Equivalent Resistance of 5×n-laddered Network

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Abstract

The basic structure of 5×n -step network was studied by matrix transform method. Thus, the resistances of both the infinite and finite network were obtained. In addition, the resistances of the finite 5×n -laddered network were measured experimentally by NI Multisim 10 when n equals to positive integer such as one,two,three ... ,and so on. It is found that the measured values of the equivalent resistances of finite 5×n-laddered networks are consistent with the calculated ones.

Keywords: Equivalent Resistance, 5×n-laddered Network, NI Multisim 10, Kirchhoff's current law

1. Introduction

Equivalent Resistance of n×n-laddered network is a general problem in introductory physics. With the development of communication technology, network technologies have become more and more important in many fields such as resistance network and self-control systems. Lots of actual network technological problems can also be simulated by resistance network. This stimulates the research of network's resistance. Up to now, studies have been devoted to the equivalent resistance of different resistance networks,[1-9] but those results have no experimental foundations. Unfortunately, some results are even wrong.[3] Therefore, it is necessary to develop an effective method (matrix transform) to calculate the equivalent resistance of resistance network.

Matrix transform is commonly used in mathematics and theoretical physics, especially in quantum physics. However, students trained in the algebra-based physics course are used to matrix transform only in linear equations, but not in mechanics,electromagnetism,and difference equations. In this paper, we will construct difference equations of closed circuits and find relation of matrix transform and difference equations to calculate the equivalent resistance of 5×n-laddered network. Furthermore, we measured its resistance by using NI Multisim 10 simulation software. Agreement is achieved between the measured values and the theoretical ones.

2. The Differential Equation Model on Current

2.1 The development of the differential equation on current

Figure 1 is a schematic diagram of 5×n-laddered resistance network ($n\to\infty$). The equivalent resistance of between **a** and **f** nodes was assumed to be R_{af} .

Suppose each resistor has the same value **r**. The direction of the steady current is from nodes **f** to **a** as shown in Fig 1. Each branch current and its flowing direction were marked in Fig. 2, which is a part of 5×n-laddered resistance network (hereafter denoted as 5×n-laddered resistance sub-networks). Assume the currents of the horizontal resistances in the second rank are I_{ak}, I_{bk}, I_{ck}, I_{dk}, I_{ek}, and I_{fk} (1≤k≤n) respectively as shown Fig. 2. Accordingly, the currents of the vertical resistance are I_k, $I_k^{'}$, $I_k^{''}$, $I_k^{'''}$, and $I_k^{''''}$ respectively.

Kirchhoff's current law [10] is used for the all nodes in the second rank. Differential current equations were obtained as follows:

$$I_{ak} - I_{ak-1} = -I_k \tag{1}$$

$$I_k - I_k^{'} = I_{bk} - I_{bk-1} \tag{2}$$

$$I_k^{'} - I_k^{''} = I_{ck} - I_{ck-1} \tag{3}$$

$$I_k^{''} - I_k^{'''} = I_{dk-1} - I_{dk} \tag{4}$$

$$I_k^{'''} - I_k^{''''} = I_{ek-1} - I_{ek} \tag{5}$$

In terms of the symmetry, we obtain the following equations

$$I_{ak} = I_{fk};\ I_{bk} = I_{ek};\ I_{ck} = I_{dk};\ I_k = I_k^{''''};\ I_k^{'} = I_k^{'''} \tag{6}$$

Trace all the meshes in the (k-1)-th rank and apply Kirchhoff's voltage law (KVL).[10]

$$I_k r + I_{ak-1} r - I_{k-1} r - I_{bk-1} r = 0 \tag{7}$$

$$I_k^{'} r + I_{bk-1} r - I_{ck-1} r - I_{k-1}^{'} r = 0 \tag{8}$$

$$I_k^{''} r + I_{ck-1} r + I_{dk-1} r - I_{k-1}^{''} r = 0 \tag{9}$$

In the same manner, trace the all meshes in the k-th rank and apply KVL.

$$I_{ak} r + I_{k+1} r - I_k r - I_{bk} r = 0 \tag{10}$$

$$I_{bk} r + I_{k+1}^{'} r - I_{ck} r - I_k^{'} r = 0 \tag{11}$$

$$I_{ck} r + I_{k+1}^{''} r + I_{dk} r - I_k^{''} r = 0 \tag{12}$$

Eq. (9) subtracted from Eq. (12) gives

$$(I_{ck} - I_{ck-1}) + (I_{k+1}^{''} + I_{k-1}^{''}) + (I_{dk} - I_{dk-1}) - (I_k^{''} - I_k^{''}) = 0$$

