On the Deformation Retractions of Frenet Curves

in Minkowski 4 - Space

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Abstract

In this paper, the position vector equation of the Frenet curves with constant curvatures in Minkowski 4-space has been presented. New types for retractions and deformation retracts of Frenet curves in \(E^4\) are deduced. The relations between the Frenet apparatus of the Frenet curves before and after the deformation retracts are obtained.

Keywords: Minkowski 4-space \(E^4\), Frenet curves, retraction, folding, deformation retracts

AMS Subject Classification(2010):


1. Introduction and Definitions

Minkowski space time in \(E^4\) is an Euclidean space provided with the standard flat metric given by \((X, Y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4\), where \((x_1, x_2, x_3, x_4)\) and \((y_1, y_2, y_3, y_4)\) are rectangular coordinate system in \(E^4\). Since \((,)\) is an indefinite metric, recall that a vector \(v \in E^4\) can have one of the three casual characters; it can be space like, if \(< u, u > > 0\) or \(u = 0\), time like, if \(< u, u > < 0\), null or light like if \(< u, u > = 0\) and \(u \neq 0\). The norm of a vector \(v\) is given by \(\| v \| = \sqrt{\langle v, v \rangle}\). Space like or time-like curve \(\alpha(s)\) is said to be parametrized by arclength function \(s\) if \(g(\alpha'(s), \alpha'(s)) = \pm 1\). The velocity of \(\alpha\) at \(t \in I\) is \(\alpha' = \frac{d \alpha(u)}{du}_t\). Next, \(v, w \in E^4\) are said to be orthogonal vectors if \(g(v, w) = 0\) (M. Turgut & S. Yilmaz.2008) (R. Lopez. 2008) (A. E. El-Ahmady. 2007).

In this paper, we introduce some characterizations of retraction and deformation retract of Frenet curves in \(E^4\) by the components of the position vector according to the Frenet equations. Also we obtain some relations among curvatures of Frenet curves and their deformation retracts.

2. Main results

Definition: Denoted by \(\{T(s), N(s), B_1(s), B_2(s)\}\) the moving Frenet frame along the curve \(\alpha(s)\) in the space \(E^4\). Then \(T, N, B_1, B_2\) are the tangent, the principal normal, the first binormal and the second binormal vector fields respectively. Let \(\alpha(s)\) is a curve in the space-time in \(E^4\) parameterized by arc length function \(s\) Lopez. Then for the unit speed curve \(\alpha(s)\) with non-null frame vectors, such that the Frenet equations are,

\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
\mu_1k_1 & 0 & k_2 & 0 \\
0 & \mu_2k_2 & 0 & k_3 \\
0 & 0 & \mu_3k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix}.
\]

Case 1. If \(\alpha\) is a time like curve in \(E^4\). Then \(T\) is a time like vector, so the Frenet equations, \(\mu_i(1 \leq i \leq 5)\) read, \(\mu_3 = \mu_5 = -1, \mu_1 = \mu_2 = \mu_4 = 1\), where \(T, N, B_1, B_2\) are mutually orthogonal vectors with \(g(T, T) = -1, \quad g(N, N) = g(B_1, B_1) = g(B_2, B_2) = 1\).

Case 2. If \(\alpha\) is a space like curve in \(E^4\).

Then \(T\) is a space like vector, so depending on \(N\), then \(B_1\) can have all three causal characters,
**Case 2.1.** If $N$ is space-like, then $B_1$ have the next subcases

**Case 2.1.1** If $B_1$ be space like, then $\mu_i (1 \leq i \leq 5)$ read

$$\mu_1 = \mu_2 = -1, \mu_3 = \mu_4 = \mu_5 = 1,$$

where $T, N, B_1, B_2$ are mutually orthogonal vectors satisfies

$$g(T, T) = g(N, N) = g(B_1, B_1) = 1, g(B_2, B_2) = -1.$$

**Case 2.1.2** If $B_1$ is time like, then $\mu_i (1 \leq i \leq 5)$ read

$$\mu_1 = -1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = 1,$$

where $T, N, B_1, B_2$ satisfying equations,

$$g(T, T) = g(N, N) = g(B_2, B_2) = 1, g(B_1, B_1) = -1.$$

**Case 2.1.3** If $B_1$ be a null vector, then the Frenet frame equations read

$$\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 \\
0 & 0 & k_3 & 0 \\
0 & -k_2 & 0 & -k_3
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},$$

