On the Deformation Retractions of Frenet Curves

in Minkowski 4 - Space

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Abstract

In this paper, the position vector equation of the Frenet curves with constant curvatures in Minkowski 4-space has been presented. New types for retractions and deformation retracts of Frenet curves in $E^4_1$ are deduced. The relations between the Frenet apparatus of the Frenet curves before and after the deformation retracts are obtained.

Keywords: Minkowski 4-space $E^4_1$, Frenet curves, retraction, deformation retracts

AMS Subject Classification(2010):


1. Introduction and Definitions

Minkowski space time in $E^4_1$ is an Euclidean space provided with the standard flat metric given by $(X,Y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$ are rectangular coordinate system in $E^4$. Since $(,)$ is an indefinite metric, recall that a vector $u \in E^4_1$ can have one of the three casual characters; it can be space like, if $u > 0$ or $u = 0$, time like, if $u < 0$, null or light like if $u = 0$ and $u \neq 0$. The norm of a vector $v$ is given by $\| v \| = \sqrt{(v,v)}$. Space like or time-like curve $\alpha(s)$ is said to be parametrized by arc length function $s$, if $g(\alpha'(s),\alpha'(s)) = \pm 1$. The velocity of $\alpha$ at $t \in I$ is $\alpha' = \frac{d\alpha(u)}{dt}$. Next, $v, w$ in $E^4_1$ are said to be orthogonal vectors if $g(v,w) = 0$ (M. Turgut & S. Yilmaz.2008) (R. Lopez. 2008) (A. E. El-Ahmady. 2007).

In this paper, we introduce some characterizations of retraction and deformation retract of Frenet curves in $E^4_1$ by the components of the position vector according to the Frenet equations. Also we obtain some relations among curvatures of Frenet curves and their deformation retracts.

2. Main results

Definition: Denoted by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space $E^4_1$. Then $T, N, B_1, B_2$ are the tangent, the principal normal, the first binormal and the second binormal vector fields respectively. Let $\alpha(s)$ is a curve in the space-time in $E^4_1$ parameterized by arc length function $s$. Lopez. Then for the unit speed curve $\alpha(s)$ with non-null frame vectors, such that the Frenet equations are,

\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} = \begin{pmatrix}
0 & k_1 & 0 & 0 \\
\mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\
0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\
0 & 0 & \mu_5 k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]

case 1. If $\alpha$ is a time like curve in $E^4_1$. Then $T$ is a time like vector, so the Frenet equations, $\mu_i (1 \leq i \leq 5)$ read, $\mu_3 = \mu_5 = -1, \mu_1 = \mu_2 = \mu_4 = 1$, where $T, N, B_1, B_2$ are mutually orthogonal vectors with $g(T,T) = -1$, $g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1$.

case 2. If $\alpha$ is a space like curve in $E^4_1$.

Then $T$ is a space like vector, so depending on $N$, then $B_1$ can have all three casual characters,
**Case 2.1.** If \( N \) is space-like, then \( B_1 \) have the next subcases

**Case 2.1.1** If \( B_1 \) be space like, then \( \mu_i (1 \leq i \leq 5) \) read

\[
\mu_1 = \mu_3 = -1, \mu_2 = \mu_4 = \mu_5 = 1,
\]

where \( T, N, B_1, B_2 \) are mutually orthogonal vectors satisfies

\[
g(T, T) = g(N, N) = g(B_1, B_1) = 1, g(B_2, B_2) = -1.
\]

**Case 2.1.2** If \( B_1 \) be time like, then \( \mu_i (1 \leq i \leq 5) \) read

\[
\mu_1 = -1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1,
\]

where \( T, N, B_1, B_2 \) satisfying equations,

\[
g(T, T) = g(N, N) = g(B_2, B_2) = 1, g(B_1, B_1) = -1.
\]

**Case 2.1.3** If \( B_1 \) be a null vector, then the Frenet frame equations read

\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 \\
0 & 0 & k_3 & 0 \\
0 & -k_2 & 0 & -k_3
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]

where \( T, N, B_1, B_2 \) satisfying equations,

\[
g(T, T) = g(N, N) = g(B_1, B_1) = 1, g(B_2, B_2) = -1.
\]

**Remark.** The curves which satisfy the previous cases called Frenet curves.

**Case 2.2** If \( N \) is time-like, then \( \mu_i (1 \leq i \leq 5) \) read

\[
\mu_2 = -1, \mu_3 = \mu_4 = \mu_5 = 1,
\]

where \( T, N, B_1, B_2 \) satisfying equations,

\[
g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, g(N, N) = -1.
\]