Considering $I_{ck} = I_{dk}$, $I_{ck-1} = I_{dk-1}$, we get

$$2(I_{ck} - I_{ck-1}) + I_{k+1}^{''} + I_{k-1}^{''} = 0 \tag{13}$$

Substituting Eq. (3) into Eq. (13) yields

$$2(I_k^{'} - I_k^{''}) + I_{k+1}^{''} + I_{k-1}^{''} = 0 \tag{14}$$

Eq. (14) is simplified as this

$$I_{k+1}^{''} + I_{k-1}^{''} = 4I_k^{''} - 2I_k^{'} \tag{15}$$

Eq. (11) subtracted from Eq. (8) gives,

$$I_{bk} + I_{k+1}^{'} - I_{ck} - I_k^{'} - I_k^{'} - I_{bk-1} + I_{ck-1} + I_{k-1}^{'} = 0 \tag{16}$$

Substituting Eqs. (2),(3) into Eq. (16) produces

$$I_{k+1}^{'} - I_k^{'} + I_k - I_k^{'} - I_k^{'} + I_{k-1}^{'} - I_k^{'} + I_k^{''} = 0 \tag{17}$$

Eq. (17) can further be simplified as

$$I_{k+1}^{'} + I_{k-1}^{'} = 4I_k^{'} - I_k - I_k^{''} \tag{18}$$

And then Eq. (10) subtracted from Eq. (7) gives,

$$I_{ak}+I_{k+1}\text{-}I_{k}\text{-}I_{bk}\text{-}I_{k}\text{-}I_{ak-1}+I_{k-1}+I_{bk-1}=0 \tag{19}$$

Similarly, substituting Eqs. (1), (2) into Eq. (19) yields

$$I_{k+1}+I_{k-1}=4I_{k}\text{-}I_{k}^{'} \tag{20}$$

Eqs. (15), (18), and (20) can be written in matrix form, therefore we get

$$\begin{bmatrix} I_{k+1} \\ I_{k+1}^{'} \\ I_{k+1}^{''} \end{bmatrix}=\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} I_{k} \\ I_{k}^{'} \\ I_{k}^{''} \end{bmatrix}-\begin{bmatrix} I_{k-1} \\ I_{k-1}^{'} \\ I_{k-1}^{''} \end{bmatrix} \tag{21}$$

Eq. (21) is differential equation model of current parameters. We provide a new method to solve such problems by using matrix transform method and constructing new differential equations model.

Multiply both sides of the matrix (21) by three order undetermined matrix on the left, we have

$$\begin{bmatrix} \lambda_1 & \lambda_4 & 1 \\ \lambda_2 & \lambda_5 & 1 \\ \lambda_3 & \lambda_6 & 1 \end{bmatrix}\begin{bmatrix} I_{k+1} \\ I_{k+1}^{'} \\ I_{k+1}^{''} \end{bmatrix}=\begin{bmatrix} \lambda_1 & \lambda_4 & 1 \\ \lambda_2 & \lambda_5 & 1 \\ \lambda_3 & \lambda_6 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} I_{k} \\ I_{k}^{'} \\ I_{k}^{''} \end{bmatrix}-\begin{bmatrix} \lambda_1 & \lambda_4 & 1 \\ \lambda_2 & \lambda_5 & 1 \\ \lambda_3 & \lambda_6 & 1 \end{bmatrix}\begin{bmatrix} I_{k-1} \\ I_{k-1}^{'} \\ I_{k-1}^{''} \end{bmatrix} \tag{22}$$

Assuming the existence of constants t_1, t_2, t_3, and let the following matrix equation be correct.

$$\begin{bmatrix} \lambda_1 & \lambda_4 & 1 \\ \lambda_2 & \lambda_5 & 1 \\ \lambda_3 & \lambda_6 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix}=\begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & \lambda_4 & 1 \\ \lambda_2 & \lambda_5 & 1 \\ \lambda_3 & \lambda_6 & 1 \end{bmatrix} \tag{23}$$

