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Application of Vandermonde Determinant in Combination Mathematics

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Abstract

Given an application of Vandermonde determinant in Combination mathematics, that is, proved several important combinatorial identities by using Vander monde determinant.

Keywords: Vandermonde determinant, Application, Combinatorial identities

1. Introduction

There are many methods about the proof of the combinatorial identities , such as direct checking computations , making use of exponential and generating function , permanent , number theory , differential and integration . In this paper, we proved several important combinatorial identities with Vandermonde determinant by using algebraic method, which is simple and clear than any other methods.

2. Lemmas

Lemma 1 Assume D_n be a Vandermonde determinant with order n composed by $1,2,\cdots,n$. M_j be confactor of D_n deleting row n and column j .

Then

$$M_j = \prod_{m=1}^{n-2} m! C_{n-1}^{n-j}, \qquad j = 1,2,\cdots,n \tag{1}$$

Proof when $j=1$; equality (1) is right.

When $1 < j < n$;

$$M_j = \frac{\prod_{m=2}^{n}(m-1)}{j-1} \cdot \frac{\prod_{m=3}^{n}(m-2)}{j-2} \cdot \cdots \cdot \frac{\prod_{m=j}^{n}(m-j+1)}{j-(j-1)} \cdot \prod_{m=j+2}^{n}(m-j-1)\cdots[n-(n-1)]$$

$$= \frac{(n-1)!(n-2)!\cdots(n-j+1)!(n-j-1)!\cdots 2!1!}{(j-1)!}$$

$$= \prod_{m=1}^{n-2} m! \frac{(n-1)!}{(j-1)!(n-1)!}$$

$$= \prod_{m=1}^{n-2} m! C_{n-1}^{n-j}.$$

Equality (1) is right.

When $j=n$; it is easy to see that equality (1) is right .

Hence: equality (1) is right .

Lemma 2 Assume D_n be a Vandermond determinant with order n composed by $1,2,\cdots,n$. S_j be cofactor of D_n deleting row 1 and column j .

Then

$$S_j = \prod_{m=1}^{n-1} m! C_n^j, j = 1,2,\cdots n. \tag{2}$$

Proof $$S_j = \frac{n!}{j} M_j = \frac{n!}{j}\prod_{m=1}^{n-2} m! C_{n-1}^{n-j} = \prod_{m=1}^{n-1} m! C_n^j.$$

3. Theorems

Theorem 1: $$\sum_{j=1}^{n} (-1)^{n+j} j^{n-1} C_n^j = n!. \qquad (3)$$

Proof assume that

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{n-1} & \cdots & n^{n-1} \end{vmatrix} = \prod_{m=1}^{n-1} m!, \qquad (4)$$

expand D_n along row n, from Lemma 1 we can derive :

$$D_n = (-1)^{n+1}\prod_{m=1}^{n-2} m! C_{n-1}^{n-1} + (-1)^{n+2} 2^{n-1}\prod_{m=1}^{n-2} m! C_{n-1}^{n-2} + \cdots$$

$$+ (-1)^{n+j} j^{n-1}\prod_{m=1}^{n-2} m! C_{n-1}^{n-j} + \cdots + (-1)^{2n-1} n^{n-1}\prod_{m=1}^{n-2} m! C_{n-1}^{0}$$

$$= \sum_{j=1}^{n} (-1)^{n+j} j^{n-1}\prod_{m=1}^{n-2} m! C_{n-1}^{n-j}$$

$$= \prod_{m=1}^{n-1} m! \quad \text{(from equality (4))},$$

Thus

$$\sum_{j=1}^{n} (-1)^{n+j} j^{n-1} C_n^j = n!.$$

Theorem 2: $$\sum_{j=1}^{n} (-1)^{j+1} C_n^j = 1. \qquad (5)$$

Proof Assume D_n be a Vandermond determinant with order n composed by $1,2,\cdots,n$ .expand D_n along row 1 , from Lemma 1 we can derive :

$$D_n = S_1 + (-1)^{1+2} S_2 + \cdots + (-1)^{1+j} S_j + \cdots + (-1)^{1+n} S_n$$

$$= \prod_{m=1}^{n-1} m! C_n^1 + (-1)^{1+2}\prod_{m=1}^{n-1} m! C_n^2 + \cdots + (-1)^{1+j}\prod_{m=1}^{n-1} m! C_n^j + \cdots + (-1)^{1+n}\prod_{m=1}^{n-1} m! C_n^n$$

$$= \sum_{j=1}^{n} (-1)^{1+j}\prod_{m=1}^{n-1} m! C_n^j = \prod_{m=1}^{n-1} m! \quad \text{(from equality (4))},$$

Therefore

$$\sum_{j=1}^{n} (-1)^{j+1} C_n^j = 1.$$

Theorem 3: $$\sum_{j=0}^{n-1} (-1)^j C_n^j (n-j)^i = 0, \quad (1 \le i \le n). \qquad (6)$$

Proof Assume D_n be a Vandermond determinant with order n composed by $1,2,\cdots,n$ .The sum of the product of both the element of row 1 of D_n and the algebra cofactor of the corresponding element of row n of D_n is zero, from Lemma 1 we can derive:

$$0 = \sum_{k=1}^{n} (-1)^{n+k} M_k K^{i-1} = \sum_{k=1}^{n} (-1)^{n+k}\prod_{m=1}^{n-2} m! C_{n-1}^{n-k} K^{i-1}.$$

Let $j = n-k$, from equality (2) we can derive:

$$0 = \sum_{j=1}^{n-1} (-1)^{2n-j} \sum_{m=1}^{n-2} m! C_{n-1}^{j} (n-j)^{i-1},$$

Thus

$$0 = \sum_{j=0}^{n-1} (-1)^{j} C_{n-1}^{j} (n-j)^{i-1}. \tag{7}$$

Since

$$C_{n-1}^{j} = \frac{(n-1)!}{j!(n-1-j)!} = \frac{n!}{j!(n-j)!} \cdot \frac{n-j}{n} = \frac{n-j}{n} C_{n}^{j}, \tag{8}$$

Substitute (8) into (7) we can derive:

$$\sum_{j=0}^{n-1} (-1)^{j} C_{n}^{j} (n-j)^{i} = 0, \qquad (1 \le i \le n).$$

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