



A General Projection Method for the System of Relaxed Cocoercive Variational Inequalities in Hilbert Spaces

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Abstract

In this paper, we consider a new algorithm for a generalized system for relaxed cocoercive nonlinear inequalities in Hilbert spaces by the convergence of projection methods. Our results extend and improve the recent ones announced by many others.

Keywords: Relaxed cocoercive nonlinear variational inequality, Projection method, Relaxed cocoercive mapping

1. Introduction and preliminaries

Variational inequalities introduced by Stampacchia in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, see [1-6] and references therein.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Let C be a closed convex subset of H and let $A: C \rightarrow H$ be a nonlinear mapping. Let P_C be the projection of H onto the convex subset C . The classical variational inequality which denoted by $VI(C, A)$ is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \forall v \in C. \quad (1.1)$$

Recall that A is said to be relaxed (u, v) -cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-u)\|Ax - Ay\|^2 + v\|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (1.2)$$

Consider a system (SNVID) of nonlinear variational inequality problems as follows:

Find $x^*, y^*, z^* \in C$ such that

$$\langle sT_1(x^*, y^*, z^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, s > 0, \quad (1.3)$$

$$\langle tT_2(x^*, y^*, z^*) + y^* - z^*, x - y^* \rangle \geq 0, \quad \forall x \in C, t > 0, \quad (1.4)$$

$$\langle rT_3(x^*, y^*, z^*) + z^* - x^*, x - z^* \rangle \geq 0, \quad \forall x \in C, r > 0, \quad (1.5)$$

One can easily see the SNVID problems (1.3), (1.4) and (1.5) are equivalent to the following projection formulas:

$$x^* = P_C[y^* - sT_1(y^*, z^*, x^*)], \quad s > 0,$$

$$y^* = P_C[z^* - tT_2(z^*, x^*, y^*)], \quad t > 0,$$

$$z^* = P_C[x^* - rT_3(x^*, y^*, z^*)], \quad r > 0,$$

respectively, where P_C is the projection of H onto C .

2. Algorithms

In this section, we consider an introduction of the general three-step models for the projection methods and its special form can be applied to the convergence analysis for the projection

methods in the context of the approximation solvability of the SNVID problems (1.3)-(1.5).

Algorithm 2.1. For any $x_0, y_0 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative processes:

$$\begin{aligned} z_{n+1} &= P_C[x_{n+1} - rT_3(x_{n+1}, y_{n+1}, z_n)], \\ y_{n+1} &= P_C[z_{n+1} - tT_2(z_{n+1}, x_{n+1}, y_n)], \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P_C[x_{n+1} - rT_3(x_{n+1}, y_{n+1}, z_n)], \end{aligned} \quad (2.1)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ for all $n \geq n_0$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 2.1. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 2.1. A mapping $T: C \times C \times C \rightarrow H$ is said to be relaxed (u, v) -cocoercive if there exist constants $u, v > 0$ such that, for all $x, x' \in C$,

$$\langle T(x, y, z) - T(x', y', z'), x - x' \rangle \geq (-u)\|T(x, y, z) - T(x', y', z')\|^2 + v\|x - x'\|^2, \quad \forall y, y', z, z' \in C.$$

Definition 2.2. A mapping $T: C \times C \times C \rightarrow H$ is said to be μ -Lipschitz continuous in the first variable if there exists a constant $\mu > 0$ such that, for all $x, x' \in C$,

$$\|T(x, y, z) - T(x', y', z')\| \leq \mu\|x - x'\|, \quad \forall y, y', z, z' \in C$$

3. Main results

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H . Let $T_1: C \times C \times C \rightarrow H$ be a relaxed (u_1, v_1) -cocoercive and μ_1 -Lipschitz continuous mapping in the first variable, $T_2: C \times C \times C \rightarrow H$ be a relaxed (u_2, v_2) -cocoercive and μ_2 -Lipschitz continuous mapping in the first variable, $T_3: C \times C \times C \rightarrow H$ be a relaxed (u_3, v_3) -cocoercive and μ_3 -Lipschitz continuous mapping in the first variable. Suppose that $x^*, y^*, z^* \in C$ are solutions of the SNVID problems (1.3)-(1.5) and $\{x_n\}, \{y_n\}, \{z_n\}$ are the sequences generated by Algorithm 2.1. If $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying the following conditions:

- (i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (ii) $0 < s, t, r < \min\left\{\frac{2(v_1 - u_1\mu_1^2)}{\mu_1^2}, \frac{2(v_2 - u_2\mu_2^2)}{\mu_2^2}, \frac{2(v_3 - u_3\mu_3^2)}{\mu_3^2}\right\}$;
- (iii) $v_1 - u_1\mu_1^2, v_2 - u_2\mu_2^2, v_3 - u_3\mu_3^2$;

then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to x^*, y^* and z^* , respectively.