**Case 2.3** If \( N \) is light-like (null), then the Frenet equations read

\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
0 & 0 & k_2 & 0 \\
0 & k_3 & 0 & -k_2 \\
-k_1 & 0 & -k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]

where \( k_1 = 0 \), when \( \alpha \) is a straight line or \( k_1 = 1 \), in all other cases. With \( T, N, B_1, B_2 \) mutually orthogonal vectors satisfying the equations,

\[
g(T, T) = g(B_1, B_1) = 1, g(N, N) = g(B_2, B_2) = 0,
\]

\[
g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, g(B_1, B_2) = 1.
\]

**Case 3.** If \( \alpha \) is light-like (null) curve in \( E^4 \);

Then \( T \) is a null vector, so the Frenet equations has the form,

\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
k_2 & 0 & -k_1 & 0 \\
0 & -k_2 & 0 & k_3 \\
-k_1 & 0 & -k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]

where \( k_1 = 0 \), when \( \alpha \) is a straight line or \( k_1 = 1 \), in all other cases. With \( T, N, B_1, B_2 \) mutually orthogonal vectors satisfying the equations,

\[
g(T, T) = g(N, N) = g(B_1, B_1) = 0, g(B_2, B_2) = 1,
\]

\[
g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, g(B_1, B_2) = 1.
\]

Where the functions \( k_1(s), k_2(s), k_3 = k_3(s) \) are called respectively the first, second and third curvature of the curve \( \alpha(s) \) (J. Walrave. 1995).

**Definition 2.1.** A subset \( A \) of a topological space \( X \) is called retract of \( X \) if there exists a continuous map \( r: X \to A \) called a retraction such that \( r(a) = a \) for any \( a \in A \) (A. E. El-Ahmady & A.T.M. Zidan. 2019).
Definition 2.2. A subset $A$ of a topological space $X$ is a deformation retracts of $X$ if there exists a retraction $r: X \to A$ and a homotopy $\varphi: X \times I \to X$ such that:
\[
\begin{cases}
\varphi(x, 0) = x, & x \in X, \\
\varphi(x, 1) = r(x), & a \in A, t \in [0, 1]
\end{cases}
\]

Definition 2.3. Time like curves and space like curves with space like or time like normal vector (curves with non-null frame vectors) are called Frenet curves, where $g(T, T) \neq 0, g(N, N) \neq 0, g(B_1, B_1) \neq 0$ and $g(B_2, B_2) \neq 0$.

3. Position vector of the Frenet curves in $E^4_1$.
Frenet equations of the Frenet curves are,
\[
\begin{pmatrix}
T' \\
N' \\
B'_1 \\
B'_2
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
\mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\
0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\
0 & 0 & \mu_5 k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix}
\]
(1.1)
Let $\eta(s)$ be a Frenet curve in $E^4_1$, whose position vector satisfies the parametric equation,
\[
\eta(s) = v_1(s)T(s) + v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
(2.1)
For some differentiable functions $v_i(s), 1 \leq i \leq 4$, and for $\mu_i (1 \leq i \leq 5), \mu_i \in \{1, -1\}$.
By differentiating equation (2.1) with respect to arc-length parameter $s$ and using the Frenet equations (1.1), for Frenet curves in $E^4_1$, we get
\[
\eta'(s) = (v'_1 + \mu_1 k_1 v_2)T(s) + (v'_2 + k_1 v_1 + \mu_3 k_2 v_3)N(s) + (v'_3 + \mu_2 k_2 v_2 + \mu_5 k_3 v_4)B_1(s) + (v'_4 + \mu_4 k_3 v_3)B_2(s)
\]
(3.1)
then we get
\[
\begin{align*}
v'_1 + \mu_1 k_1 v_2 &= 1 \\
v'_2 + k_1 v_1 + \mu_3 k_2 v_3 &= 0 \\
v'_3 + \mu_2 k_2 v_2 + \mu_5 k_3 v_4 &= 0 \\
v'_4 + \mu_4 k_3 v_3 &= 0
\end{align*}
\]
(4.1)
4. Deformation retracts of Frenet curves in $E^4_1$.
We introduce types of retraction on Frenet curves with non-zero curvature in $E^4_1$.
In the position vector equation of Frenet curve $\eta(s)$, in equation (2.1), if we put $v_1(s) = 0$, then the Frenet retraction curve defined by $\eta_{r_1}(s) = r_1(\eta(s))$ where,
\[
\eta_{r_1}(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
if we put $v_2(s) = 0$, then the Frenet retraction curve defined by $\eta_{r_2}(s) = r_2(\eta(s))$ where,
\[
\eta_{r_2}(s) = v_1(s)T(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
if we put $v_3(s) = 0$, then the Frenet retraction curve defined by $\eta_{r_3}(s) = r_3(\eta(s))$ where,
\[
\eta_{r_3}(s) = v_1(s)T(s) + v_2(s)N(s) + v_4(s)B_2(s)
\]
if we put $v_4(s) = \bar{c}$, $\bar{c} \neq 0$ is constant, then the Frenet retraction curve defined by $\eta_{r_4}(s) = r_4(\eta(s))$ where,
\[
\eta_{r_4}(s) = v_1(s)T(s) + v_2(s)N(s) + v_4(s)B_2(s)
\]
Theorem 4.1. Let $\eta_{r}(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s)$, be the position vector of the Frenet retracted curve of the Frenet curve $\eta(s)$ in $E^4_1$, by taking $v_1(s) = 0$, then $\eta_{r}(s)$ lies in the subspace $NB_1B_2$, and satisfies the differential equation
Proof. The position vector of the Frenet retracted curve $\eta_r(s)$ of the Frenet curve $\eta(s)$ in $E^4_1$, by taking $v_1(s) = 0$, in equation (2), can be written as,

