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Hamilton-Jacobi Formalism of Singular Lagrangians with Linear Accelerations

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Abstract

This paper examined a new model for solving mechanical problems of second-order linear Lagrangian systems, using the Hamilton-Jacobi formalism. Lagrangians linear in accelerations with coefficients given by functions of coordinates alone yield primary constraints. It is shown that the equations of motion can be obtained from the action integral and these equations are equivalent to the canonical method.

Keywords: Hamilton-Jacobi, linear acceleration, action integral

1. Introduction

The canonical formalism for investigating singular systems has been developed by (Rabei & Guler, 1992; Pimentel & Teixeiria, 1996, 1998). A set of Hamilton-Jacobi partial differential equations was obtained and the equations of motion were written as total differential equations.

The Hamilton-Jacobi treatment has been studied for singular Lagrangians (Rabei et al., 2004). The Hamilton-Jacobi functions in configuration space have been obtained by solving the HJPDEs. This has led to another approach for solving mechanical problems for these singular systems.

Singular Lagrangians with linear velocities have been studied (Rabei et al., 2003) by using the canonical method. In this method, the integrable action was obtained directly without considering the total variation of constraints. In this paper, we wish to extend the model for second-order linear Lagrangian.

More recently, the path integral quantization of Lagrangians with linear accelerations has been investigated (Hasan, 2014) by using the canonical method. It is shown that by calculating the integrable action and constructing the wave function, the quantization has been carried out.

This paper is organized as follow. In Section 2, a new model of singular Lagrangian with linear acceleration is proposed. In Section 3, several illustrative examples are examined. The work closes with some concluding remarks in Section 4.

2. The Model of Hamilton-Jacobi Formalism for Lagrangian with Linear Acceleration

The general form of a second-order linear Lagrangian is

$$L(q_i,\dot{q}_i,\ddot{q}_i)=a_i(q_j,\dot{q}_j)\ddot{q}_i-V(q_j,\dot{q}_j) \tag{2.1}$$

The associated Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)=0 \tag{2.2}$$

Have at most order three. Lagrangians linear in accelerations with coefficients given by functions of coordinates alone yield primary constraints. If $a_i(q,\dot{q})=a_i(q)$, and let $V(q,\dot{q})=V(q)$, then the general form of a second-order linear Lagrangian becomes

$$L(q_i,\dot{q}_i,\ddot{q}_i)=a_i(q_j)\ddot{q}_i-V(q_j) \tag{2.3}$$

The generalized momenta p_i, π_i conjugate to the generalized coordinates q_i, $\dot{q}_i$, respectively:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right);$$

$$p_i = -\frac{da_i}{dt} = b_i(\dot{q}_j) = -H_i^p \tag{2.4}$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i} = a_i(q_j) = -H_i^\pi \tag{2.5}$$

Equations (2.4) and (2.5) become

$$H_i'^\pi\left(q_i, \dot{q}_i, p_i, \pi_i\right) = \pi_i + H_i^\pi = 0;$$

$$H_i'^\pi\left(q_i, \dot{q}_i, p_i, \pi_i\right) = \pi_i - a_i = 0 \tag{2.6}$$

$$H_i'^p\left(q_i, \dot{q}_i, p_i, \pi_i\right) = p_i + H_i^p = 0$$

$$H_i'^p\left(q_i, \dot{q}_i, p_i, \pi_i\right) = p_i - b_i = 0 \tag{2.7}$$

Equations (2.6) and (2.7) are called primary constraints (Dirac, 1950).

The canonical Hamiltonian H_0 is given by:

$$H_0 = p_i\dot{q}_i + \pi_i\ddot{q}_i \; - L = b_i(\dot{q}_j)\dot{q}_i + V(q_j) \tag{2.8}$$

