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Several Theorems for the Trace of Self-conjugate Quaternion Matrix

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Abstract

The purpose of this paper is to discuss the inequalities for the trace of self-conjugate quaternion matrix. We present the inequality for eigenvalues and trace of self-conjugate quaternion matrices. Based on the inequality above, we obtain several inequalities for the trace of quaternion positive definite matrix.

Keywords: Quaternion matrix, Trace, Inequalities, Eigenvalues

1. Introduction

Quaternion was introduced by the Irish mathematician Hamilton (1805-1865) in 1843. The literature on quaternion matrices, though dating back to 1936 [1], is fragmentary. Quaternion is mostly used in computer vision because of their ability to represent rotations in 4D spaces. It is also used in programming video games and controlling spacecrafts [2, 3] and so forth. The research on mathematical objects associated with quaternion has been dynamic for years. There are many research papers published in a variety of journals each year and different approaches have been taken for different purposes, and the study of quaternion matrices is still in development. As is expected, the main obstacle in the study of quaternion matrices is the non-commutative multiplication of quaternion. The theory on eigenvalues, determinant, singular values and trace of real and complex matrices has been well established. On the contrary, little is known for the trace of quaternion matrices.

As usual, R and C are the set of the real and complex numbers. We denote by H (in honor of the inventor, Hamilton) the set of real quaternion:

$$H=\{a=a_0+a_1i+a_2j+a_3k, a_0,a_1,a_2,a_3\in R\}.$$

For $a=a_0+a_1i+a_2j+a_3k\in H$, the conjugate of a is $\bar{a}=a^*=a_0-a_1i-a_2j-a_3k$ and the norm of a is $N(a)=\sqrt{a\bar{a}}=\sqrt{\bar{a}a}=(a_0^2+a_1^2+a_2^2+a_3^2)^{1/2}$. Let $H^{n\times n}$ and $H^{n\times 1}$ be respectively the collections of all $n\times n$ matrices with entries in H and n-column vectors. Let I_n be the collections of all $n\times n$ unit matrices with entries in H. For $X\in H^{n\times 1}, X^T$ is the transpose of X. If $X=(X_1,X_2\dots X_n)^T$, then $\bar{X}=(\bar{X}_1,\bar{X}_2,\dots,\bar{X}_n)^T$ is the conjugate of X and $X^*=(\bar{X}_1,\bar{X}_2,\dots,\bar{X}_n)$ is the conjugate transpose of X. The norm of X is defined to be $N(X)=\sqrt{X^*X}$. For an

$n \times n$ matrix $A = (a_{ij})_{n\times n}$ $(a_{ij} \in H)$, the conjugate transpose of A is the $n \times n$ matrix $A^* = \bar{A}^T = (a_{ij}^*)_{n\times n}$.

The research of matrices is continuously an important aspect of algebra problems over quaternion division algebra, the subject, such as eigenvalues, singular values, congruent and positive definite of self- conjugate matrices as well as sub-determinant of self- conjugate matrices and so on, has been extensively explored [4-15], while little is known for the trace of quaternion matrices. For linear algebraists and matrix theories, some basic questions on the trace of quaternion matrix are different from real or complex matrix. For instance, if A and B are the $n \times n$ matrices, then $Tr(AB) = Tr(BA)$ and $Tr(A) = \sum_i^n \lambda_i$ are not always right. In this section, we introduce the notation and terminology. In section 2, we define some definitions and recall several lemmas. In section 3, we discuss the inequality about eigenvalues and trace of self-conjugate quaternion matrices. In section 4, we conclude the paper with several inequalities for the trace of quaternion positive definite matrix obtained by the result in section 3 and the Holder's inequality over quaternion division algebra.

2. Definitions and Lemmas

We begin this section with some basic definitions and lemmas.

Definition 2.1 Let $A \in H^{n\times n}$. A is said to be the self-conjugate quaternion matrix if $A^* = A$. $H(n,*)$ is the set of self-conjugate quaternion matrices, $H(n,>)$ is the set of quaternion positive definite matrices.

Definition 2.2 Let $A \in H^{n\times n}$. A is said to be the quaternion unitary matrix if $A^*A = AA^* = I_n$. $H(n,u)$ is the set of quaternion unitary matrices.

Definition 2.3 Let $A \in H^{n\times n}$. $\sum_{i=1}^n a_{ii}$ is said to be the trace of matrix A, remarked by $Tr(A)$. That is $Tr(A) = \sum_{i=1}^n a_{ii}$.

