



Periodic Solutions of Higher Order with Restoring Terms Delay Functional Differential Equation with Neutral Type

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Abstract

Using the theory of coincidence degree, the authors have studied the existence of periodic solutions of a type of higher order with restoring terms delay functional differential equations with neutral type, and some new results for the existence of periodic solutions have been obtained.

Keywords: Delay functional differential equation, Coincidence degree, Periodic solutions

1. Introduction and Lemma

The functional differential equation with time delay, because of its wide application, has long been a main subject of common concern. But because the argument in complex time delay may depend on condition itself or variation rate, it is difficult to predict the property of the solution to the functional differential equation with complex deviating argument. It brings such big difficulty to our research work that in a long period of time the research of this equation progressed slowly. In recent years with further development of nonlinear functional analysis and algebra-ictopology, people have already had some achievement on the research of this equation. This paper mainly study periodic solutions of the following kind:

$$x^{(m)}(t) + a(t)x^{(m)}(t - \tau) + f(x(t - \sigma))\dot{x}(t - \sigma) + b(t)g(x(t - \tau)) + c(t)x(t - \tau) = p(t) \quad (1)$$

Lemma ^[7] Let X, Y be Banach space, $L : D_L \subset X \rightarrow Y$ is a Fredholm mapping of index 0 $P : X \rightarrow X, Q : Y \rightarrow Y$ are continuous mapping projector; Ω is an open bounded set in X ,

and $N : \bar{\Omega} \times [0, 1] \rightarrow Y$ is L -Compact on $\bar{\Omega}$, further more suppose:

$$(a) Lx \neq \lambda N(x, \lambda), \forall x \in D_L \cap \partial\Omega, \lambda \in (0, 1);$$

$$(b) QN(x, 0) \neq 0, \forall x \in \text{Ker}(L) \cap \partial\Omega;$$

$$(c) \deg(QN(x, 0), \text{Ker}(L) \cap \Omega, 0) \neq 0.$$

then the equation $Lx = N(x, 1)$ has at least one solution in $\bar{\Omega}$. Where $\deg$ is Brouwer degree.

2. Main Results and Proof of Theorems

Theorem: Suppose f, a, b, c, g, p are continuous for their variables respectively; There exists T , such that $p(t + T) = p(t)$, $a(t + T) = a(t)$, $b(t + T) = b(t) > 0$, $0 < |a(t)| < a_1 < 1$,

$$c(t + T) = c(t) < 0, \quad a_m = \max_{t \in R} \{ |a^m(t)| \} < c_0 = \min_{t \in R} \{ |c(t)| \} \leq c_1 = \max_{t \in R} \{ |c(t)| \} < \frac{1 - a_1}{T^m},$$

$a(t) \in C^m(R, R)$ and further more suppose are as follows,

$$a. \exists 0 < M < \frac{1 - a_1 - c_1 T^m}{T^{m-1}}, \quad \forall x \in R, \quad |f(x)| \leq M,$$

$$b. \exists A > 0, \quad \forall x \in R, \quad |g(x)| \leq A;$$

$$c. \forall 0 \neq x \in R, \quad xg(x) < 0$$

then Eq.(1) has at least one T -periodic solution.

Proof of Theorem : In order to use continuation theorem to obtain T-periodic solution of Eq.(1), we firstly make some required preparations. Let $X = \{x \in C^{m-1}(\mathbf{R}, \mathbf{R}) | x(t+T) = x(t)\}$

$Y = \{y \in C(\mathbf{R}, \mathbf{R}) | y(t+T) = y(t)\}$ and the norm of X is $\|x\| = \max_{0 \leq i \leq m-1} \{ |x^{(i)}|_{\infty} \}$, there $|x^{(0)}|_{\infty} = |x|_{\infty} = \max_{t \in \mathbf{R}} \{ |x(t)| \}$, $|x^{(i)}|_{\infty} = \max_{t \in \mathbf{R}} \{ |x^{(i)}(t)| \}$, $i = 1, 2, \dots, m-1$, $\|y\| = \max_{t \in \mathbf{R}} \{ |y(t)| \}$, then the X and Y with this norm are Banach space.

Firstly, we study the priori bound of T-periodic solution of the following equation

$$x^{(m)}(t) + \lambda a(t)x^{(m)}(t-\tau) + \lambda f(x(t-\sigma))\dot{x}(t-\sigma) + \lambda b(t)g(x(t-\tau)) + \lambda c(t)x(t-\tau) = \lambda^2 p(t) \quad (1.1)$$