where $T, N, B_1, B_2$ satisfying equations,

$$g(T, T) = g(N, N) = g(B_2, B_2) = 1, g(B_1, B_1) = -1.$$

**Remark.** The curves which satisfy the previous cases called Frenet curves.

**Case 2.2** If $N$ is time-like, then $\mu_i (1 \leq i \leq 5)$ read

$$\mu_5 = -1, \mu_1 = \mu_2 = \mu_3 = \mu_4 = 1,$$

where $T, N, B_1, B_2$ are satisfying equations,

$$g(T, T) = g(B_1, B_1) = g(B_2, B_2) = 1, g(N, N) = -1.$$

**Case 2.3** If $N$ is light-like (null), then the Frenet equations read

$$\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
0 & 0 & k_2 & 0 \\
0 & k_3 & 0 & -k_2 \\
-k_1 & 0 & -k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},$$

where $k_1 = 0$, when $\alpha$ is a straight line or $k_1 = 1$, in all other cases. With $T, N, B_1, B_2$ are mutually orthogonal vectors satisfying the equations,

$$g(T, T) = g(B_1, B_1) = 1, g(N, N) = g(B_2, B_2) = 0, g(T, B_1) = g(B_1, B_2) = 0, g(N, B_2) = 1.$$

**Case 3.** If $\alpha$ is light-like (null) curve in $\mathbb{E}^4$.

Then $T$ is a null vector, so the Frenet equations has the form,

$$\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
k_2 & 0 & -k_1 & 0 \\
0 & -k_2 & 0 & k_3 \\
-k_1 & 0 & -k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},$$

where $k_1 = 0$, when $\alpha$ is a straight line or $k_1 = 1$, in all other cases. With $T, N, B_1, B_2$ are mutually orthogonal vectors satisfying the equations,

$$g(T, T) = g(N, N) = g(B_1, B_1) = 0, g(B_2, B_2) = 1,$$

$$g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0, g(B_1, B_2) = 1.$$

Where the functions $k_1 = k_1(s), k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called respectively the first, second and third curvature of the curve $a(s)$ (J. Walrave. 1995).

**Definition 2.1.** A subset $A$ of a topological space $X$ is called retract of $X$ if there exists a continuous map $r : X \to A$ called a retraction such that $r(a) = a$ for any $a \in A$ (A. E. El-Ahmady & A.T.M. Zidan. 2019).
A subset \( A \) of a topological space \( X \) is a deformation retracts of \( X \) if there exists a retraction \( r: X \to A \) and a homotopy \( \varphi: X \times I \to X \) such that:
\[
\begin{cases}
\varphi(x, 0) = x & x \in X, \\
\varphi(a, t) = a & a \in A, t \in [0, 1]
\end{cases}
\]

**Definition 2.3.** Time like curves and space like curves with space like or time like normal vector (curves with non-null frame vectors) are called Frenet curves, where \( g(T, T) \neq 0, g(N, N) \neq 0, g(B_1, B_1) \neq 0 \) and \( g(B_2, B_2) \neq 0 \).

**3. Position vector of the Frenet curves in \( E_4^4 \).**

Frenet equations of the Frenet curves are,
\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
\mu_1 k_1 & 0 & \mu_2 k_2 & 0 \\
0 & \mu_3 k_2 & 0 & \mu_4 k_3 \\
0 & 0 & \mu_5 k_3 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix}
\]
(1).

Let \( \eta(s) \) be a Frenet curve in \( E_4^4 \), whose position vector satisfies the parametric equation,
\[
\eta(s) = v_1(s)T(s) + v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
(2).

For some differentiable functions \( v_j(s), 1 \leq j \leq 4 \), and for \( \mu_i (1 \leq i \leq 5), \mu_i \in \{1, -1\} \).

