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Study on the Duality between MFP and ACP

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Abstract

Under the generalized weak convexity of (F, α, ρ, d), we studied the results of several sorts of duality type about the problem of multi-objective fractional programming (MFP), extended this results to the generalized arcwise connected hypothesis, established the optimized problem of arcwise connected area (ACP) and the optimal sufficient condition of $\min_{x\in S} f(x) \quad s.t.\ g(x) \le 0$ under constraint condition, and gave the duality model, and obtained the conclusions of weak duality and strong duality.

Keywords: Arcwise connected function, Generalized convexity, Weak (strong) duality, Strong Quasi-arc, Optimal solution

1. Introduction

One of important extensions about convexity was the concept of invariant convexity put forward by Hanson in 1981, after that, in twenties years, thirty sorts of generalized convexity functions are introduced, which makes the research contents of the optimized problems become very deep and abundant. The extrusive problem in these problems is the duality problem under the weak convexity concept. In the optimization theory, for an appointed mathematic programming problem, there are many types of duality, and two famous dualities are Wolfe duality (Wolfe, 1961, p.239-244) and Mond-Weir duality (Mond, 1981, p.263-28), and in recent years, the mixed duality has been thought as the type of various optimized problems, and the mixed duality (Aghezzaf, 2000, p.91-101, Aghezzah, 2001, p.617-628, Zi, 1993, p.113-118, Liang, 2001, p.446-461, Mond, 1982, p.105-124, Mukherjee, 2000, p.571-586, Preda, 1992, p.365-377, Xu, 1996, p.621-635, Yang, 2000, p.999-1005, Zhang, 1997, p.29-44).

Generally, when we solute an optimized problem, the feasible area is usually in the area with interior points, but in practical problems, it always doesn't possess this condition, for example, the feasible area of the problem is the following line-type figure without interior point which is seen Figure 1. Its feasible area is connected by curve S. So when we define the function in this feasible area, we can not consider its partial derivative or directional derivative, and the grads of the function. For these problems, in early 1970, Ortega and Rheinboldt (Ortega, 1970) put forward the concept of regional arcwise connection, after that, Avriel and Zang (Avriel, 1980, p.407-435) extended it as various generalized convexities. The arcwise connected function and various generalized functions possess very good local-global extremum property, and in this article, we mainly introduce the duality problem result under the generalized weak convexity of (F, α, ρ, d) introduced by Liang. Z. A, Huang. H. Z. and Purdulos. P. M (Liang, 2003, p.447-471), and extend it into the weak duality and strong duality of generalized arcwise connected function optimized problem (ACP), and give some results.

2. Basic conclusions of MFP duality under generalized weak convexity of (F, α, ρ, d)

Here, we will give the conclusions of several duality problems about MFP.

Supposed (MFP) $$\min \frac{f(x)}{g(x)} \triangleq \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)^T \qquad (1—1)$$

$$s.t \quad h(x) \le 0, \quad x \in X .$$

When $X \subset R^n$ is an open set, $f_i, g_i (i=1,2,\cdots p)$ is the real valued function on X, and h is m dimensional vector value function defined on X, and $M=\{1,2,\cdots m\} and M_0, M_1 \cdots M_q$ is a

partition of M, i.e. $\bigcup_{k=0}^{q} M_k = M, M_k \cap M_l = \phi$, when $k \neq l$, so the generalized Mond-Weir duality of MEP is

$$Max \frac{f(u)}{g(u)} + \lambda_{M_0}^T h_{M_0}(u) e \triangleq (\frac{f_1(u)}{g_1(u)} + \lambda_{M_0}^T h_{M_0}(u) + \cdots + \frac{f_p(u)}{g_p(u)}) + \lambda_{M_0}^T h_{M_0}(u))^T \qquad (1\text{----}2)$$

$$s.t \quad \sum_{i=1}^{q} \tau_i \nabla \frac{f_i(u)}{g_i(u)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(u) = 0$$

$$\lambda_{M_k}^T h_{M_k}(u) \geq 0, \quad k=1,2,\cdots q$$

$$\tau = (\tau_1, \tau_2, \cdots \tau_p)^T \in R_+^P, \tau > 0, \sum_{i=1}^{P} \tau_i = 1$$

$$\lambda_{M_k} \in R_+^{|M_k|} \quad k=0,1,2,\cdots q \quad u \in X$$

Where, $e=(1,1,\cdots 1)^T$ and λ_{M_k} represent column vectors, and their component subscripts belong to M_k.