$$\eta_r(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),$$

where $\eta_r(s)$ lies in the subspace $NB_1B_2$, and by taking $v_1(s) = 0$, in equations (4),

$$\mu_k k_i v_2 = 1$$

$$v'_2 + \mu_3k_2v_3 = 0$$

$$v'_3 + \mu_kk_2v_2 + \mu_kk_3v_4 = 0$$

$$v'_4 + \mu_kk_3v_3 = 0$$

(5).

By solving the system in equations (5), then the Frenet retracted curve $\eta_r(s)$ satisfies the differential equation in their curvatures and this completes the proof.

**Theorem 4.2.** The position vector equations of the Frenet retraction curves $\eta_{r1}(s)$ of the Frenet curve $\eta(s)$ with non-zero curvatures in $E^4_1$ can be written in the form,

$$\eta_{r1}(s) = \frac{1}{\mu_k k_1}N(s) + \frac{k'_1}{\mu_1 \mu_2 k_2 k_1^2} B_1(s) - \frac{1}{\mu_3 k_3} \left( \frac{\mu_2 k_2}{k_1} - \frac{d}{ds} \left( \frac{1}{\mu_1 \mu_3 k_2 ds \left( 1 \right)} \right) \right) B_2(s),$$

$$\eta_{r2}(s) = (s + c)T(s) - \left( k_1(s + c) \right) B_1(s) + \frac{1}{\mu_3 k_3} \left( k_1(s + c) \right) B_2(s),$$

$$\eta_{r3}(s) = \frac{\mu_2 \tilde{c}}{\mu_2 k_1} \left( k_1(s + c) \right) T(s) + \frac{\mu_2 k_2}{\mu_2 k_1} N(s) + \tilde{c} B_2(s),$$

$$\eta_{r4}(s) = \frac{\mu_2 \tilde{c}}{\mu_2 k_1} \left( k_1(s + c) \right) T(s) - \frac{\mu_2 \tilde{c} k_3}{\mu_2 k_2} N(s) + \tilde{c} B_1(s),$$

where $\tilde{c}$ be non-zero constant.

**Proof.** The position vector equations of the Frenet retraction curves $\eta_{r1}(s)$ of the Frenet curve $\eta(s)$ with non-zero curvatures in $E^4_1$ can be written in the form,

$$\eta_{r1}(s) = \sum_{j=1}^{4} v_j W_j, \quad i, j \in \{1, 2, 3, 4\}, \quad v_j = 0, \quad \text{when } i = j,$n

where $W_1 = T$, $W_2 = N$, $W_3 = B_1$, and $W_4 = B_2$, so we get,

$$\eta_{r1}(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),$$

From equations (5), where $v_1(s) = 0$, then we get,

$$v_2(s) = \frac{1}{\mu_1 k_1}$$

$$v_3(s) = \frac{k'_1}{\mu_1 \mu_2 k_2 k_1^2}$$

and

$$v_4(s) = -\frac{1}{\mu_1 k_1} \left( \frac{\mu_2 k_2}{k_1} - \frac{d}{ds} \left( \frac{1}{\mu_1 \mu_3 k_2 ds \left( 1 \right)} \right) \right),$$