By differentiating equation (2) with respect to arc-length parameter \( s \) and using the Frenet equations (1), for Frenet curves in \( E_4^4 \), we get
\[
\eta'(s) = (v'_1 + \mu_1 k_1 v_2)T(s) + (v'_2 + k_1 v_1 + \mu_3 k_2 v_3)N(s) + (v'_3 + \mu_2 k_2 v_2 + \mu_5 k_3 v_4)B_1(s) + (v'_4 + \mu_4 k_3 v_3)B_2(s)
\]
(3).

then we get
\[
\begin{align*}
v'_1 + \mu_1 k_1 v_2 &= 1 \\
v'_2 + k_1 v_1 + \mu_3 k_2 v_3 &= 0 \\
v'_3 + \mu_2 k_2 v_2 + \mu_5 k_3 v_4 &= 0 \\
v'_4 + \mu_4 k_3 v_3 &= 0
\end{align*}
\]
(4).

**4. Deformation retracts of Frenet curves in \( E_4^4 \).**

We introduce types of retraction on Frenet curves with non-zero curvature in \( E_4^4 \).

In the position vector equation of Frenet curve \( \eta(s) \), in equation (2),
if we put \( v_1(s) = 0 \), then the Frenet retraction curve defined by \( \eta_{r_1}(s) = r_1(\eta(s)) \) where,
\[
\eta_{r_1}(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
if we put \( v_2(s) = 0 \), then the Frenet retraction curve defined by \( \eta_{r_2}(s) = r_2(\eta(s)) \) where,
\[
\eta_{r_2}(s) = v_1(s)T(s) + v_3(s)B_1(s) + v_4(s)B_2(s)
\]
if we put \( v_3(s) = 0 \), then the Frenet retraction curve defined by \( \eta_{r_3}(s) = r_3(\eta(s)) \) where,
\[
\eta_{r_3}(s) = v_1(s)T(s) + v_2(s)N(s) + v_4(s)B_2(s)
\]
if we put \( v_4(s) = \bar{c}, \bar{c} \neq 0 \) is constant, then the Frenet retraction curve defined by \( \eta_{r_4}(s) = r_4(\eta(s)) \) where,
\[
\eta_{r_4}(s) = v_1(s)T(s) + v_2(s)N(s) + v_4(s)B_2(s)
\]

**Theorem 4.1.** Let \( \eta_r(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s) \), be the position vector of the Frenet retracted curve of the Frenet curve \( \eta(s) \) in \( E_4^4 \), by taking \( v_4(s) = 0 \), then \( \eta_r(s) \) lies in the subspace \( NB_1B_2 \), and satisfies the differential equation.
Proof. The position vector of the Frenet retracted curve \( \eta_r(s) \) of the Frenet curve \( \eta(s) \) in \( E_4^2 \), by taking \( v_1(s) = 0 \), in equation (2), cab be written as,

\[
\eta_r(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),
\]

where \( \eta_r(s) \) lies in the subspace \( NB_1B_2 \), and by taking \( v_1(s) = 0 \), in equations (4),

\[
\begin{align*}
\mu_1k_1v_2 &= 1 \\
v'_2 + \mu_3k_2v_3 &= 0 \\
v'_3 + \mu_2k_2v_2 + \mu_5k_3v_4 &= 0 \\
v'_4 + \mu_4k_3v_3 &= 0
\end{align*}
\]

(5).

By solving the system in equations (5), then the Frenet retracted curve \( \eta_r(s) \) satisfies the differential equation in their curvatures and this completes the proof.

**Theorem 4.2.** The position vector equations of the Frenet retraction curves \( \eta_{r1}(s) \) of the Frenet curve \( \eta(s) \) with non-zero curvatures in \( E_4^2 \) can be written in the form,