Lemma 1. [4] Let $A \in H(n,*)$. Then, A is unitary similar to a real diagonal matrix, that is, there exists a unitary matrix $U \in H(n,u)$, such that

$$U^*AU = diag(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where, $\lambda_1, \lambda_2, \cdots \lambda_n \in R$ are the eigenvalues of A.

Lemma 2. Let $A \in H(n,*)$ and $B \in H(n,*)$. If there exists $U \in H(n,u)$, such that $B = UAU^*$, then, $TrA = TrB$.

Proof. Since $A \in H(n,*)$, by Lemma 1, there exists $V \in H(n,u)$, such that

$$V^*AV = diag(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where, $\lambda_1, \lambda_2, \cdots \lambda_n \in R$ are the eigenvalues of A. For any $U \in H(n,u)$, we have

$$\sum_{i=1}^n u_{ij}u_{ij}^* = \sum_{i=1}^n N^2(u_{ij}) = 1, (j = 1,2,\cdots,n) \qquad (2.2)$$

$$\sum_{j=1}^n u_{ij}u_{ij}^* = \sum_{j=1}^n N^2(u_{ij}) = 1, (i = 1,2,\cdots,n) \quad . \qquad (2.3)$$

So

$$TrA = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \left\{ \sum_{k=1}^n v_{ik} \left(\sum_{l=1}^n \lambda_k \delta_{kl} \right) v_{il}^* \right\} = \sum_{i=1}^n \left\{ \sum_{k=1}^n \sum_{l=1}^n \lambda_k v_{ik} \delta_{kl} v_{il}^* \right\}$$

$$= \sum_{i=1}^n \left\{ \sum_{l=1}^n \lambda_l v_{ll} v_{ll}^* \right\} = \sum_{i=1}^n \lambda_l \left\{ \sum_{l=1}^n v_{ll} v_{ll}^* \right\} = \sum_{i=1}^n \lambda_l$$

Meanwhile, we have $B = UAU^*$, then

$B = UVdiag(\lambda_1, \lambda_2, \cdots, \lambda_n)V^*U^*$.

Since $UV \in H(n,u)$, therefore

$TrA = TrB = \sum_{i=1}^n \lambda_l$.

Thus

$TrA = TrB$ (2.4)

3. The inequality for the trace of self-conjugate quaternion matrices

It is well known that the eigenvalues and trace of any self-conjugated quaternion matrix are all real numbers. In this section, we shall discuss the inequality about eigenvalues and trace of self-conjugate quaternion matrices.

Theorem 1. Let $A \in H(n,>), B \in H(n,*)$, their eigenvalues are $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta$ respectively, if A, B are commutative, then $Tr(AB) \leq \sum_{i=1}^{n} \alpha_i \beta_i$ ·

Proof. Since $A \in H(n,*), B \in H(n,*)$, by lemma 1, there exists unitary matrices $U_1, U_2 \in H(n,u)$, such that

$$U_1^* A U_1 = diag(\alpha_1, \alpha_2, \cdots, \alpha_n) \quad (3.1)$$

$$U_2^* B U_2 = diag(\beta_1, \beta_2, \cdots, \beta_n) \quad (3.2)$$

where $\alpha_i > 0 (i = 1,2,\cdots n)$.Therefore

$$Tr(AB) = Tr\left[U_1 diag(\alpha_1, \alpha_2, \cdots, \alpha_n) U_1^* U_2 diag(\beta_1, \beta_2, \cdots, \beta_n) U_2^*\right] \cdot$$

By (3.1) and (3.2), we have

$$U_1^* AB U_1 = diag(\alpha_1, \alpha_2, \cdots, \alpha_n) U_1^* U_2 diag(\beta_1, \beta_2, \cdots, \beta_n) U_2^* U_1 \cdot$$

Let $U_1^* U_2 = U = (u_{ij})_{n\times n}$, it is easy to know $U \in H(n,u)$, then

$$\sum_{i=1}^{n} u_{ij} u_{ij}^* = \sum_{i=1}^{n} N^2(u_{ij}) = 1, (j = 1, 2, \cdots, n)$$

$$\sum_{j=1}^{n} u_{ij} u_{ij}^* = \sum_{i=1}^{n} N^2(u_{ij}) = 1, (i = 1, 2, \cdots, n) \cdot$$