The corresponding HJPDEs

$$H_\circ' = \frac{\partial S}{\partial t} + b_i\dot{q}_i + V(q_j) = 0 \tag{2.9a}$$

$$H_\circ'^\pi = \frac{\partial S}{\partial \dot{q}} - a_i = 0 \tag{2.9b}$$

$$H_i'^p = \frac{\partial S}{\partial q} - b_i = 0 \tag{2.9c}$$

The equations of motion are obtained as total differential equations follows:

$$dq_i = \frac{\partial H_0'}{\partial p_i}dt + \frac{\partial H_j'^p}{\partial p_i}dq_j + \frac{\partial H_j'^\pi}{\partial p_i}d\dot{q}_j = dq_j \tag{2.10a}$$

$$d\dot{q}_i = \frac{\partial H_0'}{\partial \pi_i}dt + \frac{\partial H_j'^p}{\partial \pi_i}dq_j + \frac{\partial H_j'^\pi}{\partial \pi_i}d\dot{q}_j = d\dot{q}_j \tag{2.10b}$$

$$dp_i = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_j'^p}{\partial q_i}dq_j - \frac{\partial H_j'^\pi}{\partial q_i}d\dot{q}_j = -\frac{\partial V}{\partial q_i}dt + \frac{\partial a_j}{\partial q_i}d\dot{q}_j \tag{2.10c}$$

$$d\pi_i = -\frac{\partial H_0'}{\partial \dot{q}_i}dt - \frac{\partial H_\alpha'^p}{\partial \dot{q}_i}dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial \dot{q}_i}d\dot{q}_\alpha = -b_i(\dot{q}_j)dt \tag{2.10d}$$

The set of Equations (2.10) are integrable (Muslih & Guler, 1998), the total variation of Equation (2.6) and Equation (2.7) can be written as:

$$\begin{aligned} dH_i'^\pi &= d\pi_i - da_i = 0 \\ &= -b_i(\dot{q}_j)dt - da_i(q_j) \end{aligned} \tag{2.11}$$

$$\begin{aligned} dH_i'^p = dp_i - db_i = 0 &= -\frac{\partial V}{\partial q_i}dt + \frac{\partial a_j}{\partial q_i}d\dot{q}_j - db_i(\dot{q}_j) \\ &= -\frac{\partial V}{\partial q_i}dt + \frac{\partial a_j}{\partial q_i}d\dot{q}_j - db_i(\dot{q}_j) \end{aligned} \tag{2.12}$$

So, we have

$$\frac{\partial b_i(\dot{q})}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial a_j(q)}{\partial q_i} d\dot{q}_j = -\frac{\partial V}{\partial q_i} dt \tag{2.13}$$

which is equivalent to

$$\frac{\partial a_i(q)}{\partial q_j} d\dot{q}_j + \frac{\partial a_j(q)}{\partial q_i} d\dot{q}_j = \frac{\partial V}{\partial q_i} dt \tag{2.14}$$

Or

$$\ddot{q}_j = f_{ij}^{-1} \frac{\partial V(q)}{\partial q_i};$$

Defining the symmetric matrix f_{ij} as

$$f_{ij} = \frac{\partial a_i(q)}{\partial q_j} + \frac{\partial a_j(q)}{\partial q_i} \tag{2.15}$$

If the inverse of the matrix f_{ij} exist, then we can solve all the dynamics q_i, while if the rank of the matrix f_{ij} is n-R, then we can solve the dynamics q_a in terms of independent parameters $(t, q_\alpha, \dot{q}_\alpha)$, $\alpha = 1, 2, ..., R$.

The total derivative of the Hamilton-Jacobi function can be obtained as:

$$dS = \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial S}{\partial t} dt \tag{2.16}$$

Using the HJPDEs Equations (2.9), we get

$$dS = a_i d\dot{q}_i - V dt \tag{2.17}$$

One can integrate the above Equation (2.17) to give

$$S = \int a_i d\dot{q}_i - \int V dt \tag{2.18}$$

We can use the fact that

$$\int d(a_i \dot{q}_i) = a_i \dot{q}_i = \int a_i d\dot{q}_i + \int \dot{q}_i da_i,$$

Equation (2.18) reduces to

$$S = \frac{1}{2}[a_i \dot{q}_i + \int a_i d\dot{q}_i - \int \dot{q}_j da_j] - \int V dt \tag{2.19}$$

By some rearragment, Equation (2.19) becomes

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \int [\dot{q}_j da_j - a_i d\dot{q}_i + 2V dt] \tag{2.20}$$

And using the fact that

$$\frac{d}{dt}(q_j da_j) = -q_j db_j + \dot{q}_j da_j,$$

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \int \frac{d}{dt}(q_j da_j) - \frac{1}{2} \int [q_j db_j - a_i d\dot{q}_i + 2V dt]$$