Expand the matrix (23) and simplify, we achieve

$$\begin{cases} 4\lambda_1-\lambda_4=t_1\lambda_1 \\ -\lambda_1-4\lambda_4-2=t_1\lambda_4 \\ -\lambda_4+4=t_1 \end{cases} \qquad \begin{cases} 4\lambda_2-\lambda_5=t_2\lambda_2 \\ -\lambda_2-4\lambda_5-2=t_2\lambda_5 \\ -\lambda_5+4=t_2 \end{cases} \qquad \begin{cases} 4\lambda_3-\lambda_6=t_3\lambda_3 \\ -\lambda_3-4\lambda_6-2=t_3\lambda_6 \\ -\lambda_6+4=t_3 \end{cases}$$

The solutions of the above equations are

$$\lambda_1=1, \qquad \lambda_2=1, \qquad \lambda_3=1, \tag{24}$$

$$\lambda_4=\sqrt{3}, \qquad \lambda_5=-\sqrt{3}, \qquad \lambda_6=\sqrt{3}, \tag{25}$$

$$t_1=4-\sqrt{3}, \qquad t_2=4+\sqrt{3}, \qquad t_3=4-\sqrt{3} \tag{26}$$

Substitute Eqs. (23), (24), (25), and (24) into Eq. (16), hence the matrix equation can be written as

$$\begin{bmatrix} I_{k+1}+\sqrt{3}I_{k+1}^{'}+I_{k+1}^{''} \\ I_{k+1}-\sqrt{3}I_{k+1}^{'}+I_{k+1}^{''} \\ I_{k+1}+\sqrt{3}I_{k+1}^{'}+I_{k+1}^{''} \end{bmatrix}=\begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}\begin{bmatrix} I_{k}+\sqrt{3}I_{k}^{'}+I_{k}^{''} \\ I_{k}-\sqrt{3}I_{k}^{'}+I_{k}^{''} \\ I_{k}+\sqrt{3}I_{k}^{'}+I_{k}^{''} \end{bmatrix}\text{-}\begin{bmatrix} I_{k\text{-}1}+\sqrt{3}I_{k\text{-}1}^{'}+I_{k\text{-}1}^{''} \\ I_{k\text{-}1}-\sqrt{3}I_{k\text{-}1}^{'}+I_{k\text{-}1}^{''} \\ I_{k\text{-}1}+\sqrt{3}I_{k\text{-}1}^{'}+I_{k\text{-}1}^{''} \end{bmatrix} \tag{27}$$

Let $I_k+\sqrt{3}I_k^{'}+I_k^{''}=x_k$, Simplify the matrix (23), we hold

$$x_{k+1}=t_1x_k-x_{k-1} \tag{28}$$

where t_1 is the known constant, based on the definition of differential equation, Eq. (28) is obviously second order linear differential equation with fixed coefficients. Let $x_k=x^k$, substituting it into Eq. (28) yields the following characteristic equation

$$x^2 = t_1 x - 1 \tag{29}$$

Note that for $I_k - \sqrt{3}I_k^{'} + I_k^{''} = y_k$, and simplify the matrix (27), we get

$$y_{k+1} = t_2 y_k - y_{k-1} \tag{30}$$

Let $y_k = y^k$, substitute it into Eq. (30), we also obtain the following characteristic equation.

$$y^2 = t_2 y - 1 \tag{31}$$

Similarly, we have

$$z^2 = t_3 z - 1 \tag{32}$$

Therefore, the characteristic equation of differential equation is

$$\begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{33}$$

Suppose the roots of the equation with respect to x are α and β, the roots of the equation with respect to y are γ and δ, and those of the equation with respect to z are μ and ν, the solutions of characteristic Eq. (33) are as follows:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4 - \sqrt{3} + \sqrt{15 - 8\sqrt{3}}) \\ \frac{1}{2}(4 - \sqrt{3} - \sqrt{15 - 8\sqrt{3}}) \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4 + \sqrt{3} + \sqrt{15 + 8\sqrt{3}}) \\ \frac{1}{2}(4 + \sqrt{3} - \sqrt{15 + 8\sqrt{3}}) \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(4 - \sqrt{3} + \sqrt{15 - 8\sqrt{3}}) \\ \frac{1}{2}(4 - \sqrt{3} - \sqrt{15 - 8\sqrt{3}}) \end{bmatrix} \tag{36}$$