\[
\begin{align*}
\eta_{r1}(s) &= \frac{1}{\mu_1k_1}N(s) + \frac{k'_1}{\mu_1\mu_3k_2k_4}B_1(s) - \frac{1}{\mu_5k_3}\left(\frac{\mu_2k_2}{k_1} - \frac{d}{ds}\left(\frac{1}{\mu_1\mu_3k_2\frac{d}{ds}\left(\frac{1}{k_1}\right)}\right)\right)B_2(s), \\
\eta_{r2}(s) &= (s+c)T(s) - \left(k_1(s+c)\right)B_1(s) + \frac{1}{\mu_3k_2}\left(k_1(s+c)\right)B_2(s), \\
\eta_{r3}(s) &= \frac{c\mu_1}{\mu_2k_1}\frac{d}{ds}\left(k'_2\right)T(s) + \frac{c\mu_1k_1}{\mu_3k_2}N(s) + cB_2(s), \\
\eta_{r4}(s) &= \frac{\mu_5c}{\mu_2k_1}\frac{d}{ds}\left(k'_2\right)T(s) - \frac{\mu_5c}{\mu_2k_2}N(s) + cB_1(s),
\end{align*}
\]

where \( \tilde{c} \) be non-zero constant.

Proof. The position vector equations of the Frenet retraction curves \( \eta_{r1}(s) \) of the Frenet curve \( \eta(s) \) with non-zero curvatures in \( E_4^2 \) can be written in the form,

\[
\eta_{r1}(s) = \sum_{j=1}^{4} v_j W_j, \quad i,j \in \{1, 2, 3, 4\}, \quad v_j = 0, \quad \text{when} \quad i = j,
\]

where \( W_1 = T \), \( W_2 = N \), \( W_3 = B_1 \), and \( W_4 = B_2 \), so we get,

\[
\eta_{r1}(s) = v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),
\]

From equations (5), where \( v_1(s) = 0 \), then we get,

\[
\begin{align*}
v_2(s) &= \frac{1}{\mu_1k_1}, \\
v_3(s) &= \frac{k'_1}{\mu_1\mu_3k_2k_4}, \quad \text{and} \quad v_4(s) = -\frac{1}{\mu_5k_3}\left(\frac{\mu_2k_2}{k_1} - \frac{d}{ds}\left(\frac{1}{\mu_1\mu_3k_2\frac{d}{ds}\left(\frac{1}{k_1}\right)}\right)\right),
\end{align*}
\]

and the position vector equations of the Frenet retraction curve \( \eta_{r1}(s) \) of the Frenet curve \( \eta(s) \) with non-zero curvatures can be written as follow,

\[
\eta_{r1}(s) = \frac{1}{\mu_1k_1}N(s) + \frac{k'_1}{\mu_1\mu_3k_2k_4}B_1(s) - \frac{1}{\mu_5k_3}\left(\frac{\mu_2k_2}{k_1} - \frac{d}{ds}\left(\frac{1}{\mu_1\mu_3k_2\frac{d}{ds}\left(\frac{1}{k_1}\right)}\right)\right)B_2(s).
\]

Similarly, we can find the Frenet retraction curves \( \eta_{r2}(s), \eta_{r3}(s), \eta_{r4}(s) \) and this completes the proof.

**Corollary 4.1.** The Frenet equations of the Frenet curves with non-zero constant curvatures in the Euclidean space \( E^4 \), are coincide with the Frenet equations of the Frenet curves of constant curvatures in Minkowski 4-space \( E_4^2 \), if \( \mu_1 = \mu_3 = \mu_5 = -1 \), and \( \mu_2 = \mu_4 = 1 \).

Proof. The proof is clear by substituting \( \mu_1 = \mu_3 = \mu_5 = -1 \) and \( \mu_2 = \mu_4 = 1 \), in equations (4), with the same constant curvatures. Then we have
\[
\begin{pmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{pmatrix} =
\begin{pmatrix}
0 & k_1 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 \\
0 & -k_2 & 0 & k_3 \\
0 & 0 & -k_3 & 0
\end{pmatrix} \begin{pmatrix}
T \\
N \\
B_1 \\
B_2
\end{pmatrix},
\]
which they have the same position vector, and this completes the proof.

5. Frenet curves with constant curvatures in $E_4^1$ and their Deformation retracts.

The deformation retract (D.R.) of $\eta(s) \in E_4^1$ into $\eta_2(s) = r_2(\eta(s))$ is given by

\[
D(x,h) = e^h(1-h)\{\eta(s)\} + \frac{h}{2}(h+1)\{\eta_0(s)\}, \ m \in \mathbb{R} - \{0\},
\]

where $D(x,0) = \{\eta(s)\}$, and $D(x,1) = \{\eta_1(s)\}$.

