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The Multi Ideal Convergence of Difference Strongly of χ^2 in P-Metric Spaces Defined by Modulus

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Abstract

The aim of this paper is to introduce multi and study a new concept of the $\chi 2$ space via ideal convergence of difference operator defined by modulus. Some topological properties of the resulting sequence spaces are also examined.

Keywords: analytic sequence, modulus function, double sequences, $\chi 2$ space, p-metric space, multi ideal

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \quad (m, n = 1,2,3,...).$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{m,n} \left|x_{mn}\right|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$\left|x_{mn}\right|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = \sup_{m,n} \{\left|x_{mn} - y_{mn}\right|^{\frac{1}{m+n}} : m, n : 1,2,3,...\}, \quad (1.1)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2. Let $\phi = \{\text{finite sequences}\}$.

Consider a double sequence $x = (x_{mn})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$$

for all m, n $\in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero otherwise.

An Orlicz function is a function $f : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with$f(0) = 0$, $f(x) > 0$, for $x > 0$ and $f(x) \to\infty$ as $x \to\infty$. If convexity of Orlicz function f is replaced by$f(x+y) \le f(x) + f(y)$, then this function is called modulus function. An modulus function f is said to satisfyΔ^2- condition for all values u, if there exists $K > 0$ such that $f(2u) \le Kf(u)$, $u \ge 0$.

Remark 1.1 An Modulus function satisfies the inequality $f(\lambda x) \le\lambda f(x)$ for all λ with $0 <\lambda< 1$.

Lemma 1.2 Let f be an modulus function which satisfies Δ^2-condition and let $0 <\delta< 1$. Then for each $t \ge\delta$, we have $f(t) < K\delta^{-1}f(2)$ for some constant $K > 0$.

Let M and Φ be mutually complementary modulus functions. Then, we have

(i) For all $u, y \ge 0$,$uy \le M(u) + \Phi(y)$, (Young's inequality) (Kamthan& Gupta, 1981).(1.2)

(ii) For all $u \ge 0$,$u\eta(u) = M(u) + \Phi(\eta(u))$. (1.3)

(iii) For all $u \ge 0$, and $0 <\lambda< 1$,$M(\lambda u) \le\lambda M(u)$. (1.4)

Lindenstrauss, J. and Tzafriri, L. (1971), used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$ $(1 \le p <\infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\{ |v|u - f_{mn}(u) : u \ge 0\},\ m, n = 1, 2, \ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined by

$$t_f = \{x \in w^3 : M_f(|x_{mnk}|)^{1/m+n+k} \to 0 \text{ as } m,n,k \to \infty\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, \quad x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric space, (i.e.))

Let (X_i, d_i), $i \in I$ be a family of metric spaces such that each two elements of the family are disjoint. Denote $X : \bigcup_{i\in I} X_i$ .If we define

$$d(x, y) = \begin{cases} d_i(x, y), & \text{if } x, y \in X_i \\ +\infty & \text{if } x \in X_i, y \in X_j, i \ne j \end{cases}$$

then the pair (X, d) is a Luxemburg metric space.The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz(1981)as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for Z = c, c_0 and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by Başar and Altay and in the case $0 < p < 1$. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\ge1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = (\textstyle\sum_{k=1}^{\infty}|x_k|^p)^{1/p}, \quad (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta_m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j}x_{m+i,n+j}.$$

2. Definitions and Preliminaries

Let $\Delta^m X$ be a non empty set. A non-void class $I \subseteq 2^{\Delta^m X}$ (power set, of $\Delta^m X$) is called an ideal if I is additive (i.e $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e $A \in I$ and $B \subseteq A \Rightarrow B \in I$). A non-empty family of sets $F \subseteq 2^{\Delta^m X}$ is said to be a filter on $\Delta^m X$ if $\phi \notin F$; $A, B \in F \Rightarrow A \cap B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$. For each ideal I there is a filter F(I) given by $F(I) = \{K \subseteq N : N \setminus K \in I\}$. A non-trivial ideal $I \subseteq 2^{\Delta^m X}$ is called admissible if and only if $\{\{x\} : x \in \Delta^m X\} \subset I$.