2.1 Mond-weir duality

$$\text{(MFD1)} \quad Max \frac{f(u)}{g(u)} = (\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \cdots, \frac{f_p(u)}{g_p(u)})^T \qquad (2\text{----}1)$$

$$s.t \quad \sum_{i=1}^{q} \tau_i \nabla \frac{f_i(u)}{g_i(u)} + \sum_{j=1}^{m} \lambda_j \nabla h_j(u) = 0$$

$$\lambda^T h(u) \geq 0$$

$$\tau = (\tau_1, \tau_2, \cdots \tau_p)^T \in R_+^P, \tau > 0, \sum_{i=1}^{P} \tau_i = 1$$

$$\lambda = (\lambda_1, \lambda_2, \cdots \lambda_p)^T \in R_+^m, u \in X$$

Theorem 2.1 (weak duality) (Liang, 2003, p.447-471): Supposed $\bar{x}$ is a feasible solution of (MFP), and $(\bar{u}, \bar{\tau}, \bar{\lambda})$ is a feasible solution of (MFD1), and if f_i and $-g_i (i=1,2,\cdots p)$ are convex $(F, \alpha_i, \rho_i, d_i)$ on $\bar{u}$, so $h_j (j=1,2,\cdots m)$ is convex (F, β, ζ_i, c_j) on $\bar{u}$, and the inequation exists.

$$\sum_{i=1}^{P} \bar{\tau}_i \bar{\rho}_i \frac{\bar{d}_i(\bar{x},\bar{u})}{\bar{\alpha}_i(\bar{x},\bar{u})} + \sum_{j=1}^{m} \bar{\lambda}_j \zeta_j \frac{c_j^2(\bar{x},\bar{u})}{\beta(\bar{x},\bar{u})} \geq 0 \qquad (2\text{---}2)$$

Where, $\bar{\alpha}_i(\bar{x},\bar{u}) = \alpha_i(\bar{x},\bar{u}) \frac{g(\bar{u})}{g(\bar{x})}$, $\bar{\rho}_i = \rho_i (1 + \frac{f_i(\bar{u})}{g_i(\bar{u})})$, $\bar{d}_i(\bar{x},\bar{u}) = \frac{d_i(\bar{x},\bar{u})}{g_i^{\frac{1}{2}}(\bar{x})}$, so $\frac{f(\bar{x})}{g(\bar{x})} \nprec \frac{f(\bar{u})}{g(\bar{u})}$.

Deduction 2.1 (weak duality) (Liang, 2003, p.447-471): Supposed $\bar{x}$ is a feasible solution of (MFP), and $(\bar{u}, \bar{\tau}, \bar{\lambda})$ is a feasible solution of (MFD1), and if f_i and $g_i (i=1,2,\cdots p)$ are strongly convex $(F, \alpha_i, \rho_i, d_i)$ (or convex (F, α_i)). On $\bar{u}$, $h_j (j=1,2,\cdots m)$ is strongly convex $\bar{u}(F, \beta, \zeta_i, c_i)$, so $\underline{\frac{f(\bar{x})}{g(\bar{x})} \nprec \frac{f(\bar{u})}{g(\bar{u})}}$.