and the position vector equations of the Frenet retraction curve $\eta_{r1}(s)$ of the Frenet curve $\eta(s)$ with non-zero curvatures can be written as follow,

$$\eta_{r1}(s) = \frac{1}{\mu_1 k_1} N(s) + \frac{k'_1}{\mu_1 \mu_2 k_2 k_1^2} B_1(s) - \frac{1}{\mu_3 k_3} \left( \frac{\mu_2 k_2}{k_1} - \frac{d}{ds} \left( \frac{1}{\mu_1 \mu_3 k_2 ds \left( 1 \right)} \right) \right) B_2(s).$$

Similarly, we can find the Frenet retraction curves $\eta_{r2}(s)$, $\eta_{r3}(s)$, $\eta_{r4}(s)$ and this completes the proof.

**Corollary 4.1.** The Frenet equations of the Frenet curves with non-zero constant curvatures in the Euclidean space $E^4$, are coincide with the Frenet equations of the Frenet curves of constant curvatures in Minkowski 4-space $E^4_1$, if $\mu_1 = \mu_2 = \mu_3 = -1$, and $\mu_4 = 1$.

**Proof.** The proof is clear by substituting $\mu_1 = \mu_3 = \mu_5 = -1$ and $\mu_2 = \mu_4 = 1$, in equations (4), with the same constant curvatures. Then we have
\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} = \begin{pmatrix}
k_1 & 0 & 0 \\
-k_1 & k_2 & 0 \\
0 & -k_2 & k_3 \\
0 & 0 & -k_3
\end{pmatrix} \begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]

which they have the same position vector, and this completes the proof.

5. Frenet curves with constant curvatures in \( E^4_1 \) and their Deformation retracts.

The deformation retract \( D.R. \) of \( \eta(s) \in E^4_1 \) into \( \eta_r(s) = r_t(\eta(s)) \) is given by

\[
D(x,h) = e^h(1-h)\{\eta(s)\} + \frac{h}{2}(h+1)\{\eta_6(s)\}, \quad m \in \mathbb{R} - \{0\},
\]

where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_1(s)\} \).

The D.R of \( \eta(s) \in E^4_1 \) into \( \eta_r(s) = r_2(\eta(s)) \) is given by

\[
D(x,h) = (1-h)\{\eta(s)\} + (h+1)\{\eta_2(s)\},
\]

where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_2(s)\} \).

The D.R of \( \eta(s) \in E^4_1 \) into \( \eta_r(s) = r_3(\eta(s)) \) is given by

\[
D(x,h) = \left( \frac{1-h}{1+h} \right)\{\eta(s)\} + (h e^{h-1})\{\eta_3(s)\},
\]

where \( D(x,0) = \{\eta(s)\} \) and \( D(x,1) = \{\eta_3(s)\} \).

The D.R of \( \eta(s) \in E^4_1 \) into \( \eta_r(s) = r_4(\eta(s)) \) is given by

\[
D(x,h) = \left( \frac{2h}{h+1} \right)\{\eta(s)\} + (|h-1|)\eta_3(s),
\]

where \( D(x,0) = \{\eta(s)\} \) and \( D(x,1) = \{\eta_4(s)\} \).

Let the Frenet curves equation with constant curvatures be represented as follows:

\[
\eta(s) = v_1(s)T(s) + v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),
\]

where \( k_1, k_2 \) and \( k_3 \) are non-zero constant curvatures.

**Theorem 5.1.** Let \( \eta(s) \) be a Frenet curve in \( E^4_1 \) in equation (2) with non-zero constant curvatures, then the position vector of \( \eta(s) \) has been presented by the curvature functions

\[
v_1(s) = -\mu_1 k_1 \left( \frac{c_1 e^{-\lambda_1 t} + c_2 e^{\lambda_1 t}}{\lambda_1} \right),
\]

\[
v_2(s) = \mu_2 k_2 \left( \frac{c_3 e^{-\lambda_2 t} + c_4 e^{\lambda_2 t}}{\lambda_2} \right),
\]

\[
v_3(s) = \mu_3 k_3 \int v_3(s) ds
\]

\[
v_4(s) = -\mu_4 k_4 \left( \frac{c_5 e^{-\lambda_4 t} + c_6 e^{\lambda_4 t}}{\lambda_4} \right),
\]

where \( c_i \) are integral constants and

\[
A = -\mu_1 k_1^2 - \mu_2 k_2^2 + \mu_4 k_4^2,
\]

\[
B = \mu_1 \mu_2 k_1 k_2,
\]

\[
\lambda_1 = \frac{\sqrt{-2A - 2\sqrt{A^2 - 4B}}}{2},
\]

\[
\lambda_2 = \frac{\sqrt{-2A + 2\sqrt{A^2 - 4B}}}{2}.
\]