A double sequence space E is said to be solid or normal if $(\alpha_{mn}\Delta^m x_{mn}) \in E$, whenever $(\Delta^m x_{mn}) \in E$ and for all double sequences $\alpha = (\alpha_{mn})$ of scalars with $|\alpha_{mn}| \le 1$. for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \le w$. A real valued function $d_p(x_1, ..., x_n) = \|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p = 0$ if and and only if$d_1(x_1, 0), ..., d_n(x_n, 0)$ are linearly dependent,

(ii) $\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1, 0), ..., d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_p$, $\alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2) \dots (x_n, y_n)) = (d_X(x_1, x_2, ... x_n)^p + d_Y(y_1, y_2, ... y_n)^p)^{1/p}$ for $1 \le p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), ... (x_n, y_n)) := \sup\{d_X(x_1, x_2, ... x_n), d_Y(y_1, y_2, ... y_n)\}$, for $x_1, x_2, ... x_n \in X$, $y_1, y_2, ... y_n \in Y$ is called the p-product metric of the Cartesian product of n-metric spaces is the p-norm of the n-vector of the norms of the n-sub spaces.

A trivial example of p-product metric of n-metric space is the p-norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p-norm:

$\|(d_1(x_1, 0), ..., d_n(x_n, 0))\|_E = \sup(|\det(d_{mn}(x_{mn}, 0))|) =$

$$\sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \cdots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \cdots & d_{2n}(x_{2n},0) \\ \vdots & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \cdots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, ... x_{in}) \in \mathbb{R}^n$ for each i = 1, 2, ... n.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p-metric. Any complete p-metric space is said to be p-Banach metric space.

3. Main Results

In this section we introduce the notion of different types of I-convergent double sequences. This generalizes and unifies different notions of convergence for χ^2. We shall denote the ideal of $2^{N\times N}$ by I_2.

Let I_2 be an ideal of $2^{N\times N}$, f be an modulus function. Let u and v be two non-negative integers and $\mu = (\mu_{mn})$ be a sequence of non-zero reals. Then for a sequence $\eta = (\eta_{mn})$ be a double analytic sequence of strictly positive real numbers and $\left(\Delta^v_{(\mu,u)}X, \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p\right)$ be an p-product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces. Further $\chi^2(p-\Delta^v_{(\mu,u)}X)$ denotes $\Delta^v_{(\mu,u)}X$-valued sequence space. Now, we define the following sequence spaces:

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = x = (\Delta^v_{(\mu,u)}x_{mn}) \in \chi^2(p-\Delta^v_{(\mu,u)}X) : \forall\, \varepsilon > 0,$$

$$\left\{(r,s)\in N\times N : \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, (d_1(x_1,0),\ldots,d_n(x_n,0))\right\|_p\right]^{\eta_{mn}} \geq \varepsilon\right\} \in I_2,$$

for every $d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \in \Delta^v_{(\mu,u)}X$.

$$\Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = x = (x_{mn}) \in \Lambda^2(p-\Delta^v_{(\mu,u)}X) : \exists\, K > 0,$$

$$\left\{(r,s)\in N\times N : \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, (d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \geq K\right\} \in I_2,$$

for every $d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \in \Delta^m X$.

$$\Lambda^{2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = x = (\Delta^m x_{mn}) \in \Lambda^2(p-\Delta^v_{(\mu,u)}X) : \exists\, K > 0,$$

$$\left\{(r,s)\in N\times N : \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, (d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \leq K\right\},$$

for every $d_1(x_1, 0), \ldots, d_n(x_{n-1}, 0) \in \Delta^v_{(\mu,u)}X$.