Theorem 2.2 (strong duality): Supposed $\bar{x}$ is an effective solution of (MFP), and $\bar{x}$ fulfills the restrain condition (GGCQ) (Avriel, 1980, p.407-435), so $(\bar{\tau}, \bar{\lambda}) \in R_+^p \times R_+^m$ exists and makes $(\bar{x}, \bar{\tau}, \bar{\lambda})$ be a feasible solution of (MFD1), and the objective function values on the corresponding points of (MFP) and (MFD1) are equal, and if it fulfills the generalized convex inequation in Theorem 2.1, so $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is an effective solution of (MFD1).

In fact, because $\bar{x}$ is an effective solution of (MFP), and (GGCQ) exists on $\bar{x}$, as a necessary and effective condition,

$(\bar{\tau},\bar{\lambda}) \in R_+^p \times R_+^m, \bar{\tau} > 0$ exists and makes $(\bar{u},\bar{\tau},\bar{\lambda})$ be a feasible solution of (MFD1). Though the corresponding objective functions of (MFP) and (MFD1) are equal, but if $(\bar{u},\bar{\tau},\bar{\lambda})$ is not the sufficient solution of (MFD1), so a feasible solution (x^*,τ^*,λ^*) of (MFD1) must exist and make $\frac{f(\bar{x})}{g(\bar{x})} \prec \frac{f(x^*)}{g(x^*)}$.

Its result is contradictive with the conclusion of weak duality in Theorem 2.1, so $(\bar{u},\bar{\tau},\bar{\lambda})$ is an effective solution of (MFD1).

2.2 Schaible duality

The extended formula of (MFP) Schaible duality (Schaible, 1976, p.452-46 & Schaible, 1976, p.858-867) is

$$\text{(MFD2)} \qquad Max\lambda = (\lambda_1,\lambda_2,\ldots\lambda_p)^T$$

$$s.t \quad \sum_{i=1}^{P}\tau_i\nabla_u(f_i(u)-\lambda_i g_i(u)) + \sum_{j=1}^{m} v_j \nabla h_j(u) = 0$$

$$f_i(u)-\lambda_i g_i(u) \ge 0 \qquad i = 1,2\cdots p$$

$$v^T h(u) \ge 0, \quad \tau > 0, \sum_{i=1}^{p}\tau_i = 1$$

$$\lambda \in R_+^p, \tau \in R_+^P, v \in R_+^{|M_k|}, u \in X$$

Theorem 2.3 (weak duality) (Liang, 2003, p.447-471): Supposed that $\bar{x}$ is a feasible solution of (MFP), and $(\bar{u},\bar{\tau},\bar{\lambda},\bar{v})$ is a feasible solution of (MFD2), if one of following equation comes into existence.

(1) f_i and $-g_i$ $(i = 1,2,\cdots p)$ are convex F,α_i,ρ_i,d_i on $\bar{u}$, $h_j(j = 1,2,\cdots m)$ is convex (F,β,ζ_j,c_j) on $\bar{u}$, and

□

(2) f_i and $-g_i$ $(i = 1,2,\cdots p)$ are convex F,α_i,ρ_i,d_i on $\bar{u}$, $h_j(j = 1,2,\cdots m)$ is convex (F,β,ζ_j,c_j) on $\bar{u}$, and these vectors $\bar{\tau},\bar{\lambda},\bar{v}$ fulfill

$$\sum_{i=1}^{P}\bar{\tau}_i\rho_i(1+\bar{\lambda}_i) + \sum_{j=1}^{m}\bar{v}_j\zeta_j \ge 0 \qquad \text{(2---3)}$$

So $$\frac{f(\bar{x})}{g(\bar{x})} \nprec \bar{\lambda}.$$

Theorem 2.4 (strong duality) (Liang, 2003, p.447-471): Supposed $\bar{x}$ is an effective solution of (MFP), and $\bar{x}$ fulfills the restrain condition (GGCQ) (Avriel, 1980, p.407-435), so $\bar{\tau} \in R_+^p, \bar{\lambda} \in R_+^P, v \in R_+^m$ exists and makes $(\bar{x},\bar{\tau},\bar{\lambda},\bar{v})$ be a feasible solution of (MFD2), and $\bar{\lambda} = \frac{f(\bar{x})}{g(\bar{x})}$. If all hypotheses in Theorem 2.3 are fulfilled, so the corresponding $(\bar{x},\bar{\tau},\bar{\lambda},\bar{v})$ is an effective solution of (MFD2).