Proof. Since (x^*, y^*, z^*) is the solution to the SNVID problems (1.3)-(1.5), we have

$$\begin{aligned} x^* &= P_C[y^* - sT_1(y^*, z^*, x^*)], \quad s > 0, \\ y^* &= P_C[z^* - tT_2(z^*, x^*, y^*)], \quad t > 0, \\ z^* &= P_C[x^* - rT_3(x^*, y^*, z^*)], \quad r > 0, \end{aligned}$$

Observing (2.1), we obtain

$$\|x_{n+1} - x^*\| = \|(1 - \alpha_n)x_n + \alpha_n P_C[y_n - sT_1(y_n, z_n, x_n) - x^*]\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - s[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|. \quad (3.1)$$

By the assumption that T_1 is relaxed (u_1, v_1) -cocoercive and μ_1 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} &\|y_n - y^* - s[T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)]\|^2 \\ &= \|y_n - y^*\|^2 - 2s\langle y_n - y^*, T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*) \rangle + s^2\|T_1(y_n, z_n, x_n) - T_1(y^*, z^*, x^*)\|^2 \\ &\leq \|y_n - y^*\|^2 + 2su_1\mu_1^2\|y_n - y^*\|^2 - 2sv_1\|y_n - y^*\|^2 + s^2\mu_1^2\|y_n - y^*\|^2 = \theta_1^2\|y_n - y^*\|^2 \end{aligned} \quad (3.2)$$

Where $\theta_1^2 = 1 + 2su_1\mu_1^2 - 2sv_1 + s^2\mu_1^2$. From the conditions (ii) and (iii), we know $\theta_1 < 1$.

Substitute (3.2) into (3.1) yields that

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| \quad (3.3)$$

Now, we estimate

$$\|y_{n+1} - y^*\| = \|P_C[z_{n+1} - tT_2(z_{n+1}, x_{n+1}, y_n)] - y^*\| \leq \|z_{n+1} - z^* - t[T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)]\| \quad (3.4)$$

By the assumption that T_2 is relaxed (u_2, v_2) -cocoercive and μ_2 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} & \|z_{n+1} - z^* - t[T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)]\|^2 \\ &= \|z_{n+1} - z^*\|^2 - 2t\langle z_{n+1} - z^*, T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*) \rangle + t^2 \|T_2(z_{n+1}, x_{n+1}, y_n) - T_2(z^*, x^*, y^*)\|^2 \\ &\leq \|z_{n+1} - z^*\|^2 + 2tu_2\mu_2^2 \|z_{n+1} - z^*\|^2 - 2tv_2 \|z_{n+1} - z^*\|^2 + t^2\mu_2^2 \|z_{n+1} - z^*\|^2 = \theta_2^2 \|z_{n+1} - z^*\|^2 \end{aligned} \quad (3.5)$$

where $\theta_2^2 = 1 + 2tu_2\mu_2^2 - 2tv_2 + t^2\mu_2^2$. From the conditions (ii) and (iii), we know $\theta_2 < 1$. Substituting (3.5) into (3.4) yields that $\|y_{n+1} - y^*\| \leq \theta_2 \|z_{n+1} - z^*\|$, which implies that $\|y_n - y^*\| \leq \theta_2 \|z_n - z^*\|$ (3.6)

Similarly, Substituting (3.6) into (3.3), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta_1 \theta_2 \|z_n - z^*\| \quad (3.7)$$

Next, we show that

$$\|z_{n+1} - z^*\| = \|P_C[x_{n+1} - rT_3(x_{n+1}, y_{n+1}, z_n)] - z^*\| \leq \|x_{n+1} - x^* - r[T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)]\| \quad (3.8)$$

By the assumption that T_3 is relaxed (u_3, v_3) -cocoercive and μ_3 -Lipschitz continuous in the first variable, we obtain

$$\begin{aligned} & \|x_{n+1} - x^* - r[T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)]\|^2 \\ &= \|x_{n+1} - x^*\|^2 - 2r\langle x_{n+1} - x^*, T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*) \rangle + r^2 \|T_3(x_{n+1}, y_{n+1}, z_n) - T_3(x^*, y^*, z^*)\|^2 \\ &\leq \|x_{n+1} - x^*\|^2 + 2ru_3\mu_3^2 \|x_{n+1} - x^*\|^2 - 2rv_3 \|x_{n+1} - x^*\|^2 + r^2\mu_3^2 \|x_{n+1} - x^*\|^2 = \theta_3^2 \|x_{n+1} - x^*\|^2 \end{aligned} \quad (3.9)$$

where $\theta_3^2 = 1 + 2ru_3\mu_3^2 - 2rv_3 + r^2\mu_3^2$. From the conditions (ii) and (iii), we know $\theta_3 < 1$. Substituting (3.9) into (3.8), we obtain $\|z_{n+1} - z^*\| \leq \theta_3 \|x_{n+1} - x^*\|$, which implies $\|z_n - z^*\| \leq \theta_3 \|x_n - x^*\|$ (3.10)

Similarly, substituting (3.10) into (3.7) yields that

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \theta_1 \theta_2 \theta_3 \|x_n - x^*\| \leq [1 - \alpha_n (1 - \theta_1 \theta_2 \theta_3)] \|x_n - x^*\| \quad (3.11)$$

Noticing that $\sum_{n=0}^{\infty} \alpha_n (1 - \theta_1 \theta_2 \theta_3) = \infty$ and Applying Lemma 2.1 into (3.11), we can get the desired conclusion easily. This completes the proof.

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