If $\eta = \eta_{mn} = 1$ for all $m, n \in \mathbb{N}$ we obtain

$$\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = \chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p],$$

$$\Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = \Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p],$$

$$\Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^\eta = \Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p].$$

The following well-known inequality will be used in this study: $0 \leq \inf_{mn}\eta_{mn} = H_0 \leq \eta_{mn} \leq \sup_{mn} = H < \infty$, $D = \max(1, 2^{H-1})$, then

$$|x_{mn} + y_{mn}|^{\eta_{mn}} \leq D\{|x_{mn}|^{\eta_{mn}} + |y_{mn}|^{\eta_{mn}}\}$$

for all $m, n \in \mathbb{N}$ and $x_{mn}, y_{mn} \in \mathbb{C}$. Also $|x_{mn}|^{\eta_{mn}/m+n} \leq \max(1, |x_{mn}|^{H/m+n})$ for all $x_{mn} \in \mathbb{C}$.

Theorem 3.1 The classes of sequences $\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta_{mn}}$, $\Lambda^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta_{mn}}$ are linear spaces over the complex field $\mathbb{C}$.

Proof: Now we establish the result for the case $\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),...,d_n(x_n,0))\|_p]^{\eta_{mn}}$ and the others can be proved similarly. Let $x,y\in\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),...,d_n(x_n,0))\|_p]^{\eta_{mn}}$ and α, β∈ ℂ. Then

$$\left\{(r,s)\in N\times N:\tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\geq\frac{\varepsilon}{2}\right\}\in I_2,$$

and

$$\left\{(r,s)\in N\times N:\tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\geq\frac{\varepsilon}{2}\right\}\in I_2.$$

Since $\|(d_1(x_1,0),...,d_n(x_n,0))\|_p$ be an p-product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces and f is an modulus function, the following inequality holds:

$$\tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\frac{\left(\left|\alpha\Delta^v_{(\mu,u)}x_{mn}+\beta\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}},d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\leq$$

$$\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\alpha|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}}f\left(\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}\right),d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}+$$

$$\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\alpha|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}}f\left(\left\|\left(\left|\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}\right),d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\leq$$

$$\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}\right),d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}+$$

$$\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|\left(\left|\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}\right),d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}.$$

From the above inequality we get

$$\left\{(r,s)\in N\times N:\tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\frac{\left|\alpha\Delta^v_{(\mu,u)}x_{mn}+\beta\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}},d_1(x_1,0),...,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\right\}$$

$$\subset\left\{(r,s)\in N\times N:\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}x_{mn}\right|\right)^{1/m+n},d_1(x_1,0),...,d_n(x_n,0))\right\|_p\right]^{\eta_{mn}}\geq\frac{\varepsilon}{2}\right\}\in I_2$$

$$\cup\left\{(r,s)\in N\times N:\tfrac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^v_{(\mu,u)}y_{mn}\right|\right)^{1/m+n},d_1(x_1,0),...,d_n(x_n,0))\right\|_p\right]^{\eta_{mn}}\geq\frac{\varepsilon}{2}\right\}\in I_2.$$

This completes the proof.

Theorem3.2 The class of sequence $\chi^{2I_2}_{\Delta^v_{(\mu,u)}f}[\|(d_1(x_1,0),...,d_n(x_n,0))\|_p]^{\eta}$ is a paranormed space with respect to the paranorm defined by

$$gr_s(x) = \inf\left\{\left(\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_n,0))\right\|_p\right]^{\eta_{mn}}\right)^{\frac{1}{H}} \le 1\right\},$$

for every $d_1(x_1, 0), ..., d_n(x_{n-1}, 0) \in X$.

Proof: $g_{rs}(\theta) = 0$ and $g_{rs}(-x) = g_{rs}(x)$ are easy to prove, so we omit them.

Let us take $x, y \in \chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta_{mn}}$.

Let

$$gr_s(x) = \inf\left\{\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le 1, \forall x\right\},$$

and

$$gr_s(y) = \inf\left\{\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le 1, \forall y\right\},$$

Then we have

$$\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn} + \Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le$$

$$\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} +$$

$$\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}.$$

Thus

$$\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn} + \Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le 1$$

and $g_{rs}(x + y) = g_{rs}(x) + g_{rs}(y)$.