Due to $x_{k+1} = t_1 x_k - x_{k-1}$, and $\alpha + \beta = 4 - \sqrt{3} = t_1$, $\alpha \cdot \beta = 1$, hence

$$x_{k+1} = (\alpha + \beta) x_k - \alpha \cdot \beta \cdot x_{k-1} \tag{37}$$

Rewriting x_{k+1} expression and transposing yields

$$x_{k+1} - \alpha x_k = \beta(x_k - \alpha \cdot x_{k-1}) = \beta^{k-1}(x_2 - \alpha x_1)$$

$$x_{k+1} - \beta x_k = \alpha(x_k - \beta \cdot x_{k-1}) = \alpha^{k-1}(x_2 - \beta x_1) \tag{38}$$

From the matrix equation we obviously see that $\alpha \neq \beta$, and then subtracting Eq. (38) from Eq. (37) gives

$$x_k = \frac{1}{\alpha - \beta}[(x_2 - \beta x_1)\alpha^{k-1} - (x_2 - \alpha x_1)\beta^{k-1}] \tag{39}$$

In the same manner, we get

$$y_k = \frac{1}{\gamma-\delta}[(y_2-\delta y_1)\gamma^{k-1}-(y_2-\gamma y_1)\delta^{k-1}] \tag{40}$$

$$z_k = \frac{1}{\mu-\nu}[(z_2-\nu z_1)\mu^{k-1}-(z_2-\mu z_1)\nu^{k-1}] \tag{41}$$

Eqs. (39), (40), and (41) can be written in matrix form

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha-\beta}[(x_2-\beta x_1)\alpha^{k-1}-(x_2-\alpha x_1)\beta^{k-1}] \\ \frac{1}{\gamma-\delta}[(y_2-\delta y_1)\gamma^{k-1}-(y_2-\gamma y_1)\delta^{k-1}] \\ \frac{1}{\mu-\nu}[(z_2-\nu z_1)\mu^{k-1}-(z_2-\mu z_1)\nu^{k-1}] \end{bmatrix} \tag{42}$$

where $x_k = I_k + \sqrt{3}I_k^{'} + I_k^{''}$, $y_k = I_k - \sqrt{3}I_k^{'} + I_k^{''}$, $z_k = I_k + \sqrt{3}I_k^{'} + I_k^{''}$ (k=1, 2, 3,…). Characteristic equation (42) gives the current properties of vertical resistance **r** in any sub-network.

2.2 The Properties of Boundary Current

When the current flows toward node **a** and flows away from node **b** in Fig.1, applying the current continuity equation, we have

$$\sum_{i=1}^{n+1} I_i = I, \quad \sum_{i=1}^{n+1} I_i^{'} = I, \quad \sum_{i=1}^{n+1} I_i^{''} = I \tag{43}$$

If we consider Eq. (43), summation of Eq. (42) (k=1, 2, 3, … $n+1$) gives

$$(x_2-\beta x_1)\frac{1-\alpha^{n+1}}{1-\alpha}-(x_2-\alpha x_1)\frac{1-\beta^{n+1}}{1-\beta}=(\alpha-\beta)(1+\sqrt{3}+1)I \tag{44}$$

$$(y_2-\delta y_1)\frac{1-\gamma^{n+1}}{1-\gamma}-(y_2-\gamma y_1)\frac{1-\delta^{n+1}}{1-\delta}=(\gamma-\delta)(1-\sqrt{3}+1)I \tag{45}$$

$$(z_2-\nu x_1)\frac{1-\mu^{n+1}}{1-\mu}-(z_2-\mu z_1)\frac{1-\nu^{n+1}}{1-\nu}=(\mu-\nu)(1+\sqrt{3}+1)I \tag{46}$$

Up to now, we obtain differential equations model of current parameters under boundary conditions by analyzing 5×n-laddered resistance network. Based on Kirchhoff's current law and mesh analysis, [11] we also find the following formula from Fig. 3,

$$I_{a1} = I - I_1 \tag{47}$$

$$I_{b1} = I_1 - I_1^{'} \tag{48}$$

$$I_{c1} = I_1^{'} - I_1^{''} \tag{49}$$

Due to the symmetry, we have

$$I_{c1} = I_{d1} \tag{50}$$

Applying Kirchhoff's voltage law to the first loop produces

$$I_2 r + I_{a1} r - I_{b1} r - I_1 r = 0 \tag{51}$$

$$I_2^{'} r + I_{b1} r - I_{c1} r - I_1^{'} r = 0 \tag{52}$$

$$I_2^{''} r + I_{c1} r - I_1^{''} r + I_{d1} r = 0 \tag{53}$$

Substitute Eqs. (47), (48), (49), and (50) into Eqs. (51), (52), and (53), and then simplify them, we obtain

$$I_2 = 3I_1 - I - I_1^{'} \ , \quad I_2^{'} = 3I_1^{'} - I_1^{''} - I_1 \ , \quad I_2^{''} = 3I_1^{''} - 2I_1^{'} \tag{54}$$