2.3 Extended Bector duality

Supposed $G(x) = \prod_{i=1}^{p} g_i(x), \qquad G_i(x) = \frac{G(x)}{g_i(x)} \qquad (i = 1,2,\ldots p)$,

so (MFP) can be written as the following form

$$(\overline{MFP}) \qquad \min \frac{f(x)}{g(x)} = \left(\frac{G_1(x)f_1(x)}{G(X)}, \frac{G_2(x)f_2(x)}{G(x)}, \cdots \frac{G_p(x)f_p(x)^T}{G(*)}\right)$$

$$s.t \quad h(x) \le 0, x \in X.$$

We use the equation of (MFP) from the form of ($\overline{MFP}$) to establish the following duality which is called as the extended Bector duality,

$$\text{(MFD3) Max}\left(\frac{G_1(u)f_1(u)+v_{M_0}^T(h_{M_0}(u))}{G(u)},\cdots\cdots,\frac{G_P(u)f_p(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)^T \qquad (2—4)$$

$$s.t \quad \sum_{i=1}^{p}\tau_i\nabla_u\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}+\sum_{k=1}^{q}\nabla_u v_{M_k}^t h_{M_k}(u)=0$$

$$v_k^T h_{M_k}(u)\ge 0 \quad u=1,2,\cdots q$$

$$G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)\ge 0, \quad i=1,2,\cdots p$$

$$\sum_{i=1}^{p}\tau_i=1,\tau=(\tau_1,\tau_2\cdots\tau_p)^T\in R_+^P,\tau>0$$

$$u\in X, v_{M_K}\in R_+^{|M_k|}, k=1,2,\cdots q$$

Theorem 2.5 (weak duality) (Liang, 2003, p.447-471): Supposed that $\bar{x}$ is a feasible solution of (MFP), and (u,τ,v) is a feasible solution of (MFD3), and –G is convex (F,α,ρ,d) on u point, $G_i f_i(i=1,2\cdots p)$ is convex F,α,ρ_i,d on u point, and $h_j(j=1,2,\cdots m)$ is convex (F,α,ξ_j,d) on u point, and if $\rho\ge\max_{1\le i\le\rho}\rho_i$ and the following inequation exists.

$$\sum_{i=1}^{P}\tau_i\rho_i\left(1+\frac{G_i(u)f_i(u)+v_{M_0}^T h_{M_0}(u)}{G(u)}\right)+\sum_{j\in M_0}v_j\zeta_j+G(u).\sum_{k=1}^{q}\sum_{j\in M_0}v_j\zeta_j\ge 0 \qquad (2—5)$$

$$\text{so} \quad \frac{f(x)}{g(x)}\not<\frac{\overline{G(u)}f(u)+u_{M_0}^T h_{M_0}(u)e}{G(u)}$$

Where, $\overline{G(u)}=diag\{G_1(u)\cdots G_P(u)\},e\in R^p,e=(1,1,\cdots 1)\}$.

Theorem 2.6 (strong duality) (Liang, 2003, p.447-471): Supposed $\bar{x}$ is an effective solution of (MFP), and $\bar{x}$ fulfills the restrain condition (GGCQ) (Avriel, 1980, p.407-435), so $(\bar{\tau},\bar{v})$ exists and makes $\bar{x},\bar{\tau},\bar{v}$ be a feasible solution of (MFD3), and the objective function values of (MFP) and (MFD3) are respectively equal on $\bar{x}$ and ($\bar{x},\bar{\tau},\bar{v}$), If the hypotheses and conditions in Theorem 2.5 are fulfilled, so the ($\bar{x},\bar{\tau},\bar{v}$) is an effective solution of (MFD3).