Suppose that $x = x(t) \in X$ is an arbitrary T-periodic solution of Eq.(1.1), put $x(t)$ into (1.1) and then integrate both sides of (1.1) on $[0, T]$, so yield

$$\int_0^T [a(t)x^{(m)}(t-\tau) + b(t)g(x(t-\tau)) + c(t)x(t-\tau)]dt = \lambda \int_0^T p(t)dt$$

we easily get :

$$(-1)^m \int_0^T a^{(m)}(t)x(t-\tau)dt + \int_0^T b(t)g(x(t-\tau))dt + \int_0^T c(t)x(t-\tau)dt = \lambda \int_0^T p(t)dt$$

so there exists a $t_0 \in [0, T]$, such that:

$$(-1)^m a^{(m)}(t_0)x(t_0-\tau) + b(t_0)g(x(t_0-\tau)) + c(t_0)x(t_0-\tau) = \lambda p(t_0)$$

$$\text{i.e. } |c(t_0)x(t_0-\tau)| = |\lambda p(t_0) - (-1)^m a^{(m)}(t_0)x(t_0-\tau) - b(t_0)g(x(t_0-\tau))|$$

$$\text{so } |c_0 x(t_0-\tau)| \leq p_1 + a_m |x(t_0-\tau)| + b_1 A$$

$$\text{i.e. } |x(t_0-\tau)| \leq \frac{b_1 A + p_1}{c_0 - a_m} \triangleq A_1 \quad \text{where } b_1 = \max_{t \in \mathbf{R}} \{ |b(t)| \}, \quad p_1 = \max_{t \in \mathbf{R}} \{ |p(t)| \}$$

$$\text{let } t_0 - \tau = nT + t_1, \quad n \in \mathbf{N}, \quad t_1 \in [0, T], \quad \text{so } |x(t_1)| = |x(t_0-\tau)| \leq A_1$$

In view of $\forall t \in [0, T], x(t) = x(t_1) + \int_{t_1}^t \dot{x}(s)ds$, we have

$$|x(t)| = \left| x(t_1) + \int_{t_1}^t \dot{x}(s)ds \right| \leq A_1 + \int_{t_1}^t |\dot{x}(s)|ds \leq A_1 + \int_0^T |\dot{x}(t)|dt$$

$$\text{i.e. } |x^{(0)}|_{\infty} = |x|_{\infty} \leq A_1 + \int_0^T |\dot{x}(t)|dt \quad (1.2)$$

Noting $x(t) = x(t+T)$, so there must exist numbers $\xi_i \in (0, T)$, such that $x^{(i)}(\xi_i) = 0$, there $i = 1, 2, 3, \dots, m-1$.

For $\forall t \in [0, T], x^{(i)}(t) = x^{(i)}(\xi_i) + \int_{\xi_i}^t x^{(i+1)}(s)ds = \int_{\xi_i}^t x^{(i+1)}(s)ds$, we have

$$|x^{(i)}(t)| = \left| \int_{\xi_i}^t x^{(i+1)}(s)ds \right| \leq \int_0^T |x^{(i+1)}(t)|dt \leq T \int_0^T |x^{(i+2)}(t)|dt$$

$$\leq T^2 \cdot \int_0^T |x^{(i+3)}(t)|dt \leq \dots \leq T^{m-(i+1)} \int_0^T |x^{(i+m-i)}(t)|dt = T^{m-(i+1)} \int_0^T |x^{(m)}(t)|dt,$$

$$\text{i.e. } |x^{(i)}|_{\infty} \leq T^{m-(i+1)} \int_0^T |x^{(m)}(t)|dt, \quad i = 1, 2, \dots, m-1 \quad (1.3)$$

Combining (1.2), (1.3), we get

$$|x^{(0)}|_{\infty} = |x|_{\infty} \leq T^{m-1} \int_0^T |x^{(m)}(t)|dt \quad (1.4)$$

By (1.1), we get

$$\int_0^T |x^{(m)}(t)|dt \leq \int_0^T |-\lambda a(t)x^{(m)}(t-\tau) - \lambda f(x(t-\sigma))\dot{x}(t-\sigma)|dt + \int_0^T |-\lambda b(t)g(x(t-\tau)) - \lambda c(t)x(t-\tau) + \lambda^2 p(t)|dt$$

Combining the condition a and b, we easily obtain

$$\int_0^T |x^{(m)}(t)|dt \leq a_1 \int_0^T |x^{(m)}(t-\tau)|dt + M \int_0^T |\dot{x}(t-\sigma)|dt + c_1 \int_0^T |x(t-\tau)|dt + TAb_1 + Tp_1$$

$$\text{i.e. } \int_0^T |x^{(m)}(t)| dt \leq a_1 \int_0^T |x^{(m)}(t)| dt + MT|\dot{x}|_\infty + c_1 T|x|_\infty + TAb_1 + Tp_1 \quad (1.5)$$

Noting (1.3), (1.4) and (1.5), we observe

$$\int_0^T |x^{(m)}(t)| dt \leq a_1 \int_0^T |x^{(m)}(t)| dt + MT \cdot T^{m-2} \int_0^T |x^{(m)}(t)| dt + c_1 T(A_1 + T^{m-1} \int_0^T |x^{(m)}(t)| dt) + TAb_1 + Tp_1 \quad \text{i.e.}$$

$$(1 - a_1 - MT^{m-1} - c_1 T^m) \int_0^T |x^{(m)}(t)| dt \leq TAb_1 + Tp_1 + c_1 TA_1$$

$$\text{so } \int_0^T |x^{(m)}(t)| dt \leq \frac{TAb_1 + Tp_1 + c_1 TA_1}{1 - a_1 - MT^{m-1} - c_1 T^m} \quad (1.6)$$

