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A Note on Two Theorems of C. Dong and J. Wang Concerning Combinatorial Identities

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Abstract

In a recent paper C. Dong and J. Wang rederived three classical combinatorial identities by applying a special Vandermonde determinant. Two of their results, however, turn out to be incorrectly stated. This note presents counterexamples along with revised versions of the results mentioned.

Keywords: Vandermonde determinant, Identities involving binomial coefficients

1. Introduction

C. Dong and J. Wang applied a special Vandermonde determinant in order to establish a couple of well-known combinatorial identities in an elementary way (Dong & Wang, 2007). Unfortunately, two of these are incorrectly stated. In the following we present counterexamples as well as revised versions of the respective theorems in the cited paper.

Let V_n denote the n-square Vandermonde matrix (cf. Lancaster & Tismenetsky, 1985, p. 35, Exercise 6) of the integers $1, 2, \ldots, n,$

$$V_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{bmatrix},$$

and let $D_n = \det(V_n)$ be the determinant of V_n. Furthermore, let M_j be the minor of the entry in the nth row and jth column of V_n, and let S_j be the minor of the entry in the first row and jth column of V_n.

Lemma 1 (Dong & Wang, 2007, p. 24, Eq. (1)).

$$M_j = \left(\prod_{m=1}^{n-2} m!\right)\binom{n-1}{n-j}, \quad j = 1,2,\ldots,n. \tag{1}$$

Lemma 2 (Dong & Wang, 2007, p. 24, Eq. (2)).

$$S_j = \left(\prod_{m=1}^{n-1} m!\right)\binom{n}{j}, \quad j = 1,2,\ldots,n. \tag{2}$$

Using (1) and (2) Dong and Wang obtained, among others, the following identities for positive integers n (Dong & Wang, 2007, p. 25, Eqs. (3) and (6), resp.):

$$\sum_{j=1}^{n} (-1)^{n+j} j^{n-1}\binom{n}{j} = n!. \tag{3}$$

$$\sum_{j=0}^{n-1}(-1)^j (n-j)^i \binom{n}{j} = 0, \quad 1 \le i \le n, \tag{4}$$

2. Counterexamples to Eqs. (3) and (4)

In their current form, Eqs. (3) and (4) turn out to be incorrect.

Example 1. Eq. (3) is false. For if, e. g., $n = 4$, (3) would imply

$$\sum_{j=1}^{4}(-1)^{4+j} j^3 \binom{4}{j} = -4 + 48 - 108 + 64 = 0 \ne 4!.$$

Example 2. Eq. (4) is false in the case $i = n$. For if, e. g., $i = n = 4$, (4) would imply

$$\sum_{j=0}^{3}(-1)^j (4-j)^4 \binom{4}{j} = 256 - 324 + 96 - 4 = 24 \ne 0.$$

3. Restatements of Eqs. (3) and (4) with proofs

The original Eqs. (3) and (4) have to be replaced by the following statements.

Theorem 1 (cf. Gould, 1972, p. 2, Eq. (1.13), first part). *For every nonnegative integer* n *we have*

$$\sum_{j=1}^{n}(-1)^{n+j} j^n \binom{n}{j} = n!. \tag{5}$$

Proof. According to the original proof (Dong & Wang, 2007, p. 25, Theorem 1) the following holds:

$$D_n = \sum_{j=1}^{n}(-1)^{n+j} j^{n-1}\left(\prod_{m=1}^{n-2} m!\right)\binom{n-1}{n-j} = \prod_{m=1}^{n-1} m!.$$

This gives

$$\sum_{j=1}^{n}(-1)^{n+j} j^{n-1}\binom{n-1}{n-j} = (n-1)!.$$

Using the identity (cf. Gould, 1972, p. iv)

$$\binom{n-1}{n-j} = \frac{j}{n}\binom{n}{j}, \quad 1 \le j \le n, \tag{6}$$

we eventually obtain (5). □

Theorem 2 (cf. Gould, 1972, p. 2, Eq. (1.13), second part). *For every nonnegative integer* n *we have*

$$\sum_{j=0}^{n-1}(-1)^j (n-j)^i \binom{n}{j} = 0, \quad 1 \le i \le n-1. \tag{7}$$

Proof. Let $W_n^{(i)}$ denote the n-square matrix obtained from the Vandermonde matrix V_n by replacing the nth row by the ith row, $1 \le i \le n - 1$,

$$W_n^{(i)} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-2} & 3^{n-2} & \cdots & n^{n-2} \\ 1 & 2^{i-1} & 3^{i-1} & \cdots & n^{i-1} \end{bmatrix}.$$

Since $W_n^{(i)}$ is singular by construction, expansion of $\det(W_n^{(i)})$ by the nth row gives

$$0 = \det(W_n^{(i)}) = \sum_{j=1}^{n} (-1)^{n+j} j^{i-1} M_j .$$

Using Lemma 1 and Eq. (6) we get

$$0 = \sum_{j=1}^{n} (-1)^{n+j} j^{i-1} \left(\prod_{m=1}^{n-2} m! \right) \binom{n-1}{n-j} =$$
$$= \sum_{j=1}^{n} (-1)^{n+j} j^{i} \left(\prod_{m=1}^{n-2} m! \right) \frac{1}{n} \binom{n}{j}.$$

Replacing j by $n - j$ and by the symmetry of the binomial coefficients we eventually obtain (7). □

Remark. The arguments used in the proof of Theorem 2 still hold when $W_n^{(i)}$ is replaced by the matrix

$$X_n^{(i)} = \begin{bmatrix} 1 & 2^{i-1} & 3^{i-1} & \cdots & n^{i-1} \\ 1 & 2 & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-2} & 3^{n-2} & \cdots & n^{n-2} \\ 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{bmatrix}, \quad 2 \le i \le n,$$

followed by the expansion of $\det(X_n^{(i)})$ by the first row (which involves the numbers S_j given in Lemma 2).

References

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