime cordial labeling is as shown in Fig 6.

**Theorem 2.9** The graph obtained by switching of an arbitrary vertex in cycle \( C_n \) admits prime cordial labeling except \( n = 5 \).

**Proof:** Let \( v_1, v_2, \ldots, v_n \) be the successive vertices of \( C_n \) and \( G_v \) denotes the graph obtained by switching of a vertex \( v \). Without loss of generality let the switched vertex be \( v_1 \) and we initiate the labeling from the switched vertex \( v_1 \).

To define \( f: V(G_v) \to \{1, 2, 3, \ldots, |V(G)|\} \) we consider following four cases.

**Case 1:** \( n = 4 \)

The case when \( n=4 \) is to be dealt separately. The graph \( G_v \) and its prime cordial labeling is shown in Fig 7.

**Case 2:** \( n \) even, \( n \geq 6 \)

\[
\begin{align*}
  f(v_1) &= 2, f(v_2) = 1, f(v_3) = 4 \\
  f(v_{i+2}) &= 2(i + 3), \quad 1 \leq i \leq n/2 - 3 \\
  f(v_{n/2+i}) &= 6, f(v_{n/2+i}) = 3 \\
  f(v_{n/2+2+i}) &= 2i + 3, \quad 1 \leq i \leq n/2 - 2
\end{align*}
\]

Using above pattern we have

\[
e_f(0) = e_f(1) + 1 = n - 2
\]

**Case 3:** \( n = 5 \)

For the graph \( G_v \) the possible pairs of labels of adjacent vertices are \((1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\). Then obviously \( e_f(0) = 1, e_f(1) = 4 \). That is, \(|e_f(0) - e_f(1)| = 3\) and in all other possible arrangement of vertex labels we have \(|e_f(0) - e_f(1)| > 3\). Thus, \( G_v \) is not a prime cordial graph.

**Case 4:** \( n \) odd, \( n \geq 7 \)

\[
\begin{align*}
  f(v_1) &= 2, f(v_2) = 1, f(v_3) = 4 \\
  f(v_{i+2}) &= 2(i + 3), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 3 \\
  f(v_{2n/2+i}) &= 6, f(v_{2n/2+i}) = 3
\end{align*}
\]
\[ f(v_{\lfloor n/2 \rfloor + 1}) = 2i + 3, \ 1 \leq i \leq \lfloor n/2 \rfloor - 1 \]

Using above pattern we have
\[ e_f(0) + 1 = e_f(1) = n - 2 \]

Thus in cases 1, 2 and 4 \( f \) satisfies the condition for prime cordial labeling. That is, \( G_v1 \) is a prime cordial graph.

**Illustration 2.10** Consider the graph obtained by switching the vertex in \( C_7 \). The prime cordial labeling is as shown in Fig 8.

### 3. Concluding Remarks

It is always interesting to investigate whether any graph or graph families admit a particular type of graph labeling? Here we investigate five results corresponding to prime cordial labeling. Analogous work can be carried out for other graph families and in the context of different graph labeling problems.

**References**


Figure 4. $T(C_6)$ and its prime cordial labeling

Figure 5. $P_2 \cup P_3$ and its prime cordial labeling

Figure 6. Two cycles $C_5$ join by $P_7$ and its prime cordial labeling

Figure 7. Vertex switching in $C_4$ and its prime cordial labeling
Figure 8. Vertex switching in $C_7$ and its prime cordial labeling