3. The optimal condition and duality of generalized arcwise connected function

After we give the weak duality and strong duality of (MFP) under some very weak generalized functions, now we consider the optimized problem which area is arcwise connection.

$$\text{(ACP)} \qquad \min f(x)$$
$$s.t \quad x\in X$$

here, $X=\{x\in S; g_j(x)\le 0, j=1,2,\cdots m\}.f(x),g_j(x),(j=1,2,\cdots m)$ is the real valued function on the set of arcwise connection $S\subseteq R^n$, and to any $x_1,x_2\in S$ and the arc $H_{x_1x_2}, f(x) and g_j(x)$ $(j=1,2,\cdots m)$ connecting $x_1 and x_2$ are arcwise derivative about $H_{x_1x_2}$ on x_2.

Definition 3.1: Supposed $H_{x_1x_2}$ is a continual vector value function, i.e. $H_{x_1x_2}:[0,1]\to S, and H_{x_1x_2}(0)=x_1$, $H_{x_1x_2}(1)=x_2$. $x_1,x_2\in S\subset R^n$ are arcwise connections. If a vector $\nabla^- H_{x_1x_2}(\lambda_0)\in R^n$ and a vector value function $\alpha:[0,1]\to R^n$ exist and fulfill $\lim_{t\to 0}\alpha(t)=0$ to make following equation come into existence when $0\le\lambda\le 1$

$$H_{x_1x_2}(\lambda)-H_{x_1x_2}(\lambda_0)=(\lambda-\lambda_0)\nabla^-.H_{x_1x_2}(\lambda_0)+(\lambda-\lambda_0).\alpha(\lambda-\lambda_0). \qquad (3—1)$$

So vector $\nabla^- H_{x_1x_2}(\lambda_0)$ is called as the directional derivative of $H_{x_1x_2}$ on the point of $\lambda=\lambda_0$, which is got from the following equation

$$\nabla^{-} H_{x_1x_2}(\lambda_0) = \lim_{\lambda \to \lambda_0} \{[H_{x_1x_2}(\lambda) - H_{x_1x_2}(\lambda_0)]/(\lambda - \lambda_0)\} \cdot \quad (3—2)$$

Thus, we can define the arcwise derivative concept of arcwise connected function.

Definition 3.2: Supposed f(x) is the continual real valued function on the arcwise connected set $S \subseteq R^n$, to any one point x in S, $x_0 \in S$, H_{x,x_0} is the arcwise connecting x and x_0. If x tends towards x_0 along H_{x,x_0}, the following limitation exists.

$$J_{H_{x,x0}}(x_0) = (\nabla^{-}.H_{X_1,X_2}(1))^T \nabla f(x_0) = \lim_{\lambda \to 1} \frac{f(H_{x,x_0}(\lambda)) - f(x_0)}{\lambda - 1} \quad (3---3)$$

So we call $f(x)$ is arcwise derivative about H_{x,x_0} on the point of x_0, and it is marked as $f_{H_{x,x0}}(x_0)$.

In this way, to $0 \le \lambda \le 1$, a continual arcwise connected function (ACF) f(x) on S can be denoted as $f(H_{x_1x_2}(\lambda)) = f(x_2) + (\lambda - 1) f_{H_{x_1x_2}}(x_2) + (1-\lambda).\overline{\alpha}.(1-\lambda)$. (3----4)

Here, $\overline{\alpha} : [0,1] \to R$, and fulfills $\lim_{t \to 0} \overline{\alpha}(t) = 0$.

Definition 3.3: Supposed f(x) is the continual real valued function on the arcwise connected set $S \subseteq R^n$, to any one point x in S, $x_0 \in S$, the arcwise H_{x,x_0} connecting x and x_0 exists and makes the following containment relationship come into existence.

$$f(x) \le f(x_0) \Rightarrow f_{H_{x,x0}}(x_0) \ge 0 \quad (3—5)$$

So we call that f(x) is the puppet arcwise connected function on x_0 which is marked as PACF.