Eq. (54) can be written in matrix form,consequently the differential equations model of 5×n-laddered resistance network under boundary conditions are obtained.

$$\begin{bmatrix} I_2 \\ I_2^{'} \\ I_2^{''} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_1^{'} \\ I_1^{''} \end{bmatrix} - \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

For Eq. $x_2 = I_2 + \sqrt{3}I_2^{'} + I_2^{''}$, substitute Eq. (54) into it and simplify, we achieve

$$x_2 = (3-\sqrt{3})(I_1^{''} + \sqrt{3}I_1^{'} + I_1) - I \tag{56}$$

In the same manner for $x_1 = I_1 + \sqrt{3}I_1^{'} + I_1^{''}$, and $\alpha + \beta =$ 4- $\sqrt{3}$, Eq. (56) can be expressed as

$$x_2 = (\alpha + \beta - 1)x_1 - I \tag{57}$$

Similarly, we have

$$y_2 = (\gamma + \delta - 1)y_1 - I \tag{58}$$

$$z_2 = (\mu + \nu - 1)z_1 - I \tag{59}$$

Eqs. (57), (58), (59) can be written in matrix form

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} (\alpha + \beta - 1)x_1 - I \\ (\gamma + \delta - 1)y_1 - I \\ (\mu + \nu - 1)z_1 - I \end{bmatrix} \tag{60}$$

Substitute Eq. (60) into Eqs. (44), (45), (46), and simplify them, we get

$$\begin{bmatrix} x_1 \\ \\ y_1 \\ \\ z_1 \end{bmatrix} = \begin{bmatrix} (1+\sqrt{3}+1)(1-\dfrac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}})I \\ \\ (1-\sqrt{3}+1)(1-\dfrac{\gamma^n - \delta^n}{\gamma^{n+1} - \delta^{n+1}})I \\ \\ (1+\sqrt{3}+1)(1-\dfrac{\mu^n - \nu^n}{\mu^{n+1} - \nu^{n+1}})I \end{bmatrix} \tag{61}$$

where $x_1 = I_1 + \sqrt{3}I_1^{'} + I_1^{''}$, $y_1 = I_1 - \sqrt{3}I_1^{'} + I_1^{''}$, $z_1 = I_1 + \sqrt{3}I_1^{'} + I_1^{''}$, therefore, these matrix equations give the current properties of 5×n-laddered resistance network under boundary conditions.

2.3 General Properties of the Equivalent Resistance

2.3.1 The equivalent resistance $R_{af}(n)$ in finite network

From the above discussion, we obtained

$$I_1 + \sqrt{3}I_1^{'} + I_1^{''} = x_1, I_1 - \sqrt{3}I_1^{'} + I_1^{''} = y_1, I_1 + \sqrt{3}I_1^{'} + I_1^{''} = z_1$$

These formulas give the relationship of between $I_1, I_1^{'}, \text{and } I_1^{''}$. Therefore we easily solve $I_1, I_1^{'}, I_1^{''}$ with respect to $x_1, y_1,$ and z_1.

$$I_1^{'} = \frac{1}{2\sqrt{3}}(x_1 - y_1); \quad I_1 = \frac{1}{\sqrt{5}+1}(x_1 + y_1); \quad I_1 + I_1^{''} = \frac{1}{2}(x_1 + y_1)$$