Noting(1.3),(1.5)and(1.6), we have

$$|x^{(0)}|_\infty = |x|_\infty \leq A_1 + T^{m-1} \cdot \frac{TAb_1 + Tp_1 + c_1 TA_1}{1 - a_1 - MT^{m-1} - c_1 T^m} = A_1 + \frac{T^m (Ab_1 + p_1 + c_1 A_1)}{1 - a_1 - MT^{m-1} - c_1 T^m} \triangleq \omega_0$$

$$|x^{(i)}|_\infty \leq T^{m-(i+1)} \cdot \frac{TAb_1 + Tp_1 + c_1 TA_1}{1 - a_1 - MT^{m-1} - c_1 T^m} = \frac{T^{m-i} (Ab_1 + p_1 + c_1 A_1)}{1 - a_1 - MT^{m-1} - c_1 T^m} \triangleq \omega_i, i = 1, 2, \dots, m-1$$

Let $\omega = \max_{0 \leq i \leq m} \{\omega_i + 1\}$, and we take $\Omega = \{x | x \in X : \|x\| < \omega\}$, then Ω is an open and bounded set in X .

Let $L : D_L \subset X \rightarrow Y : x \rightarrow Lx = x^{(m)}(t)$;

$N : X \times I \rightarrow Y : x \rightarrow N(x, \lambda) = -a(t)x^{(m)}(t - \tau) - f(x(t - \sigma))\dot{x}(t - \sigma) - b(t)g(x(t - \tau)) - c(t)x(t - \tau) + \lambda p(t)$, then the corresponding equation of $Lx = \lambda N(x, \lambda)$ is Eq.(1.1).

Now, we define projection operators as follows,

$$P : X \rightarrow \text{Ker}(L) : x \rightarrow Px = \frac{1}{T} \int_0^T x(t) dt; \quad Q : Y \rightarrow Y / \text{Im}(L) : y \rightarrow Qy = \frac{1}{T} \int_0^T y(t) dt$$

obviously, P, Q are continuous operator, $\text{Im}(P) = R = \text{Ker}(L)$, $\text{Ker}(Q) = \text{Im}(L)$, and it is easy to prove that L is a Fredholm mapping of index 0 and N is L -Compact on $\bar{\Omega}$.

From the above discussion and the construction of Ω , we have known that $\forall x \in D_L \cap \partial\Omega$

$\lambda \in (0, 1)$, therefore the condition (a) of lemma holds.

For arbitrary $x \in \text{Ker}(L) \cap \partial\Omega$, $\|x\| = \omega$, by the definition of Q, N , we have

$$\begin{aligned} QN(x, 0) &= \frac{1}{T} \int_0^T [-a(t)x^{(m)}(t - \tau) - f(x(t - \sigma))\dot{x}(t - \sigma) - b(t)g(x(t - \tau)) - c(t)x(t - \tau)] dt \\ &= -\frac{1}{T} \int_0^T b(t)g(x(t - \tau)) dt - \frac{1}{T} \int_0^T c(t)x(t - \tau) dt \end{aligned}$$

$$\begin{aligned} \text{so } xQN(x, 0) &= -\frac{1}{T} x \int_0^T b(t)g(x(t - \tau)) dt - \frac{1}{T} x \int_0^T c(t)x(t - \tau) dt \\ &= -\frac{1}{T} xg(x) \int_0^T b(t) dt - \frac{1}{T} x^2 \int_0^T c(t) dt > 0 \end{aligned}$$

therefore the condition (b) of lemma holds.

Make a transformation $H(x, \mu) = \mu x + (1 - \mu)QN(x, 0)$, $\forall x \in \partial\Omega \cap \text{Ker}(L)$, $\mu \in [0, 1]$

We have $xH(x, \mu) = \mu x^2 + x(1 - \mu)QN(x, 0)$

$$= \mu x^2 - (1 - \mu) \frac{1}{T} g(x)x \int_0^T b(t) dt - (1 - \mu) \frac{1}{T} x^2 \int_0^T c(t) dt > 0$$

so $H(x, \mu) \neq 0$, i.e. $H(x, \mu) \neq 0$ is a homotopy, $\deg(QN(x, 0), \text{Ker}(L) \cap \Omega, 0) = \deg(I,$

$\text{Ker}(L) \cap \Omega, 0) = \deg(I, R \cap \Omega, 0) \neq 0$, where I is identity mapping and the condition (c) of lemma holds.

From above all, the requirements of lemma are all met, so Eq. (1) has at least one T-periodic solution under the condition of theorem 1, so far the proof of theorem is completed.

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