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_j da_j - \frac{1}{2} \int [q_j db_j - a_i d\dot{q}_i + 2V dt] \tag{2.21}$$

Assuming that the function $a_i(q)$ and $V(q)$ satisfy the following conditions

$$q_j \frac{\partial a_i}{\partial q_j} = a_i, \quad \frac{\partial V}{\partial q_j} q_j = 2V$$

Equation (2.21) becomes

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_j da_j - \frac{1}{2} \int q_j [db_j - \frac{\partial a_i}{\partial q_j} d\dot{q}_i + \frac{\partial V}{\partial q_j} dt] \tag{2.22}$$

The action S to be an integrable function, the terms in the brackets must be zero, i.e.

$$db_j - \frac{\partial a_i}{\partial q_j} d\dot{q}_i + \frac{\partial V}{\partial q_j} dt = 0 \tag{2.23}$$

Equation (2.23) gives the equation of motion for the coordinates q_j.

3. Examples

3.1 The First Example

Consider the following singular Lagrangian:

$$L = -q_1\ddot{q}_1 + q_2\ddot{q}_2 - \frac{1}{2}(q_2^2 + q_2^2) \tag{3.1}$$

The potential of this Lagrangian is given by

$$V = \frac{1}{2}(q_1^2 + q_2^2)$$

and the coefficients a_1 and a_2 are

$$a_1 = -q_1, \quad a_2 = q_2$$

The generalized momenta by using Equation (2.4) and Equation (2.5) are:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_1}\right) = \dot{q}_1 = -H_1^p; \tag{3.2a}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_2}\right) = -\dot{q}_2 = -H_2^p; \tag{3.2b}$$

$$\pi_1 = \frac{\partial L}{\partial \ddot{q}_1} = -q_1 = -H_1^\pi; \tag{3.2c}$$

$$\pi_2 = \frac{\partial L}{\partial \ddot{q}_2} = q_2 = -H_2^\pi \cdot \tag{3.2d}$$

From Equation (2.6) and Equation (2.7) the primary constraints are given as

$$H_1'^\pi = \pi_1 + q_1; \tag{3.3a}$$

$$H_2'^\pi = \pi_2 - q_2; \tag{3.3b}$$

$$H_1'^p = p_1 - \dot{q}_1; \tag{3.3c}$$

$$H_2'^p = p_2 + \dot{q}_2. \tag{3.3d}$$

The canonical Hamiltonian H_0 is given by

$$H_\circ = \dot{q}_1^2 - \dot{q}_2^2 + \frac{1}{2}(q_1^2 + q_2^2) \cdot \tag{3.4}$$

Making use of (2.23), we can obtain the equation of motion for q_1 and q_2

$$d\dot{q}_1 + q_1 dt + d\dot{q}_1 = 0, \tag{3.5a}$$

$$-d\dot{q}_2 + q_2 dt - d\dot{q}_2 = 0. \tag{3.5b}$$

These equations are given by

$$2\ddot{q}_1 + q_1 = 0; \tag{3.6a}$$

$$2\ddot{q}_2 - q_2 = 0. \tag{3.6b}$$

Equations (3.6) have the following solutions

$$q_1 = A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}; \tag{3.7a}$$

$$q_2 = Ae^{t/\sqrt{2}} + Be^{-t/\sqrt{2}}. \tag{3.7b}$$

3.2 The Second Example

Let consider the singular Lagrangian:

$$L = q_2\ddot{q}_1 - q_1\ddot{q}_2 - q_3\ddot{q}_3 - \frac{1}{2}(q_1^2 + q_2^2 + q_3^2)\cdot \tag{3.8}$$

The potential of this Lagrangian is given by

$$V = \frac{1}{2}(q_1^2 + q_2^2 + q_3^3), \tag{3.9}$$

and the coefficients a_1, a_2 and a_3 are

$$a_1 = q_2, \quad a_2 = -q_1, \quad a_3 = -q_3$$

The generalized momenta by using Equation (2.4) and Equation (2.5) are:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_1}\right) = -\dot{q}_2 = -H_1^p; \tag{3.10a}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_2}\right) = \dot{q}_1 = -H_2^p; \tag{3.10b}$$

$$p_3 = \frac{\partial L}{\partial \dot{q}_3} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_3}\right) = \dot{q}_3 = -H_3^p; \tag{3.10c}$$

$$\pi_1 = \frac{\partial L}{\partial \ddot{q}_1} = q_2 = -H_1^\pi; \tag{3.10d}$$

$$\pi_2 = \frac{\partial L}{\partial \ddot{q}_2} = -q_1 = -H_2^\pi; \tag{3.10e}$$

$$\pi_3 = \frac{\partial L}{\partial \ddot{q}_3} = -q_3 = -H_3^\pi\cdot \tag{3.10f}$$