Under the same condition, if the containment relationship is $f(x) \le f(x_0) \Rightarrow f_{H_{x,x_0}}(x_0) > 0$, so we call f(x) is the strong puppet arcwise connected function on x_0 which is marked as SPACF, and if $f(x) < f(x_0) \Rightarrow f_{H_{x,x_0}}(x_0) > 0$, so we call f(x) is the strict strong puppet arcwise connected function on x_0 which is marked as STPACF.

If f(x) is PACF, SPACF and STPACF on any point of S, so we call f(x) is PACF, SPACF and STPACF on S.

Theorem 3.1 (Zhiun, 2001): Supposed f(x) is the quasi-arcwise connected function QACF on an arcwise connected set $S \subseteq R^n$, if $x_0 \in S$ is a strict local minimum point of f(x), so x_0 is a strict global minimum point of f(x) on S.

Theorem 3.2 (Zhiun, 2001): Supposed f(x) is the strong quasi-arcwise connected function SQACF on an arcwise connected set $S \subseteq R^n$, if $x_0 \in S$ is a strict local minimum point of f(x), so x_0 is the only strict global minimum point of f(x) on S.

Prove: counterevidence. Supposed f(x) is SQACF and $x_0 \in S$ is a local minimum point of f(x), if $\overline{x} \in S$ exists and makes $f(\overline{x}) < f(x_0)$, so the arcwise $H_{\overline{x},x_0}$ connecting $\overline{x}$ with x_0 exists, to any $0 \le \lambda < 1$, there is

$$f(H_{\overline{x},x_0}(\lambda)) < f(x_0) \cdot \quad (3---6)$$

To any neighbor area of x_0, we can always find λ_0 to make $H_{\overline{x},x_0}(\lambda)$ in this neighbor area when $\lambda_0 \le \lambda < 1$, that is contradictive with that x_0 is a local minimum point of f(x), so the theorem is proved.

To STQACF, there are following theorems.

Theorem 3.3: Supposed f(x) is the STQACF defined on an arcwise connected set $S \subseteq R^n$, and if $x_0 \in S$ is a strict local minimum point of f(x), so x_0 is the global minimum point of f(x) on S.

Theorem 3.4: Supposed f(x) is the real valued continual funciton on an arcwise connected set $S \subseteq R^n$, $x_0 \in S$ is the point to fulfill $\nabla f(x_0) = 0$, and if f(x) is STPACF, so x_0 is the global minimum point of f(x) on S. If f(x) is SPACF, so x_0 is the only strict global minimum point of f(x) on S.

Prove: supposed f(x) is STPACF, $x_0 \in S$ is the point to fulfill $\nabla f(x_0) = 0$, so to any $x \in S$ and corresponding arcwise $H_{x,x}$, there is

$$f_{H_{x,x_0}}(x_0) = (\nabla^{-} H_{x,x_0}(v)\nabla f(x_0)) = 0.$$

Thus, from the definition of STPACF, we can obtain $f(x) \geq f(x_0)$, i.e. x_0 is the global minimum point of f(x) on S, and if f(x) is SPACF, so from definition, we can obtain $f(x) > f(x_0)$. To any $x \in S$, $x \neq x_0$ comes into existence, i.e. x_0 is the only global minimum point of f(x) on S.

Theorem 3.5: supposed in the problem (ACP), $X = \{x \in S, g_j(x) \leq 0, j = 1,2,\cdots m\}$ is the feasible area, $f(x), g_j(x), j = 1,2,\cdots m$ is arcwise derivative on the arcwise connected set $S \subseteq R^n$, and if x^* is the optimal solution of (ACP), and $f_{H_{xx^*}}(x^*) and (g_I)_{H_{x,x^*}}(x^*)$ are convex functions about x, so $r_0^* \in R, r^* \in R^m$ exists and makes following equation come into existence to any $x \in S$.

$$r_0^* f_{H_{xx^*}}(x^*) + r^{*T}(g_I)_{H_{x,x^*}}(x^*) \leq 0 \qquad (3\text{---}7)$$

$$r^{*T} g(x^*) \leq 0 \qquad (3\text{---}8)$$

$$(r_0^* . r^*) \geq 0 \qquad (3\text{---}9)$$

Here, $I := I(x^*) = \{i \mid g_i(x^*) = 0\}, J := J(x^*) = \{j \mid g_j(x^*) < 0\}$.

