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The Shapley Value for Stochastic Cooperative Game

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Abstract

In this paper we extend the notion of Shapley value to the stochastic cooperative games. We give the definition of marginal vector to the stochastic cooperative games and we define the Shapley value for this game. Furthermore, we discuss the axioms of the Shapley value and give the proofs of these axioms.

Keywords: Stochastic cooperative game, Shapley value, Marginal vector, Carrier

1. Introduction

In general, the payoffs of a coalition in cooperative games are assumed to be known with certainty. In many cases, however, payoffs to coalitions are uncertain. If the formation of coalitions and allocations has to take place before the payoffs are realized, standard cooperative game theory can not be applied.

Suijs et al. (1995) considered cooperative games with stochastic payoffs, the model introduced by Suijs et al. (1995) is explicitly incorporates preferences on stochastic payoffs for each agent and allows each coalition to choose from several actions.

Suijs et al. (1999) continue on the model introduced by Suijs et al. (1995). They extend the definitions of superadditivity and convexity for TU games to stochastic cooperative games. Furthermore, they show that a stochastic cooperative game has a nonempty core.

In this paper we take the model introduced by Suijs et al. (1999) as a basis. We define the Shapley value of stochastic cooperative games. Furthermore, we discuss the axioms of the Shaley value.

This paper is organized as follows. In section2 we introduce basic definitions concerning stochastic cooperative games. Section3 presents our main results. The axioms for Shapley value of stochastic cooperative game.

2. Stochastic cooperative games

Let us first recall some of the definitions concerning stochastic cooperative games as introduced by Suijs et al. (1999). A stochastic cooperative game is described by a tuple

$\Gamma = \left(N, \{A_S\}_{S\subseteq N}, \{X_S\}_{S\subseteq N}, \{\geq_i\}_{i\in N}\right)$

where N is the set of agents, A_S the nonempty and finite set of actions a coalition S can take, $X_S : A_S \to L^1(R)$ the payoff function of coalition S, assigning to each action $a \in A_S$ a stochastic payoff $X_S(a) \in L^1(R)$ with finite expectation, and $\geq_i$ the preference relation of agent i over the set $L^1(R)$ of stochastic payoffs with finite expectation. We assume that for each player the preferences are complete, transitive and continuous. Furthermore, we assume that $P(X_\varnothing(a) = 0) = 1$ for all $a \in A_\varnothing$. The class of all cooperative games with stochastic payoffs with agent set N is denoted by $SG(N)$.To simplify notation, however, we restrict our attention to the case that each coalition only has one action to take, that is, $|A_S| = 1$ for all $S \subseteq N$. So we can denote a stochastic cooperative game by $\Gamma = (N, \{X_S\}_{S\subseteq N}, \{\geq_i\}_{i\in N})$.

For our definition of the Shapley value of stochastic cooperative game, we first give the definition of marginal vector.

Let (N,v) be a game and let Π_N be the set of all permutations of N. Then the kth coordinate of the marginal vector $m^{\pi}(v)$, $\pi\in\Pi_N$, is defined by

$$m_k^{\pi}(v)=v(\langle j\,|\,\pi(j)\le\pi(k)\rangle)-v(\langle j\,|\,\pi(j)<\pi(k)\rangle)$$

Now we extend the notion of marginal vector to the stochastic cooperative games.

Let $\Gamma=(N,\{X_S\}_{S\subseteq N},\{\ge_i\}_{i\in N})$ be a stochastic cooperative game, and let Π_N be the set of all permutations of N. Then the kth coordinate of the marginal vector $m^{\pi}(\Gamma)$, $\pi\in\Pi_N$ is defined by

$$m_k^{\pi}(\Gamma)=X_{\langle j|\pi(j)\le\pi(k)\rangle}-X_{\langle j|\pi(j)<\pi(k)\rangle}$$

Before we give the definition of Shspley value of stochastic cooperative game, we define some useful notions.

1. (Carrier) Let $\Gamma=(N,\{X_S\}_{S\subseteq N},\{\ge_i\}_{i\in N})$ be a stochastic cooperative game, $T\subseteq N$ is called a carrier for the game if $X_S=X_{S\cap T}$ for all $S\subseteq N$.

There have two properties of the carrier:

(1). Let T be the carrier of Γ, then for all $T\subseteq T'\subseteq N$, T' is the carrier of Γ too.

Because, for all $S\subseteq N$, we have

$$\begin{aligned}X_{S\cap T'}&=X_{(S\cap T')\cap T}\\&=X_{S\cap T}\\&=X_S\end{aligned}$$

(2). Let T be the carrier of Γ, $i\notin T$, then for all $S\subseteq N$, we have

$$\begin{aligned}X_{S\cup\{i\}}&=X_{(S\cup\{i\})\cap T}\\&=X_{S\cap T}\\&=X_S\end{aligned}$$

2. (Dummy player) The player $k\in N$ is a dummy in the stochastic cooperative game $\Gamma=(N,\{X_S\}_{S\subseteq N},\{\ge_i\}_{i\in N})$ if for every $S\subseteq N$ such that $k\notin S$ and $S\bigcup\{k\}\in N$, we have

$$X_{S\cup\{k\}}=X_S=0$$

We can describe the Shapley value as the average of the marginal vectors of the player i to the coalitions.