Now, let $\lambda^{u}_{mn} \to \lambda$, where $\lambda^{u}_{mn}, \lambda \in \mathbb{C}$ and $g_{rs}\left(\Delta^{v}_{(\mu,u)}x^{u}_{mn} - \Delta^{v}_{(\mu,u)}x_{mn}\right) \to 0$ as $u \to \infty$. We have to prove that $g_{rs}\left(\lambda_{mn}\Delta^{v}_{(\mu,u)}x^{u}_{mn} - \lambda\Delta^{v}_{(\mu,u)}x_{mn}\right) \to 0$ as $u \to \infty$. Let

$$gr_s(x^u) = \left\{\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x^{u}_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le 1, \forall x \in X\right\}$$

and

$$gr_s(x^u - x) = \left\{\sup_{rs} \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left|\Delta^{v}_{(\mu,u)}x^{u}_{mn} - \Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \le 1,\right\}$$

for all $x \in X$. We observe that

$$f\left(\left\|\left(\frac{\left|\lambda^{u}_{mn}\Delta^{v}_{(\mu,u)}x^{u}_{mn} - \lambda\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}}{\left|\lambda^{u}_{mn} - \lambda\right|^{1/m+n} + \left|\lambda\right|^{1/m+n}}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right) \le$$

$$f\left(\left\|\left(\frac{\left|\lambda_{mn}^{u}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}^{u}\right|)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+\left|\lambda\right|^{1/m+n}}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right)+$$

$$f\left(\left\|\left(\frac{\left|\lambda\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn}\right|)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+\left|\lambda\right|^{1/m+n}}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right)\leq$$

$$\frac{\left|\lambda_{mn}^{u}-\lambda\right|}{\left|\lambda_{mn}^{u}-\lambda\right|+\left|\lambda\right|}f\left(\left\|(\left|\Delta_{(\mu,u)}^{v}x_{mn}^{u}\right|)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)+$$

$$\frac{\left|\lambda\right|}{\left|\lambda_{mn}^{u}-\lambda\right|+\left|\lambda\right|}f\left(\left\|(\left|\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\Delta_{(\mu,u)}^{v}x_{mn}\right|)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right).$$

From this inequality, it follows that

$$f\left(\left\|\left(\frac{\left|\lambda_{mn}^{u}\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta^{2}x_{mn}\right|)^{1/m+n}}{\left|\lambda_{mn}^{u}-\lambda\right|^{1/m+n}+\left|\lambda\right|^{1/m+n}}, d_1(x_1,0),\ldots,d_n(x_{n-1},0)\right)\right\|_p\right)^{\eta_{mn}}\leq 1$$

and consequently

$$g_{rs}(\lambda_{mn}^{u}\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\lambda\Delta_{(\mu,u)}^{v}x_{mn})\leq(\left|\lambda_{mn}^{u}-\lambda\right|)^{\frac{\eta_{mn}}{H}}\inf\left\{g_{rs}(\Delta_{(\mu,u)}^{v}x_{mn}^{u})\right\}+$$

$$(\left|\lambda\right|)^{\frac{\eta_{mn}}{H}}\inf\left\{g_{rs}(\Delta_{(\mu,u)}^{v}x_{mn}^{u}-x)\right\}\leq\max\left\{\left|\lambda\right|,(\left|\lambda\right|)^{\frac{\eta_{mn}}{H}}\right\}g_{rs}(\Delta_{(\mu,u)}^{v}x_{mn}^{u}-\Delta^{m}x_{mn}).$$

Hence by our assumption the right hand side tends to zero as u, m and n $\to\infty$. This completes the proof.

Theorem 3.3(i) If $0<\inf_{mn}\eta_{mn}=H_0\leq\eta_{mn}<1$, then

$\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}\subset\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]$.

(ii) If $1\leq\eta_{mn}\leq\sup_{mn}\eta_{mn}=H<\infty$, then

$\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}\subset\chi^{2I_2}_{f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$.