**Proof.** Let \( \eta(s) \) be a constant curvatures Frenet curve in \( E^4_1 \), by differentiating the second and third equations in equations (4), for \( \mu_i (1 \leq i \leq 5), \mu_i \in \{1, -1\} \), so we can get the system,
Corollary 5.1. Let $\eta(s)$ be a constant curvature time like curve in (2). Then the position vector of $\eta(s)$ has been presented by the curvature functions in (6), when $\mu_i(1 \leq i \leq 5)$ read, $\mu_3 = \mu_5 = -1$, $\mu_4 = 1$.

Corollary 5.2. The position vector of the Frenet retraction curves $\eta_{ri}(s)$ of the Frenet curve $\eta(s)$ with non-zero constant curvatures in $E_4^1$ can be written in the form,

$$\eta_{r1}(s) = \frac{1}{\mu_1 k_1} N(s)$$
$$\eta_{r2}(s) = (s + c) T(s) - \left( \frac{k_1(s+c)}{\mu_3 k_2} \right) B_1(s)$$
$$\eta_{r3}(s) = \frac{c \mu_1 k_1}{\mu_3 k_2} N(s) + c B_2(s)$$
$$\eta_{r4}(s) = \frac{\mu_3 \bar{c} k_3}{\mu_2 k_2} N(s) + \bar{c} B_2(s),$$

where $\bar{c}$ be non-zero constant.

Now we introduce the retraction for the position vector of Frenet curves $\eta(s)$ as follow:

$$\eta(s) = v_1(s) T(s) + v_2(s) N(s) + v_3(s) B_1(s) + v_4(s) B_2(s),$$

for some differentiable functions $v_j(s), 1 \leq j \leq 4$.

Let $r_i: [\eta(s) - \delta] \rightarrow [\eta(s) - \delta]^*$. Where $[\eta(s) - \delta]$ be open Frenet curve in $E_4^1$ and $[\eta(s) - \delta]^*$ be the retraction of the position vector $\eta(s)$.

The retraction $r_5(\eta(s)) = \eta_5(s)$, by substituting $c_1 = 0$ in equations (6),

$$r_5(\eta(s)) = \eta_5(s) = v_{r5_1}(s) T(s) + v_{r5_2}(s) N(s) + v_{r5_3}(s) B_1(s) + v_{r5_4}(s) B_2(s).$$

The retraction $r_6(\eta(s)) = \eta_6(s)$, by substituting $c_2 = 0$ in equations (6),

$$r_6(\eta(s)) = \eta_6(s) = v_{r6_1}(s) T(s) + v_{r6_2}(s) N(s) + v_{r6_3}(s) B_1(s) + v_{r6_4}(s) B_2(s).$$

The retraction $r_7(\eta(s)) = \eta_7(s)$, by substituting $c_3 = 0$ in equations (6),

$$r_7(\eta(s)) = \eta_7(s) = v_{r7_1}(s) T(s) + v_{r7_2}(s) N(s) + v_{r7_3}(s) B_1(s) + v_{r7_4}(s) B_2(s).$$

The retraction $r_8(\eta(s)) = \eta_8(s)$, by substituting $c_4 = 0$ in equations (6),

$$r_8(\eta(s)) = \eta_8(s) = v_{r8_1}(s) T(s) + v_{r8_2}(s) N(s) + v_{r8_3}(s) B_1(s) + v_{r8_4}(s) B_2(s).$$