Prove: first, we prove the equation group

$$f_{H_{xx^*}}(x^*) > 0 \qquad (3\text{---}10)$$

$$(g_I)_{H_{x,x^*}}(x^*) > 0 \qquad (3\text{---}11)$$

has no solution in S.

counterevidence, if $x \in S$ exists and is a solution of the equation group, and because $f_{H_{xx^*}}(x^*)$ and $(g_I)_{H_{x,x^*}}(x^*)$ exist, so to any $0 \leq \lambda \leq 1$

$$f(H_{x,x^*}(\lambda)) = f(x^*) + (\lambda - 1) f_{H_{X,X^*}}(x^*) + (1-\lambda)\alpha(1-\lambda) \qquad (3\text{---}12)$$

$$g_i(H_{x,x^*}(\lambda)) = g_i(x^*) + (\lambda - 1)(g_i)_{H_{X,X^*}}(x^*) + (1-\lambda)\alpha_i(1-\lambda) \qquad (3\text{---}13)$$

here, $\alpha : [0,1] \to R.\ \lim_{t \to 0} \alpha(t) = 0$ $\qquad (3.--14)$

$$\alpha_i : [0,1] \to R.\ \lim_{t \to 0} \alpha_i(t) = 0 \qquad (3.--15)$$

from (3.10), (3.11), (3.14) and (3.15), we can obtain, to enough big $\lambda markas \lambda_0 < \lambda < 1$

$$f_{H_{x,x^*}}(x^*) - \alpha(1-\lambda) > 0$$

$$(g_i)_{H_{x,x^*}}(x^*) - \alpha_i(1-\lambda) > 0 \qquad i \in I$$

thus, from (3.10) and (3.11), to $\lambda_0 < \lambda < 1$, there are

$$f_{H_{X,X^*}}(\lambda) - f(x^*) < 0 \qquad (3.--16)$$

$$(g_i)_{H_{x,x^*}}(x^*) - g_i(x^*) < 0 \qquad i \in I \qquad (3.--17)$$

Because $g_j, j \in I$ is arcwise derivative and continual on x^*, and the arcwise $H_{x,x^*}(\lambda)$ is also the continual function about λ, so $\lim_{\lambda \to 1} g_j(H_{x,x^*}(\lambda)) = g_j(x^*) < 0$.

That means $\lambda_j^*, j \in J$ exists, and when $\lambda_j^* \leq \lambda < 1$, $g_j(H_{x,x^*}(\lambda)) < 0$ $\qquad (3.--18)$.

Supposed $\lambda^* = \max\{\lambda_0, \lambda_j^*\}$, so from (3.16), (3.11), (3.18), to $\lambda^* < \lambda < 1$, we can obtain $H_{x,x^*}(\lambda) \in X$, and

$f(H_{x,x^*}(\lambda)) - f(x^*) < 0$. That is contradictive with that x^* is the optimal solution of (ACP), so the equation group (3.10) and (3.11) has no solution.

Because $f_{H_{X,X^*}}(x^*) and (g_I)_{H_{X,X^*}}(x^*)$ are convex function about x, so $r_0^* \in R, r_i^* \in R^m$ which are not zero completely exist and make flowing equation come into existence to any $x \in S$.

$$r_0^* f_{H_{x,x^*}}(x^*) + r_I^{x^T}(g_I)_{H_{x,x^*}}(x^*) \le 0.$$

Let $r_J^* = 0$, so the theorem is proved.

Now, we establish the Mond-Weir duality of (ACP), and give the theorems of weak duality and strong duality.

