



Logarithmic Barrier Function Method for Convex Quadratic Programming Problem

Xinghua Wang (Corresponding author)
college of Science, Yanshan University
Qinhuangdao 066004, China
E-mail: wxh_811129@163.com

Wen Liu
college of Science, Yanshan University
Qinhuangdao 066004, China

Pingping Qin & Lifeng Sun
college of Science, Yanshan University
Qinhuangdao 066004, China

Abstract

In this paper, we present a kind of interior point algorithm for solving Convex Quadratic Programming problem using Logarithmic Barrier Function. In which Newton Method and Interior Penalty Function Method are combined to obtain a simple construct and easily calculating algorithm. In each iterate it needs only to solve an equality constrained Quadratic Programming problem.

Keywords: Quadratic Programming, Logarithmic Barrier Function, Newton Method, Convergence

Introduction

Solution of Quadratic Programming problem is an important subject in Mathematical Programming and Industry domain, which has wide practice background. There are many algorithms for Quadratic Programming problem, such as Potential Function Projection Method; Affine Scaling Method; Primal-dual Interior Point Algorithm and so on. In this paper, we improved the Methods of document , we obtain the search direction of d_k by Newton Method and then solve the optimal solution of equality constrained problem.

1. Convergence

To Nonlinear Programming problem: $(P) \min\{f(x)|Ax = b, x \geq 0\}$

In which $f(x) = \frac{1}{2}x^T Qx + c^T x$ $c \in R^n, b \in R^m, A \in R^{n \times m}, Q \in R^{n \times n}, x \in R^n$

Let $\Omega_{int} = \{x \in R^n | Ax = b, x > 0\}$, $Rank(A) = m$, Q is a symmetric positive matrix.

Considering corresponding Logarithmic Barrier Function problem $:(P_{u_k}) \min\left\{f(x) - u_k \sum_{i=1}^n \ln x_i | Ax = b, x > 0\right\}$

In which $u_k > 0$ is a penalty factor .

So to the optimal solution of (P) and (P_{u_k}) , we give such theorem:

Theorem 1: If $f(x)$ is an Convex Function, $\Omega_{int} = \{x | Ax = b, x > 0\} \neq \emptyset$, $\{u_k\}$ is a strict monotone decreasing Sequence of positive numbers which goes to zero, If we let x^* is the optimal solution of (P) ,

$x(u_k)$ is the optimal solution of (P_{u_k}) , Then $x(u_k)$ is the optimal solution of (P) , too.

Proof: First proof $\{P(x_{u_k}, u_k)\}$ is a strict monotone decreasing Sequence with lower-bound.

Because $x_{u_k} \in \Omega_{\text{int}}$

So

$$P(x_{u_{k+1}}, u_{k+1}) \leq P(x_{u_k}, u_{k+1}) = f(x_{u_k}) + u_{k+1}B(x_{u_k}) \leq f(x_{u_k}) + u_k B(x_{u_k}) = P(x_{u_k}, u_k)$$

and

$$P(x_{u_k}, u_k) = f(x_{u_k}) + u_k B(x_{u_k}) \geq f(x_{u_k}) \geq f(x^*)$$

that $\{P(x_{u_k}, u_k)\}$ is strict monotone decreasing and has a lower-bound,

So there exists limiting (P_0) , Next Proof

$$P_0 = f(x^*) \quad (1)$$

We use evidence to the contrary

If $P_0 > f(x^*)$, Then there must exists

$$\varepsilon > 0$$

$$\text{Such that } P_0 > f(x^*) + 2\varepsilon \quad (2)$$

We can see from the continuity of $f(x)$, there exists

$$\bar{x} \in \Omega_{\text{int}} \text{ let } f(\bar{x}) < f(x^*) + \varepsilon$$

Because

$$P(x_{u_k}, u_k) \leq P(\bar{x}, u_k) = f(\bar{x}) + u_k B(\bar{x}) < f(x^*) + \varepsilon + u_k B(\bar{x})$$

Also when k is sufficiently large, there has $u_k B(\bar{x}) < \varepsilon$, So we have

$$\lim_{k \rightarrow \infty} P(x_{u_k}, u_k) \leq f(x^*) + 2\varepsilon$$

That $P_0 \leq f(x^*) + 2\varepsilon$ it is contrary to (2)