Then we extend the Shapley value to the stochastic cooperative games.

If $\Gamma=(N,\{X_S\}_{S\subseteq N},\{\ge_i\}_{i\in N})$ is a stochastic cooperative game, then the Shapley value for the player i $(i\in N)$ is

$$\varphi_i(\Gamma)=\sum_{S\subseteq N\setminus\{i\}}\frac{|S|!(n-|S|-1)!}{n!}m_i^{\pi}(\Gamma) \qquad (1)$$

3. Axioms for the Shapley value

We give one axiomatization for the Shapley value of stochastic cooperative game. We consider the axioms:

1. (Efficiency) If T is a carrier of Γ, then $\sum_{i\in T}\varphi_i(\Gamma)=X_T(\Gamma)\cdot$

2. (Symmetry) If π is a permutation of N, such that $X_{\pi S}=X_S$ for all $S\subseteq N$. (2)

Then $\varphi_{\pi i}(\Gamma)=\varphi_i(\Gamma)$ for all $i\in N$.

3. (Additivity) If Φ and $\Psi\in SG(N)$, we have $\varphi_i(\Phi+\Psi)=\varphi_i(\Phi)+\varphi_i(\Psi)$

for all $i\in N$.

Then the Shapley value of stochastic cooperative game is the vector that satisfies the axioms above.

$$\varphi(\Gamma)=(\varphi_1(\Gamma),\varphi_2(\Gamma),\cdots,\varphi_n(\Gamma)) \qquad (3)$$

Lemma1. The definition of (1)

$\varphi(\Gamma)=(\varphi_1(\Gamma),\varphi_2(\Gamma),\cdots,\varphi_n(\Gamma))$

is the Shapley value of $SG(N)$.

Proof. (1) Efficiency axiom. Let T be the carrier of Γ, according to the property (2) of the carrier, we have $X_{S\cup\{i\}}=X_S$,for all $i\in N\setminus T$, for all $S\subseteq N$,then for all $i\in N\setminus T$,we get

$$\varphi_i(\Gamma)=\sum_{S\subseteq N\setminus\{i\}}\frac{|S|!(n-|S|-1)!}{n!}m_i^{\pi}(\Gamma)=0$$

such that

$$\begin{aligned}X_T&=X_{N\cap T}\\&=X_N\\&=\sum_{i\in N}\varphi_i(\Gamma)\\&=\sum_{i\in T}\varphi_i(\Gamma)\end{aligned}$$

(2). Symmetry axiom. Let π be a permutation of N, which satisfies formula (2), then we have $|\pi S|=|S|$, and for each $i\in N$, such that

$$\begin{aligned}\varphi_{\pi i}(\Gamma)&=\sum_{\pi S\subseteq N\setminus\{\pi i\}}\frac{|\pi S|!(n-|\pi S|-1)!}{n!}m_{\pi i}^{\pi}(\Gamma)\\&=\sum_{S\subseteq N\setminus\{i\}}\frac{|S|!(n-|S|-1)!}{n!}m_i^{\pi}(\Gamma)\\&=\varphi_i(\Gamma)\end{aligned}$$

(3). Additivity axiom. Because $\varphi_i(\Gamma)$ is the linear function of Γ, $i=1,2,\cdots,n$, $\varphi(\Gamma)$ satisfies the additivity axiom obviously.

Above all, $\varphi(\Gamma)$ is the Shapley value of Γ.

Lemma2. Define the function X^T:

$$X_S^T=\begin{cases}1 & S\supseteq T\\ 0 & otherwise\end{cases}\quad \text{for all } T\subseteq N,\ T\neq\varnothing.$$

then for each constant $c\geq 0$, cX^T is the characteristic function, and each Shapley value Φ of cX^T satisfies

$$\varphi_i(cX^T)=\begin{cases}0 & i\notin T\\ \dfrac{c}{|T|} & i\in T\end{cases}$$

Proof. It is easy to prove that cX^T is the characteristic function, and T is a carrier of cX^T.

While $i\in N\setminus T$, T and $T\bigcup\{i\}$ are carriers of cX^T, then according to efficiency axiom, we have

$$\begin{aligned}\sum_{j\in T}\varphi_j(cX^T)=cX_T^T=cX_{T\cup\{i\}}^T&=\sum_{j\in T\cup\{i\}}\varphi_j(cX^T)\\&=\sum_{j\in T}\varphi_j(cX^T)+\varphi_i(cX^T)\end{aligned}$$

Hence $\varphi_i(cX^T)=0$, for all $i\notin T$.