Substitute $I_1^{'}, I_1$ and $I_1 + I_1^{"}$ into $(2I_1 + 2I_1^{'} + I_1^{"})$ yields

$$2I_1 + 2I_1^{'} + I_1^{"} =$$

$$= \frac{3\sqrt{3}+\sqrt{15}+2\sqrt{5}+2}{2(\sqrt{15}+\sqrt{3})}(2+\sqrt{3})\ (1-\frac{\alpha^n-\beta^n}{\alpha^{n+1}-\beta^{n+1}})I + \frac{3\sqrt{3}+\sqrt{15}-2\sqrt{5}-2}{2(\sqrt{15}+\sqrt{3})}(2-\sqrt{3})$$
$$(1-\frac{\gamma^n-\delta^n}{\gamma^{n+1}-\delta^{n+1}})I$$

$$= \frac{8\sqrt{3}+4\sqrt{15}+7\sqrt{5}+13}{2(\sqrt{15}+\sqrt{3})}(1-\frac{\alpha^n-\beta^n}{\alpha^{n+1}-\beta^{n+1}})I + \frac{8\sqrt{3}+4\sqrt{15}-7\sqrt{5}-13}{2(\sqrt{15}+\sqrt{3})}(1-\frac{\gamma^n-\delta^n}{\gamma^{n+1}-\delta^{n+1}})I \quad (62)$$

From Fig. 3, we have

$$U_{af} = (2I_1 + 2I_1^{'} + I_1^{"})r$$

Applying Ohm's law yields

$$R_{af} = U_{af} / I = r(2I_1 + 2I_1^{'} + I_1^{"}) / I$$

Therefore, we find

$$\frac{R_{af}(n)}{r} = \frac{8\sqrt{3}+4\sqrt{15}}{\sqrt{15}+\sqrt{3}} - \frac{8\sqrt{3}+4\sqrt{15}+7\sqrt{5}+13}{2(\sqrt{15}+\sqrt{3})}(\frac{\alpha^n-\beta^n}{\alpha^{n+1}-\beta^{n+1}}) - \frac{8\sqrt{3}+4\sqrt{15}-7\sqrt{5}-13}{2(\sqrt{15}+\sqrt{3})}$$
$$(\frac{\gamma^n-\delta^n}{\gamma^{n+1}-\delta^{n+1}}) \quad (63)$$

2.3.2 The equivalent resistance $R_{af}(\infty)$ in infinite network

As n tends to infinity, Fig. 1 becomes a 5×n-laddered resistance network. From Eqs. (29), (30), and (31), we easily obtain that

$$0 < \frac{\alpha}{\beta} = \frac{\frac{1}{2}(4-\sqrt{3}+\sqrt{15-8\sqrt{3}})}{\frac{1}{2}(4-\sqrt{3}+\sqrt{15-8\sqrt{3}})} < 1\ , 0 < \frac{\gamma}{\delta} = \frac{\frac{1}{2}(4+\sqrt{3}+\sqrt{15+8\sqrt{3}})}{\frac{1}{2}(4+\sqrt{3}+\sqrt{15+8\sqrt{3}})} < 1 ,$$

Consequently, we have

$$\lim_{n\to\infty}(\frac{\alpha}{\beta})^n = 0, \ \lim_{n\to\infty}(\frac{\gamma}{\delta})^n = 0 \quad (64)$$

Eq. (63) is taken limit, and then applying the formula (64) gives

$$\frac{R_{af}(\infty)}{r} = \frac{8\sqrt{3}+4\sqrt{15}}{\sqrt{15}+\sqrt{3}} - \frac{8\sqrt{3}+4\sqrt{15}+7\sqrt{5}+13}{2(\sqrt{15}+\sqrt{3})}\beta - \frac{8\sqrt{3}+4\sqrt{15}-7\sqrt{5}-13}{2(\sqrt{15}+\sqrt{3})}\delta \quad (65)$$

where $\beta = \frac{1}{2}(4-\sqrt{3}-\sqrt{15-8\sqrt{3}}), \delta = \frac{1}{2}(4+\sqrt{3}-\sqrt{15+8\sqrt{3}})$.

Eq. (65) is a general expression which denotes the equivalent resistance R_{af} of 5×n-laddered resistance network between nodes a and f, and R_{af} has a limited value in this case.

2.3.3 Measurements of $R_{af}(n)$ by Simulation Experiments

The equivalent resistance R_{af} of 5×n-laddered resistance network is measured by NI Multisim 10 when n is a series of positive integer. In the meantime,the equivalent resistance R_{af} of 5×n-laddered resistance network is calculated from Eq. (63). The relationship curves of the equivalent resistance R_{af} versus n are plotted in Fig. 4. It is found that agreement is achieved between the experimental values and the theoretical ones. These results indicate that the equivalent resistances of n×n-laddered resistance network were not only calculated by matrix transform and but also the calculated results are reliable.