The D.R. of $\eta(s) \in E_4^1$ into $\eta_3(s) = r_3(\eta(s))$ is given by

\[
D(x,h) = \left(\frac{1-h}{1+h}\right)\{\eta(s)\} + \left(h e^{h-1}\right)\{\eta_3(s)\},
\]

where $D(x,0) = \{\eta(s)\}$ and $D(x,1) = \{\eta_3(s)\}$.

The D.R. of $\eta(s) \in E_4^1$ into $\eta_4(s) = r_4(\eta(s))$ is given by

\[
D(x,h) = \left(\frac{2h}{h+1}\right)\{\eta(s)\} + \left(|h-1|\eta_3(s)\right),
\]

where $D(x,0) = \{\eta(s)\}$ and $D(x,1) = \{\eta_4(s)\}$.

Let the Frenet equation with constant curvatures be represented as follows:

\[
\eta(s) = v_1(s)T(s) + v_2(s)N(s) + v_3(s)B_1(s) + v_4(s)B_2(s),
\]

where $k_1, k_2$ and $k_3$ are non-zero constant curvatures.

**Theorem 5.1.** Let $\eta(s)$ be a Frenet curve in $E_4^4$ in equation (2) with non-zero constant curvatures, then the position vector of $\eta(s)$ has been presented by the curvature functions

\[
\begin{align*}
v_1(s) &= -\mu_1 k_1 \left( -\frac{c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s}}{\lambda_1} + \frac{-c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s}}{\lambda_2} \right) + c_0, \\
v_2(s) &= c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s} + c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s} + \frac{1}{\mu_1 k_1}, \\
v_3(s) &= \frac{k_1}{k_2} \left( \frac{\lambda_1^2 + k_1^2}{\lambda_1} \right) \left( -c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s} \right) + \frac{k_1}{k_2} \left( \frac{\lambda_2^2 + k_1^2}{\lambda_2} \right) \left( -c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s} \right) + k_1 k_2 c_5, \\
v_4(s) &= -\mu_4 k_3 \int v_3(s) ds \\
&= -\mu_4 k_3 \left( \frac{\lambda_1^2 + k_1^2}{\lambda_1} \right) \left( c_1 e^{-\lambda_1 s} + c_2 e^{\lambda_1 s} \right) + \frac{k_1}{k_2} \left( \frac{\lambda_2^2 + k_1^2}{\lambda_2^2} \right) \left( c_3 e^{-\lambda_2 s} + c_4 e^{\lambda_2 s} \right) + k_1 k_2 c_5 + c_6.
\end{align*}
\]

Where $c_i, (0 \leq l \leq 6)$ are integral constants and

\[
\begin{align*}
A &= -\left( \mu_1 k_1^2 + \mu_2 k_5 k_2^2 + \mu_4 k_5 k_3^2 \right), \\
B &= \mu_1 \mu_4 k_5 k_1 k_2 k_3, \\
\lambda_1 &= \frac{\sqrt{-2A - 2\sqrt{A^2 - 4B}}}{2}, \\
\lambda_2 &= \frac{\sqrt{-2A + 2\sqrt{A^2 - 4B}}}{2},
\end{align*}
\]

**Proof.** Let $\eta(s)$ be a constant curvatures Frenet curve in $E_4^4$, by differentiating the second and third equations in equations (4), for $\mu_i (1 \leq i \leq 5), \mu_i \in \{1, -1\}$, so we can get the system,
where $\mathcal{E} = 1$.

**Corollary 5.2.** The retraction of the Frenet curve is defined as:

$$v_1' = -\mu_1 k_1 v_2$$
$$v_2' = -\mu_2 k_2 v_1 - k_1 (1 - \mu_1 k_1 v_2)$$
$$v_3'' = \mu_4 \mu_5 k_3^2 v_3 - \mu_2 k_2 v_2'$$
$$v_4' + \mu_4 k_3 v_3 = 0$$

By solving the system in equations (8), which has non-trivial solution (6), and this completes the proof.

**Corollary 5.1.** Let $\eta(s)$ be a constant curvature time like curve in (2). Then the position vector of $\eta(s)$ has been presented by the curvature functions in (6), when $\mu_1 (1 \leq i \leq 5) \text{ read, } \mu_3 = \mu_5 = -1, \mu_4 = \mu_5 = 1$.