While $i,j\in T$, and $i\neq j$, let π be a permutation of N, such that

$$\pi k=\begin{cases}j, & k=i,\\ i, & k=j,\\ k, & k\neq i,j.\end{cases}$$

First we prove $cX_{\pi S}^T=cX_S^T$ for all $S\subseteq N$.

The reason is: If $S \supseteq T$, then $\pi S \supseteq \pi T = T$,such that $cX^T_{\pi S} = c = cX^T_S$.

Otherwise, then while $i \notin S$, we have $j \notin \pi S$;

while $j \notin S$,we have $i \notin \pi S$;

while $k \notin S$ and $k \in T \setminus \{i, j\}$, we have $k \notin \pi S$, such that $\pi S \supseteq N \setminus T$.

Hence $cX^T_{\pi S} = 0 = cX^T_S$.

According to the symmetry axiom, we have $\varphi_j(cX^T) = \varphi_{\pi i}(cX^T) = \varphi_i(cX^T)$, then from the efficiency axiom, we know $|T|\varphi_i(cX^T) = \sum_{i\in T}\varphi_i(cX^T) = cX^T = c$, for all $i \in T$,hence

$$\varphi_i(cX^T) = \frac{c}{|T|}, \text{ for all } i \in T$$

Lemma3. The characteristic function of stochastic cooperative game Γ can describe as

$$X^{cT} = \sum_{\varnothing \neq T \subseteq N} c_T X^T, \tag{4}$$

where X^T is defined in Lemma2, and $c_T = \sum_{U\subseteq T}(-1)^{|T|-|U|}X_U$. (5)

Proof. According to the definitions of X^T and c_T, for all $S \subseteq N$, we have

$$\begin{aligned}
(\sum_{\varnothing \neq T \subseteq N} c_T X^T)(S) = \sum_{\varnothing \neq T \subseteq N} c_T X^T_S
&= \sum_{\varnothing \neq T \subseteq N} c_T \\
&= \sum_{\varnothing \neq T \subseteq N}\sum_{U\subseteq T}(-1)^{|T|-|U|}X_U \\
&= \sum_{U\subseteq S}(\sum_{U\neq T\subseteq S}(-1)^{|T|-|U|})X_U \\
&= \sum_{U\subseteq S}\sum_{t=|U|}^{|S|}(-1)^{t-|U|}\binom{|S|-|U|}{t-|U|}X_U
\end{aligned}$$

Then according to the binomial theorem of combinatorial mathematics, we have

$\sum_{i=0}^{r}(-1)^i\binom{r}{i} = 0$ for all $r \in N_+$, then $\sum_{t=|U|}^{|S|}(-1)^{t-|U|}\binom{|S|-|U|}{t-|U|} = 0$, for all $U \subset S$.

So $(\sum_{\varnothing \neq T \subseteq N} c_T X^T)(S) = X_S$, for all $S \subseteq N$. We have proved formula (4).

Theorem1. If $\Gamma \in SG(N)$ is a stochastic cooperative game, then the definition of $\varphi(\Gamma)$ in formula (3) is the only Shapley value of Γ.

Proof. From Lemma1. we know that $\varphi(\Gamma)$ which defined in formula (3) is indeed the Shapley value of Γ.

Then by Lemma3. we have that

$$X_S = \sum_{\substack{\varnothing \neq T \subseteq N \\ c_T \geq 0}} c_T X^T_S - \sum_{\substack{\varnothing \neq T \subseteq N \\ c_T < 0}} (-c_T) X^T_S.$$

According to Lemma2. the two polynomials of the upper formula both are characteristic functions. And their difference X_S is also a characteristic function. Hence it can be proved by the additivity axiom that each Shapley value $\Phi(\Gamma)$ of Γ satisfies

$$\varphi_i(\Gamma) = \sum_{\substack{\varnothing \neq T \subseteq N \\ c_T \geq 0}} \varphi_i(c_T X^T) - \sum_{\substack{\varnothing \neq T \subseteq N \\ c_T < 0}} \varphi_i(-c_T X^T)$$

for all $i \in N$. So from Lemma2. we know that

$$\varphi_i(\Gamma) = \sum_{\substack{i\in T\subseteq N \\ c_T\geq 0}} \frac{c_T}{|T|} - \sum_{\substack{i\in T\subseteq N \\ c_T<0}} \frac{-c_T}{|T|} = \sum_{i\in T\subseteq N} \frac{c_T}{|T|}$$

for all $i \in N$. And we know the number c_T is unique derived by T and X_S, hence $\varphi_i(\Gamma)$ is unique derived by X_S, N and i, that is to say the Shapley value $\Phi(\Gamma)$ is unique derived by X_S and N.

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