Since $(AB)^* = B^* A^* = BA$, and A, B are commutative, then $(AB)^* = AB$, so $AB \in H(n,*)$. Hence, by Lemma 2, $Tr(AB) = Tr(U_1^* AB U_1)$, then

$$Tr(AB) = Tr\left[diag(\alpha_1, \alpha_2, \cdots, \alpha_n) U diag(\beta_1, \beta_2, \cdots, \beta_n) U^*\right]$$

$$= (\alpha_1, \alpha_2, \cdots, \alpha_n) \begin{pmatrix} N^2(u_{11}) & N^2(u_{12}) & \cdots & N^2(u_{1n}) \\ N^2(u_{21}) & N^2(u_{22}) & \cdots & N^2(u_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ N^2(u_{n1}) & N^2(u_{n2}) & \cdots & N^2(u_{nn}) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}.$$

Let

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} N^2(u_{11}) & N^2(u_{12}) & \cdots & N^2(u_{1n}) \\ N^2(u_{21}) & N^2(u_{22}) & \cdots & N^2(u_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ N^2(u_{n1}) & N^2(u_{n2}) & \cdots & N^2(u_{nn}) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

then

$$\sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} \beta_i \cdot \quad (3.3)$$

For any $k(1 \leq k < n)$, we have

$$\sum_{i=1}^{k} \xi_i = \sum_{i=1}^{k} \sum_{j=1}^{n} N^2(u_{ij}) \beta_j = \sum_{i=1}^{k} \beta_i - \sum_{i=1}^{k} \left(1 - \sum_{j=1}^{k} N(u_{ij})\right) \beta_i + \sum_{i=1}^{k} \sum_{j=k+1}^{n} N^2(u_{ij}) \beta_j$$

$$\leq \sum_{i=1}^{k} \beta_i - \beta_k \sum_{i=1}^{k} \left(1 - \sum_{j=1}^{k} N^2(u_{ij})\right) + \beta_k \sum_{i=1}^{k} \sum_{j=k+1}^{n} N^2(u_{ij})$$

$$= \sum_{i=1}^{k} \beta_i - \beta_k \sum_{i=1}^{k} \left(1 - \sum_{j=1}^{k} N^2(u_{ij})\right) + \beta_k \left(1 - \sum_{j=1}^{k} N^2(u_{ij})\right)$$

$$=\sum_{i=1}^{k}\beta_i \quad . \tag{3.4}$$

By (3.3), (3.4) ,and $\alpha_i > 0(i=1,2,\cdots n)$, then

$$Tr(AB)=\sum_{i=1}^{n}\alpha_i\xi_i \le \sum_{i=1}^{n}\alpha_i\beta_i \cdot$$

4. The inequality for the trace of quaternion positive definite matrix

In this section, we first obtain an inequality for the trace of two quaternion positive definite matrices based on Theorem 1.Then by Theorem 1 and the Holder's inequality over quaternion division algebra, the inequality for trace of the sum and multiplication of quaternion positive definite matrices is obtained.

Theorem 2. Let $A\in H(n,>), B\in H(n,>)$, their eigenvalues are $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$, $\beta_1 \ge \beta_2 \ge \cdots \ge \beta$ respectively, if A, B are commutative, then

$$\sqrt{Tr(AB)} \le \sqrt{Tr(A)Tr(B)} \le \frac{Tr(A)+Tr(B)}{2} \le \sqrt{\frac{Tr^2(A)+Tr^2(B)}{2}} .$$

Proof. Since $Tr(A)=\sum_{i=1}^{n}a_{ii}=\sum_{i=1}^{n}\alpha_i>0$ and $Tr(B)=\sum_{i=1}^{n}b_{ii}=\sum_{i=1}^{n}\beta_i>0$, by Theorem 1,we have

$$Tr(AB)\le\sum_{i=1}^{n}\alpha_i\beta_i \cdot$$

Because of $A\in H(n,>)$, then there exists $U\in H(n,u)$, such that

$$U^*AU=diag(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

where $\alpha_i > 0(i=1,2,\cdots n)$.So,

$$U^*ABU=U^*AUU^*BU=diag(\alpha_1,\alpha_2,\cdots,\alpha_n)U^*BU \cdot \tag{4.1}$$

Since $B\in H(n,>)$, then B and I_n are self-congruent, hence U^*BU and I_n are self-congruent, that is , U^*BU is quaternion positive definite matrix. For any main sub-matrix L of U^*ABU, by (4.1), we know that L can be obtained by the main sub-matrix G of U^*BU .Then

$\|L\|=|L|^{\text{row}}=\alpha_i\alpha_j\cdots|G|^{\text{row}}=\alpha_i\alpha_j\cdots\|G\|>0$,

where $\|\cdot\|$ is the determinant of quaternion matrix defined by Chen L X [10]. So, U^*ABU is quaternion positive definite matrix, hence $Tr(AB)>0$ .Since