By Equation (2.6) and Equation (2.7) the primary constraints are given as

$$H_1'^\pi = \pi_1 - q_2; \tag{3.11a}$$

$$H_2'^\pi = \pi_2 + q_1; \tag{3.11b}$$

$$H_3'^\pi = \pi_3 + q_3; \tag{3.11c}$$

$$H_1'^p = p_1 + \dot{q}_2; \tag{3.11d}$$

$$H_2'^p = p_2 - \dot{q}_1; \tag{3.11e}$$

$$H_3'^p = p_3 - \dot{q}_3\cdot \tag{3.11f}$$

The canonical Hamiltonian H_0 is given by

$$H_\circ = \dot{q}_3^2 + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2)\cdot \tag{3.12}$$

Making use of (2.23), we can obtain the equation of motion for q_3

$$d\dot{q}_3 + q_3 dt + d\dot{q}_3 = 0\cdot \tag{3.13}$$

This equation can be written as

$$2\ddot{q}_3 + q_3 = 0, \tag{3.14}$$

Which have the following solution

$$q_3 = A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}\cdot \tag{3.15}$$

4. Conclusion

This paper investigated the Hamilton-Jacobi formalism for singular Lagrangian with linear acceleration. Lagrangians linear in accelerations with coefficients given by functions of coordinates alone yield primary constraints. It is proven that the total derivative of the Hamilton-Jacobi function has been constructed using the HJPDEs and Hamilton-Jacobi function is integrable. It is shown that both the equations of motion and the integrable action are obtained from the integrability conditions and the number of independent parameters are determined from the rank of matrix f_{ij} .

References

Dirac, P. A. M. (1950). Generalized Hamiltonian Dynamics. *Canadian Journal of Mathematical Physics, 2*, 129-148. http://dx.doi.org/10.4153/CJM-1950-012-1

Hasan, E, H. (2014). Path Integral Quantization of Lagrangians with Linear Accelerations. *European Scientific Journal, 10*(3), 331-345.

Muslih, S. I., & Guler, Y. (1998). Is gauge fixing of constrained systems necessary. *Il Nuovo Cimento B, 113*, 277-290.

Pimentel, R. G., & Teixeira, R. G. (1996). Hamilton-Jacobi Formulation for Singular Systems with Second-Order Lagrangians. *Il Nuovo Ciemento B, 111*, 841-854. http://dx.doi.org/10.1007/BF02749015

Pimentel, R. G., & Teixeira, R. G. (1998). Generalization of the Hamilton-Jacobi Approach for Higher-Order Singular Systems. *Il Nuovo Ciemento B, 113*, 805-820.

Rabei, E. M., & Guler, Y. (1992). Hamilton-Jacobi Treatment of Second-Class Constraints. *Physical Review A, 46*(6), 3513-3515. http://dx.doi.org/10.1103/PhysRevA.46.3513

Rabei, E. M., Hasan, E. H., & Ghassib, H. B. (2004). Hamilton-Jacobi Treatment of Constrained Systems with Second-Order Lagrangians. *International Journal of Theoretical Physics, 43*(4), 1073-1096. http://dx.doi.org/10.1023/B:IJTP.0000048601.92005.fe

Rabei, E. M., Nawafleh K. I., Abdelrahman, Y. S., & Omari, H. Y. R. (2003). Hamilton-Jacobi treatment of lagrangians with linear velocities. *Modern Physics Letters A, 18*(23), 1591-1596. http://dx.doi.org/10.1142/S0217732303011277

Rabei, E. M., Nawafleh K. I., & Ghassib, H. B. (2004). Hamilton-Jacobi Treatment of Constrained Systems. *International Journal of Modern Physics A, 19*, 347-354. http://dx.doi.org/10.1142/S0217751X04017719