4. Conclusion

General formula $\frac{R_{af}(\infty)}{r}$ and $\frac{R_{af}(n)}{r}$ for 5×n-laddered resistance network in infinite and finite networks are achieved by the matrix transform method to solve a set of differential equations. In addition, the equivalent resistance R_{af} of 5×n-laddered resistance network is calculated from Eq. (63). Moreover,the equivalent resistance R_{af} is measured by NI Multisim 10 when n is a series of positive integer. The result exhibits that the theoretical values are consistent with the experimental ones for the equivalent resistance R_{af} of 5×n-laddered resistance network. This study also reveals that the matrix transform method may be extended to calculate the equivalent resistances of n×n-step resistance network.

References

Zhang, D. T.(2008). Effective cultivating migration ability of students. *Phys Teaching,* vol.30, pp.37-38, Feb. 2008.

Cao, X. H., & Tan, Z. Z. (2008). A general study on equivalent resistance of 2×N-laddered network. *College Phys*, vol.27, pp.25-26, May 2008

Tan, Z. Z., & Luo, D. F. (2004). The study on the Diagonal Equivalent Resistance of 2×N Ordered Network, *Guangxiwuli*,vol.*25*, pp. 43-46, May 2004.

Li, Y. A. (2002). Net work analysis on equivalent resistance of a ladder-shaped network *College Phys*, vol. 21, pp.9-10, Dec. 2002;. Li, Y. A . (2003). Study on equivalent resistance of a ladder-shaped network. *College Phys,* vol. 22, pp.9-10,March 2003.

YLi Y A . (2003). The restudy is on equivalent resistance of ladder-shaped network *College Phys,* vol. 22, pp.12-14, Oct. 2003.

Tan, Z. Z. (2004). Study on the Equivalent resistance of 2×N-laddered network. *J. Hebei Normal University (Natural Science Edition),* vol. 28, pp. 258-262, March 2004;Tan, Z .Z (2004). Restudy on the Equivalent resistance of 2×N-laddered network. *J. Hebei Normal University (Natural Science Edition,)* vol.28, pp.480-482, May 2004.

Carpentieri, E. (1996). Model characterizes transmission – line transformers. *Microwaves & RF,* vol.35, pp.73-80, Nov. 1996

Lu, J. L., & Tan, Z. Z.(2001). A general study on equivalent resistance of ladder-shaped network. *College Phys,* vol.20, pp.26-28, Oct. 2001.

Shao, J. X. (2006). The General Formula of the Equivalent Resistance of Ladder Type Network. *Phys and Eng.*, vol.16, 29-33,Apr. 2006

Halliday, D, Resnick R., & Walker, J. (2001). *Fundamentals of Physics,*6-th ed(John&Sons Inc.)

pp.677-785.

Nilsson, J. W., & Riedel, S. A. (2000). *Introduction to PspiceManual—Electric circuits using or H CAD release 9.1.*The fourth edition (Prentice-Hill).