Then proof \forall limiting point (x_0) of $x(u_k)$ is the optimal solution of (P)

$$\text{Let } \lim_{k \rightarrow \infty} x_{u_k} = x_0$$

Because $x_{u_k} \in \Omega_{\text{int}}$

So $x_0 \in \Omega_{\text{int}}$

$$f(x_0) \geq f(x^*) \text{ if } f(x_0) > f(x^*) \text{ and } \lim_{k \rightarrow \infty} [f(x_{u_k}) - f(x^*)] = f(x_0) - f(x^*) > 0$$

$$P(x_{u_k}, u_k) - f(x^*) = f(x_{u_k}) - f(x^*) + u_k B(x_{u_k}) \geq f(x_{u_k}) - f(x^*)$$

$$\text{So } \lim_{k \rightarrow \infty} [P(x_{u_k}, u_k) - f(x^*)] \neq 0 \quad P_0 \neq f(x^*)$$

It is contrary to (1)

$$\text{So } f(x_0) = f(x^*)$$

x_0 is the optimal solution of (P)

To the given initial inner point x_1 , when the iteration step get to k ,

$$x = x_k \in \Omega_{\text{int}}$$

the search direction of problem (P_{u_k}) can obtain by such that solution way:

We can consider using proper Quadratic Function to approach target function f in every iteration and then we construct the search direction by the direction which the iteration point points to the minimal point of Quadratic Function, let x_k is the approximate minimal point of f , we take f Taylor expansion at point x_k and use its second-order approximate.

We have

$$f(x) \approx f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{1}{2} (x - x_k)^T \nabla^2 f(x_k) (x - x_k)$$

Because $\nabla^2 f(x_k)$ is a symmetric matrix, if we want to solve the minimal point of $f(x)$, we may let

$$\begin{aligned} \nabla f(x) &= 0 \quad \text{that is} \\ \nabla f(x_k) + \nabla^2 f(x_k) (x - x_k) &= 0 \end{aligned}$$

We take x_{k+1} as the $k+1$ times of approximate of the minimal point of $f(x)$,

$$\begin{aligned} x_{k+1} &= x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \\ d_k &= -[\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \quad (3) \end{aligned}$$

2. Related theorem

Theorem 2 :If $d \neq 0$, then d is the reduction direction of problem (P_{u_k})

Proof: Because $\nabla^2 f(x_k)$ is positive, $[\nabla^2 f(x_k)]^{-1}$ is positive,

$$\text{So } \nabla f(x_k)^T d_k = -\nabla f(x_k)^T [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) < 0$$

d_k is the reduction direction of f at point x_k

Now if we want to ensure the strict feasibility of iteration solution x_{k+1} , we must choose proper step size λ_k

$$\text{Because } x_k \in \Omega_{\text{int}} \quad x_{k+1} = x_k + \lambda_k d_k$$

$$A x_k = A x_k + \lambda_k A d_k = A x_k = b$$

$$\text{And } x_{k+1} = x_k + \lambda_k d_k > 0$$

$$\text{Let } \lambda_k = r \min \left\{ 1, -\frac{x_k^i}{d_k^i} \mid d_k^i < 0 \right\} \quad r \in (0, 1)$$

$$\text{Generally we let } r = 0.995 \quad (4)$$

3. Logarithmic Description

Step0: give the strict initial inner point $x_1 \in \Omega_{\text{int}}$ and tolerance $\varepsilon > 0$

$$u_1 \in (0, 1), \delta \in \left(0, \frac{1}{2} \right), \beta \in (0, 1), \text{ set } k := 1$$

Step1: Compute the solution d_k of (3), if $\|d_k\| < \varepsilon$ terminate, x_k is the optimal solution of problem (P) , otherwise, go to Step2.

Step2: Compute the solution step size λ_k of (4), check if it satisfies :

$$P(x_{u_k}, u_k) - P(x_k + \lambda_k d_k, u_k) \geq -\delta \lambda_k d_k^T \nabla P(x_{u_k}, u_k), \text{ if it satisfies, go to}$$

Step3, otherwise, let $\lambda_k := \frac{\lambda_k}{2}$, go to Step2.

$$\text{Step3: let } x_{k+1} = x_k + \lambda_k d_k, u_{k+1} = \beta u_k.$$

Step4: set $k := k+1$, go to Step1.

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