$Tr(A)Tr(B)-Tr(AB)\ge\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n}\alpha_i\beta_i\ge 0$

so

$$\sqrt{Tr(AB)}\le\sqrt{Tr(A)Tr(B)} \cdot \tag{4.2}$$

Because of

$$Tr^2(A)+Tr^2(B)\ge 2Tr(A)Tr(B)$$

then

$$2Tr^2(A)+2Tr^2(B)\ge Tr^2(A)+2Tr(A)Tr(B)+Tr^2(B)$$

namely

$$\frac{Tr^2(A)+Tr^2(B)}{2}\ge\left(\frac{Tr(A)+Tr(B)}{2}\right)^2$$

so

$$\frac{Tr(A)+Tr(B)}{2}\le\sqrt{\frac{Tr^2(A)+Tr^2(B)}{2}} . \tag{4.3}$$

By (4.2) and (4.3), the conclusion holds.

Theorem 3. Let $A \in H(n,>), B \in H(n,>)$. Their eigenvalues are $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta$ respectively, if $p > 1, \frac{1}{p} + \frac{1}{q} = 1$,and A, B are commutative, then

$$Tr(AB) \leq \left(Tr\left(A^p\right)\right)^{\frac{1}{p}} \left(Tr\left(B^q\right)\right)^{\frac{1}{q}}.$$

Proof. By Theorem 1 and the Holder's inequality, we have

$$Tr(AB) \leq \sum_{i=1}^{n} \alpha_i \beta_i \leq \left(\sum_{i=1}^{n} \alpha_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \beta_i^q\right)^{\frac{1}{q}} = \left(Tr(A^P)\right)^{\frac{1}{p}} \left(Tr(B^q)\right)^{\frac{1}{q}}.$$

Specially, when $p = q = 2$, we have

$$Tr(AB) \leq \sqrt{\left(Tr(A^2)\right)} \sqrt{\left(Tr(B^2)\right)}.$$

Theorem 4. Let $A \in H(n,>), B \in H(n,>)$. Their eigenvalues are $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta$ respectively, if $p > 1$,and A, B are commutative, then

$$\left(Tr\left((A+B)^p\right)\right)^{\frac{1}{p}} \leq \left(Tr\left(A^p\right)\right)^{\frac{1}{p}} + \left(Tr\left(B^p\right)\right)^{\frac{1}{p}}.$$

Proof. Let $q = \frac{p}{p-1}$, then $p > 1, q > 0, \frac{1}{p} + \frac{1}{q} = 1$ .Since

$$(A+B)^p = A(A+B)^{p-1} + B(A+B)^{p-1}$$

then, by Theorem 3 and the Holder's inequality, we have

$$Tr\left[(A+B)^p\right] = Tr\left[A(A+B)^{p-1} + B(A+B)^{p-1}\right]$$

$$\leq \left(TrA^P\right)^{\frac{1}{p}} \left[Tr\left(\left((A+B)^{p-1}\right)^q\right)\right]^{\frac{1}{q}} + \left(TrB^P\right)^{\frac{1}{p}} \left[Tr\left(\left((A+B)^{p-1}\right)^q\right)\right]^{\frac{1}{q}}$$

$$= \left[Tr\left(\left((A+B)^{p-1}\right)^q\right)\right]^{\frac{1}{q}} \left[\left(TrA^P\right)^{\frac{1}{p}} + \left(TrB^P\right)^{\frac{1}{p}}\right]$$

$$= \left[Tr\left(\left((A+B)^{p-1}\right)^{\frac{p}{p-1}}\right)\right]^{\frac{p-1}{p}} \left[\left(TrA^P\right)^{\frac{1}{p}} + \left(TrB^P\right)^{\frac{1}{p}}\right]$$

$$= \left[Tr\left((A+B)^p\right)\right]^{\frac{p-1}{p}} \left[\left(TrA^P\right)^{\frac{1}{p}} + \left(TrB^P\right)^{\frac{1}{p}}\right].$$

That is

$$Tr\left[(A+B)^p\right] \leq \left[Tr\left((A+B)^p\right)\right]^{\frac{p-1}{p}} \left[\left(TrA^P\right)^{\frac{1}{p}} + \left(TrB^P\right)^{\frac{1}{p}}\right].$$

So

$$\left(Tr\left((A+B)^p\right)\right)^{\frac{1}{p}} \leq \left(Tr\left(A^p\right)\right)^{\frac{1}{p}} + \left(Tr\left(B^p\right)\right)^{\frac{1}{p}}.$$

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