(iii) If $0<\eta_{mn}<\mu_{mn}<\infty$ and $\left\{\frac{\mu_{mn}}{\eta_{mn}}\right\}$ is double analytic, then

$\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}\subset\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\mu}$.

Proof: The proof can be established using standard technique.

The following result is well known.

Lemma 3.4 If a sequence space E is solid, then it is monotone.

Theorem 3.5 The class of sequence $\chi^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$ is not solid and hence not monotone.

Proof: It is routine verification. Therefore we omit the proof.

Theorem 3.6 Let f, f_1 and f_2 be modulus functions. Then we have

(i) $\chi^{2I_2}_{\Delta^v_{(\mu,u)} f_1}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta} \subset \chi^{2I_2}_{\Delta^v_{(\mu,u)} f\circ f_1}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$

(ii) $\chi^{2I_2}_{\Delta^v_{(\mu,u)} f_1}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta} \cap \chi^{2I_2}_{\Delta^v_{(\mu,u)} f_2}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta} \subset \chi^{2I_2}_{\Delta^v_{(\mu,u)} f_1+\Delta^v_{(\mu,u)} f_2}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$

Proof: (i) Let $\inf_{mn}\eta_{mn} = H_0$. For given $\varepsilon > 0$, we first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of f, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $f(t) < \varepsilon_0$.

$$\text{Let } \Delta^v_{(\mu,u)} x \in \chi^{2I_2}_{\Delta^v_{(\mu,u)} f_1}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$$

We observe that

$$A(\delta) = \left\{(r,s) \in N \times N : \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} \geq \delta^H\right\} \in I_2.$$

Thus if $(r, s) \notin A(\delta)$ then

$$\tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} < \delta^H$$

$$\Rightarrow \sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} < rs\delta^H,$$

$$\Rightarrow \left[f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}} < \delta^H, \text{ for all m, n = 1, 2, ...}$$

$$\Rightarrow f_1\left(\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right) < \delta, \text{ for all m, n = 1, 2, ...}$$

Hence from above inequality and using continuity of f, we must have

$$f\left(f_1\left(\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right) < \varepsilon_0, \text{ for all m, n = 1, 2, ...}$$

$$\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}} < rs\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < rs\,\varepsilon$$

$$\Rightarrow \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}} < \varepsilon.$$

Hence we have

$$\left\{(r,s) \in N \times N : \tfrac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left(\left|\Delta^v_{(\mu,u)} x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}}\right\} \geq \varepsilon \subset A(\delta) \in I_2.$$

(ii) Let $x \in \chi^{2I_2}_{\Delta^v_{(\mu,u)} f_1}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta} \cap \chi^{2I_2}_{\Delta^v_{(\mu,u)} f_2}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$.

Then the fact that

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[(f_1+f_2)\left(\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}} \leq$$

$$\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left(\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}} +$$

$$\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_2\left(\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right)\right]^{\eta_{mn}}.$$

This completes the proof.

Theorem 3.7 The class of sequence $\Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$ is a sequence algebra

Proof: Let $\left(\Delta^{v}_{(\mu,u)}x_{mn}\right),\left(\Delta^{v}_{(\mu,u)}x_{mn}\right)\in \Lambda^{2I_2}_{\Delta^{v}_{(\mu,u)}f}[\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p]^{\eta}$ and 0 <ε< 1. Then the result follows from the following inclusion relation:

$$\left\{(r,s)\in N\times N:\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\otimes\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}\right\}\in I_2$$

$$\supseteq\left\{\left\{(r,s)\in N\times N:\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^{v}_{(\mu,u)}x_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}<\varepsilon\right\}\in\right\}$$

$$\cap\left\{\left\{(r,s)\in N\times N:\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left|\Delta^{v}_{(\mu,u)}y_{mn}\right|\right)^{1/m+n}, d_1(x_1,0),\ldots,d_n(x_{n-1},0))\right\|_p\right]^{\eta_{mn}}<\varepsilon\right\}\in I_2\right\}$$

Similarly we can prove the result for other cases.

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