**Corollary 5.2.** The position vector of the Frenet retraction curves $\eta_j(s)$ of the Frenet curve $\eta(s)$ with non-zero constant curvatures in $E_4^+$ can be written in the form,

$$\eta_j(s) = \frac{1}{\mu k_1} N(s)$$

$$\eta_r(\mathcal{E}) = (s + c) T(s) - \frac{k_1(s+c)}{\mu k_2} B_1(s)$$

$$\eta_r(\mathcal{E}) = \frac{c_\mu k_1}{\mu k_2} N(s) + c B_2(s)$$

$$\eta_r(\mathcal{E}) = \frac{c_\mu k_1}{\mu k_2} N(s) + c B_2(s)$$

where $c \neq 0$ be non-zero constant.

Now we introduce the retraction for the position vector of Frenet curves $\eta(s)$ as follow:

$$\eta(s) = v_1(s) T(s) + v_2(s) N(s) + v_3(s) B_1(s) + v_4(s) B_2(s)$$

for some differentiable functions $v_j(s), 1 \leq j \leq 4$.

Let $r_j: \{\eta(s) - \delta\} \to \{\eta(s) - \delta\}^*$. Where $\{\eta(s) - \delta\}$ be open Frenet curve in $E_4^+$ and $\{\eta(s) - \delta\}^*$ be the retraction of the position vector $\eta(s)$.

The retraction $r_5(\eta_j(s)) = \eta_5(s)$, by substituting $c_1 = 0$ in equations (6),

$$r_5(\eta(s)) = \eta_5(s) = v_{r_5} T(s) + v_{r_5} N(s) + v_{r_5} B_1(s) + v_{r_5} B_2(s).$$

The retraction $r_6(\eta(s)) = \eta_6(s)$, by substituting $c_2 = 0$ in equations (6),

$$r_6(\eta(s)) = \eta_6(s) = v_{r_6} T(s) + v_{r_6} N(s) + v_{r_6} B_1(s) + v_{r_6} B_2(s).$$

The retraction $r_7(\eta(s)) = \eta_7(s)$, by substituting $c_3 = 0$ in equations (6),

$$r_7(\eta(s)) = \eta_7(s) = v_{r_7} T(s) + v_{r_7} N(s) + v_{r_7} B_1(s) + v_{r_7} B_2(s).$$

The retraction $r_8(\eta(s)) = \eta_8(s)$, by substituting $c_4 = 0$ in equations (6),

$$r_8(\eta(s)) = \eta_8(s) = v_{r_8} T(s) + v_{r_8} N(s) + v_{r_8} B_1(s) + v_{r_8} B_2(s).$$

The retraction $r_9(\eta(s)) = \eta_9(s)$, by substituting $B_1 = 0$ in equation (2),

$$r_9(\eta(s)) = \eta_9(s) = v_1 T(s) + v_2 N(s) + v_3 B_2(s).$$

The retraction $r_{10}(\eta(s)) = \eta_{10}(s)$, by substituting $B_2 = 0$ in equation (2),

$$r_{10}(\eta(s)) = \eta_{10}(s) = v_1 T(s) + v_2 N(s) + v_3 B_1(s).$$

The retraction $r_{11}(\eta(s)) = \eta_{11}(s)$, by substituting $B_1 = 0$ and $B_2 = 0$, in equation (2),

$$r_{11}(\eta(s)) = \eta_{11}(s) = v_1 T(s) + v_2 N(s).$$

The deformation retracts of Frenet curves with constant curvatures in Minkowski 4-space, where the deformation retract of the Frenet curve is defined as:

$$\varphi: \{\eta(s) - \delta\} \times I \to \{\eta(s) - \delta\}^*$$

where $\{\eta(s) - \delta\}$ is open Frenet curve in $E_4^+$ and $\{\eta(s) - \delta\}^*$ is the retraction of the position vector $\eta(s)$ and $I$ is the closed interval $[0, 1]$, is presented by