(ACPD) Max f(u)

$$s.t \quad r_0 f_{H_{x,u}}(u) + r^T g_{H_{x,u}}(u) \le 0 \qquad (3.--19)$$

$$\sum_{j=0}^{m} r_j g_j(u) \ge 0 \qquad (3.--20)$$

$$u \in S \qquad (r_0, r) \ge 0, r_0 \in R, r \in R^m \qquad (3.--21)$$

Theorem 3.6 (weak duality): supposed x is the feasible solution of (ACP), (u, r_0, r) is the feasible solution of (ACPD), and if f(x) is STPACF on u point, $\sum_{J=0}^{m} r_j g_j(u)$ is SPACF on u,

so $$f(x) \ge f(u).$$

Prove: reduction to absurdity, if $f(x) < f(u)$, because f(x) is PACF on u point, from definition, there is

$$r_0 f_{H_{x,u}}(u) \ge 0 \qquad (3---.22).$$

If $r_0 > 0$, so the inequation strictly comes into existence, and because x is the feasible solution of (ACP), (u, r_0, r) is the feasible solution of (ACPD), and we can obtain

$$\sum_{j=1}^{m} r_j g_j(x) \le \sum_{j=1}^{m} r_j g_j(u) \qquad (3.--23)$$

$\sum_{j=1}^{m} r_j g_j(u)$ is SPACF on u, so

$$(\sum_{j=1}^{m} r_j g_j(u)_{H_{x,u}}(u)) \ge 0 \qquad (3.--24).$$

If some $r_j > 0, j = 1,2,3,\cdots m,$ so the inequation strictly comes into existence, thus, from (3.22) and (3.24), we can obtain

$$r_0 f_{H_{x,u}}(u) + r^T g_{H_{x,u}}(u) > 0 \qquad (3.--25).$$

That is contradictive with (3.19), so $f(x) \ge f(u)$.

Theorem 3.7 (strong duality): supposed x^* is the optimal solution of (ACP), $f_{H_{x,x^*}}(x^*) and (g_I)_{H_{x,x^*}}(x^*)$ are convex functions about x, so $r_0^* \in R, r^* \in R^m$ which are not zero completely exist and make (x^*, r_0^*, r^*) be the feasible solution of (ACPD), and the objective function values of (ACP) and (ACPD) are equal on x^*. If to every feasible (u, r_0, r) of (ACPD), f(x) is STPACF on u point, $\sum_{j=1}^{m} r_j g_j(u)$ is SPACF on u, so (x^*, r_0^*, r^*) is the optimal solution of (ACPD).

Prove: because x^* is the optimal solution of (ACP), so from theorem 3.1, $r_0^* \in R, r^* \in R^m$ exist and make (x^*, r_0^*, r^*) is the feasible solution of (ACPD), so the objective function values of (ACP) and (ACPD) are equal on x^*. If (x^*, r_0^*, r^*) is not the optimal solution of (ACPD), so the feasible solution of (ACPD) (u, r_0, r) exists and makes

$$f(u) > f(x^*) \qquad (3\text{----}.26).$$

Because f(x) is STPACF on u point, $\sum_{j=1}^{m} r_j g_j(u)$ is SPACF on u, (3.22) is contradictive with theorem 3.4, so (x^*, r_0^*, r^*) is the optimal solution of (ACPD).

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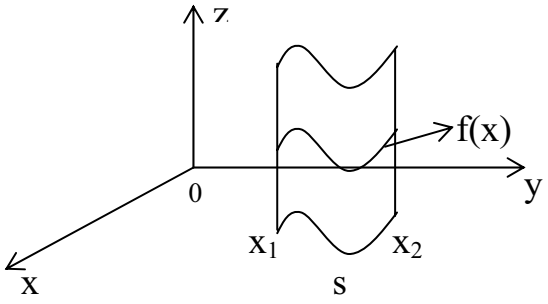

Figure 1. A Line-type Figure without Interior Point