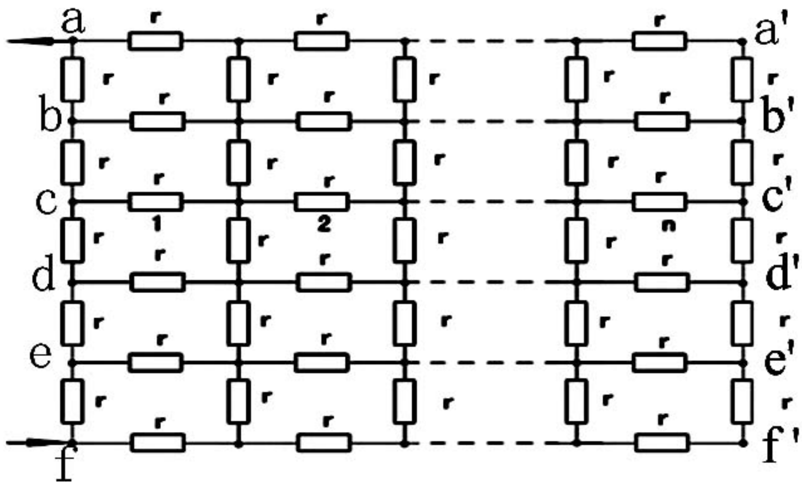

Figure 1. Schematic diagram of 5×n-laddered resistance network

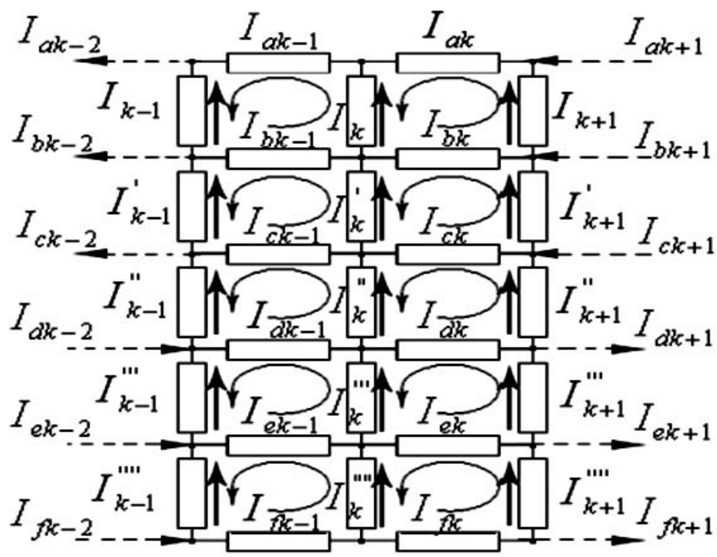

Figure 2. Schematic diagram of 5×n-laddered resistance sub-networks

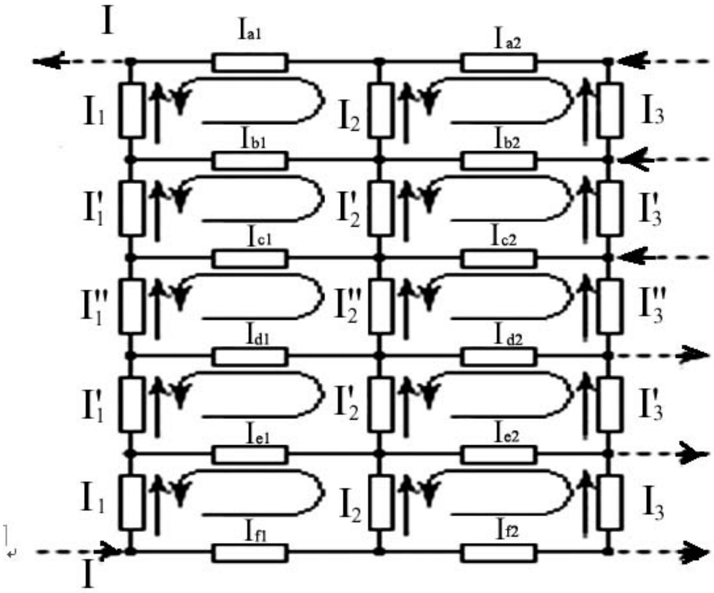

Figure 3. Current parameters of the network under boundary conditions

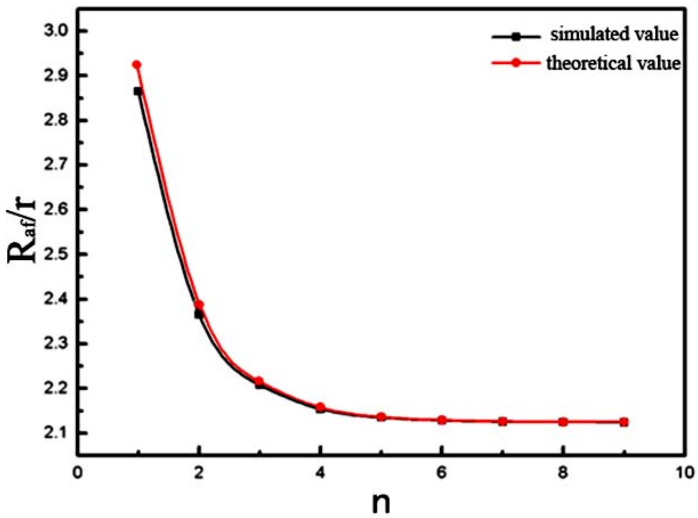

Figure 4. The relationship curves of the equivalent resistance R_{af} and n of 5×n-laddered resistance network