The deformation retracts of Frenet curves with constant curvatures in Minkowski 4-space, where the deformation retract of the Frenet curve is defined as:

$$\varphi: [\eta(s) - \delta] \times I \rightarrow [\eta(s) - \delta],$$

where $[\eta(s) - \delta]$ is open Frenet curve in $E_4^1$ and $[\eta(s) - \delta]^*$ is the retraction of the position vector $\eta(s)$ and $I$ is the closed interval $[0, 1]$, is presented by

$$\varphi(x, h): [\eta(s) - \delta] \times I \rightarrow [\eta(s) - \delta].$$
The deformation retract (D.R) of \( \eta(s) \subset E_1^4 \) into the retraction \( r_1(\eta) = \eta_1(s) \) is
\[
D(x,h) = (1 - h)\pi_1(\eta(s)) + mh\pi_1(\eta_1(s)),
\]
where \( D(x,0) = \eta(s) \), and \( D(x,1) = \eta_1(s) \), \( m, n \in \mathbb{N} - \{1\} \).
The D.R of \( \eta(s) \subset E_1^4 \) into \( r_2(\eta) = \eta_2(s) \) be
\[
D(x,h) = \sin\left(\frac{(n-1-h)}{2}\right)\left\{\eta(s)\right\} + \cos\left(\frac{(n-1-h)}{2}\right)\left\{\eta_2(s)\right\}. n \in \mathbb{N},
\]
where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_2(s)\} \).

The D.R of \( \eta(s) \subset E_1^4 \) into \( r_3(\eta) = \eta_3(s) \) is
\[
D(x,h) = (\eta - 1)[\{\eta(s)\}] + \frac{mh}{m-1+h}\{\eta_3(s)\}. m \in \mathbb{R} - \{0\},
\]
where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_3(s)\} \).

The D.R of \( \eta(s) \subset E_1^4 \) into \( r_4(\eta) = \eta_4(s) \) be
\[
D(x,h) = (1 - h)[\{\eta(s)\}] + h\{\eta_4(s)\},
\]
where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_4(s)\} \).

The D.R of \( \eta(s) \subset E_1^4 \) into \( r_5(\eta) = \eta_5(s) \) is
\[
D(x,h) = \sqrt{\text{sign}}[\{\eta(s)\}] + \frac{m\sqrt{\text{sign}}[\{\eta_5(s)\}]}{m-1+h}, m \in \mathbb{N},
\]
where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_5(s)\} \).

The D.R of \( \eta(s) \subset E_1^4 \) into \( r_6(\eta) = \eta_6(s) \) be
\[
D(x,h) = (\eta - 1)[\{\eta(s)\}] + \frac{2he^{(1-h)}}{1+h}\{\eta_6(s)\},
\]
where \( D(x,0) = \{\eta(s)\} \), and \( D(x,1) = \{\eta_6(s)\} \).

Theorem 5.2. The deformation retract of any Frenet curve in \( E_1^4 \) be a Frenet curve if and only if the Frenet apparatus \( \{T_c, N_c, B_c, k_{1r}, k_{2r}, k_{3r}\} \) of the retracted curve \( \Omega(s) = r(\eta(s)) \) can be formed by the Frenet apparatus \( \{T, N, B, k_1, k_2, k_3\} \) of \( \eta(s) \).

Proof. Let \( D(s,h) = p(h)n(\eta) + q(h)r(\eta) \) be a deformation retract of the Frenet curve \( \eta(s) \) where \( D(s,0) = \eta(s) \) and \( D(s,1) = r(\eta) \).

\[
D'(s,h) = p(h)n'(\eta) + q(h)r'(\eta) = p(h)\tau(s) + q(h)r'(\eta)\tau(s),
\]
\[
\langle D'(s,h), D'(s,h) \rangle = \langle T'_D, T'_D \rangle = \langle p(h)\tau(s) + q(h)r'(\eta)\tau(s), p(h)\tau(s) + q(h)r'(\eta)\tau(s) \rangle \neq 0.
\]

Then the deformation retract of any Frenet curve in \( E_1^4 \) be Frenet curve, since we can find that \( \langle N'_D, N'_D \rangle \neq 0 \), and \( \langle B'_D, B'_D \rangle \neq 0 \). Conversely this is clear by assume that the Frenet apparatus of the retracted curve \( \phi(s) = r(\eta(s)) \) can be formed by the Frenet apparatus of \( \eta(s) \) and by using the Frenet equations for the Frenet curves.

Conclusion. In this paper, the position vector equation of the Frenet curves with constant curvatures and non-zero curvatures in Minkowski 4-space has been presented. The retractions and Frenet frame of Frenet curves in \( E_1^4 \) are deduced. The relations between the deformation retracts and Frenet Frame of Frenet curves are obtained.

References


R. Lopez. (2008). Differential geometry of curves and surfaces in Lorentz – Minkowski space, Instituto de Matematica Estatistica, University of Sao Paulo, Brazil.

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