$$\varphi(x, h): \{\eta(s) - \delta\} \times I \to \{\eta(s) - \delta\}.$$
The deformation retract \((D.R)\) of \(\eta(s) \in E^4_1\) into the retraction \(r_1(\eta) = \eta_1(s)\) is
\[
D(x,h) = (1 - h)\eta + h\eta_1(s),
\]
where \(D(x,0) = \eta(s),\) and \(D(x,1) = \eta_1(s), m, n \in \mathbb{N} - \{1\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_2(\eta) = \eta_2(s)\) be
\[
D(x,h) = \sin\left(\frac{m(1-h)}{2}\right)\{\eta(s)\} + \cos\left(\frac{m(1-h)}{2}\right)\{\eta_2(s)\}, n \in \mathbb{N},
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_2(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_3(\eta) = \eta_3(s)\) is
\[
D(x,h) = |h - 1|\{\eta(s)\} + \frac{m}{m-1+h}\{\eta_3(s)\}, m \in \mathbb{R} - \{0\},
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_3(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_4(\eta) = \eta_4(s)\) be
\[
D(x,h) = (1 - h)|\eta(s)| + h|\eta_4(s)|,
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_4(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_5(\eta) = \eta_5(s)\) is
\[
D(x,h) = \sqrt{1-h}\{\eta(s)\} + \frac{m}{m-1+h}\{\eta_5(s)\}, m \in \mathbb{N},
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_5(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_6(\eta) = \eta_6(s)\) is given by
\[
D(x,h) = |h - 1|\{\eta(s)\} + \frac{2he^{(t-h)}}{1+h}|\eta_6(s)|,
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_6(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_7(\eta) = \eta_7(s)\) be
\[
D(x,h) = \left(\frac{1-h}{1+h}\right)|\eta| + \left(\frac{2h}{1+h}\right)|\eta_7(s)|,
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_7(s)\} \).
The D.R of \(\eta(s) \in E^4_1\) into \(r_8(\eta) = \eta_8(s)\) be
\[
D(x,h) = \cos\left(\frac{n}{2} + 2n\pi\right)|\eta(s)| - \sin\left(\frac{n}{2} + 2n\pi\right)|\eta_8(s)|, n \in \mathbb{N},
\]
where \(D(x,0) = \{\eta(s)\},\) and \(D(x,1) = \{\eta_8(s)\} \).

**Theorem 5.2.** The deformation retract of any Frenet curve in \(E^4_1\) be a Frenet curve if and only if the Frenet apparatus \(\{T_r, N_r, B_r, k_{1r}, k_{2r}, k_{3r}\}\) of the retracted curve \(\Omega(s) = r(\eta(s))\) can be formed by the Frenet apparatus \(\{T, N, B, k_1, k_2, k_3\}\) of \(\eta(s)\).

**Proof.** Let \(D(s,h) = p(h)|\eta(s)| + q(h)r(\eta)\) be a deformation retract of the Frenet curve \(\eta(s)\) where \(D(s,0) = \eta(s)\) and \(D(s,1) = r(\eta)\).
\[
D'(s,h) = p(h)|\eta'(s)| + q(h)r'(\eta)\eta'(s) = p(h)T' + q(h)r'(\eta)T(s),
\]
\[
\langle D'(s,h), D'(s,h) \rangle = \langle T'_r, T'_r \rangle = \langle p(h)T' + q(h)r'(\eta)T(s), p(h)T' + q(h)r'(\eta)T(s) \rangle \neq 0.
\]

Then the deformation retract of any Frenet curve in \(E^4_1\) be Frenet curve, since we can find that \(\langle N'_p, N'_p \rangle \neq 0,\) and \(\langle B'_p, B'_p \rangle \neq 0.\) Conversely this is clear by assume that the Frenet apparatus of the retracted curve \(\phi(s) = r(\eta(s))\) can be formed by the Frenet apparatus of \(\eta(s)\) and by using the Frenet equations for the Frenet curves.

**Conclusion.** In this paper, the position vector equation of the Frenet curves with constant curvatures and non-zero curvatures in Minkowski 4-space has been presented. The retraction and Frenet frame of Frenet curves in \(E^4_1\) are deduced. The relations between the deformation retracts and Frenet Frame of Frenet curves are obtained.

**References**


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