On Null Curves in Minkowski 3-Space and Its Fractal Folding

A.E. El-Ahmady¹, Malak E. Raslan² & A.T. M-Zidan³

¹ Department of Mathematics, Faculty of Science, Tanta University, Egypt
² Department of Mathematics, Faculty of Science, Damietta University, Egypt
Correspondence: A.T. M-Zidan, Mathematics Department, Faculty of Science, Damietta University, Egypt.
E-mail: atm_zidan@yahoo.com

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Abstract
In this paper, a form for Frenet equations of all null curves in Minkowski 3-space has been presented. New types of foldings of curves are obtained. The connection between folding, deformation and Frenet equations of curves are also deduced.

Keywords: Minkowski 3-space, null curves, conditional fractal folding, deformation, Frenet equations


1. Introduction

The Minkowski 3-space $E^3$ is the Euclidean 3-space $E^3$ provided with the standard flat metric given by

$$ g = dx_1^2 + dx_2^2 - dx_3^2, $$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system in $E^3$. Since $g$ is an indefinite metric, recall that a vector $v \in E^3$ is said space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $E^3$ can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively, space-like, time-like or null (light-like) respectively. Space-like or time-like curve $\alpha(s)$ is said to be parameterized by arc length function $s$, if $\alpha'(s) \neq 0$, $s \in I$, $\alpha(0), \alpha(1), \alpha(2)$ are mutually orthogonal vectors satisfying the equations,

$$ = \alpha(0), \alpha(1), \alpha(2) = 0, \alpha'(t) \neq 0, \text{ recall the norm of a vector } v \text{ is given by } ||v|| = \sqrt{g(v, v)}. $$

Given a unit speed curve $\alpha(s)$ in Minkowski space $E^3$ we can possible define a Frenet frame $(T(s), N(s), B(s))$ associated for each point $s$. Where $T(s), N(s)$ and $B(s)$ are the tangent, normal and binormal vector field (A. E. El-Ahmady & A.T.M. Zidan. 2019) (A. E. El-Ahmady & E. Al-Hesiny. 2013) (R. Lopez. 2008) (R. Aslaner, A. Ilshin Boran. 2009).

2. Preliminary Notes

Let $\alpha(s)$ be a curve in $E^3$. Then for the unit speed curve $\alpha(s)$ with non-null frame vectors, we distinguish three cases depending on the causal character of $T(s)$ and its Frenet equations are as follows,

$$ \left( \begin{array}{c} T' \\ N' \\ B' \end{array} \right) = \left( \begin{array}{ccc} 0 & k & 0 \\ \mu_1 k & 0 & \mu_2 \tau \\ 0 & \mu_3 \tau & 0 \end{array} \right) \left( \begin{array}{c} T \\ N \\ B \end{array} \right). $$

We write the following subcases,

Case 1. If $\alpha(s)$ is time-like curve in $E^3$, then $T$ is time-like vector and $T'$ is space-like vector. Then $\mu_1 < i < 3$, read $\mu_1 = \mu_2 = 1, \mu_3 = -1, T, B$ and $N$ are mutually orthogonal vectors satisfying the equations, $g(N, N) = g(B, B) = 1, g(T, T) = -1$.

Case 2. If $\alpha(s)$ is space like curve in $E^3$, then $T$ is space like vector, since $T'(s)$ is orthogonal to the space like vector $T'(s), T'(s)$ may be space like, time-like or light like. Thus we distinguish three cases according to $T'(s)$. 

Case 2.1. If the vector $T'(s)$ is space-like, $N$ is space-like vector and $B$ is time-like vector. Then $\mu_i(1 < i < 3)$ read $\mu_1 = -1, \mu_2 = \mu_3 = 1, T, N$ and Bare mutually orthogonal vectors satisfying $g(T,T) = g(N,N) = 1, g(B,B) = -1$.

Case 2.2. If the vector $T'(s)$ is time-like, $N$ is time-like vector and $B$ is space-like vector. Then $\mu_i(1 < i < 3)$ read $\mu_1 = \mu_2 = \mu_3 = 1$, where the orthogonal vectors $T, N$ and $B$ are satisfying $g(T,T) = g(B,B) = 1, g(N,N) = -1$.

Case 2.3. If the vector $T'(s)$ is light-like for all $s$, $N(s) = T'(s)$ is light-like vector and $B(s)$ is unique light-like vector such that $g(N,B) = -1$ and it is orthogonal to $T$. The Frenet equations are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ 1 & 0 & -\tau \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$  

Case 3. If $\alpha(s)$ is light-like curve in $E^3_1$, $g(N,N) > 0$, when the parameterization is pseudo-are so $g(N,N) = 1$ with $g(T,T) = 0$, $g(B,B) = 0$, $g(T,N) = 0$, and $B(s)$ is unique light-like vector such that $g(T,B) = -1$ and it is orthogonal to $N$ the pseudo torsion of $\alpha(s)$ be $\tau = -\langle N', B \rangle$, then the Frenet equations of $\alpha(s)$ are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & k \\ 0 & \tau & 0 \end{pmatrix}.$$  

Where the curvature $k$ can take only two values, 0 when $\alpha$ is a straight null line, or 1 in all other cases (J. Walrave. 1995).

A regular curve $\alpha : I \to E^3_1$ is called a null curve if $\alpha'$ is light-like, that is $\langle \alpha', \alpha' \rangle = 0$ (M. P. Docarmo. 1992).

Let $M$ and $N$ be two smooth manifolds of dimensions $m$ and $n$ respectively. A map $f : M \to N$ is said to be an isometric folding of $M$ into $N$ if and only if for every piece-wise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piece-wise geodesic and of the same length as $\gamma$, if $f$ does not preserve the length it is called topological folding (A. E. El-Ahmady & E. El-Hesiny. 2013). A map $d : M \to M^*$ such that $M^* = d(M)$ where $M$ and $M^*$ are two smooth Riemannian manifolds is called deformation map if $d$ is differentiable and has differentiable inverse. A deformation map $d : M \to M^*$ where $M$ and $M^*$ are two smooth Riemannian manifolds is called regular deformation if $\forall \, x,y \in M$, $K(x) = K(y) \iff K(d(x)) = K(d(y))$, $K(x)$ is the curvature at the point $x \in M$, when $(x) = K(d(x)) \forall x \in M$, it is the identity deformation which is regular deformation (M. P. Docarmo. 1992).

**Definition 2.1.** Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, be vectors in $E^3_1$, the vector product in Minkowski space-time $E^3_1$ is defined by the determinant

$$u \wedge v = \begin{vmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$  

where $e_1, e_2$ and $e_3$ are mutually orthogonal vectors (coordinate direction vectors).

**3. Form of Frenet Equations of Null Curves in Minkowski 3-Space**

**Theorem 3.1.** Let $\xi(s)$ be a null curve in $E^3_1$ with the standard flat metric given by $g = dx_1^2 + dx_2^2 - dx_3^2$. Then the bi-normal vector of $\xi(s)$ can be calculated by the form,

$$B(s) = (\frac{-1}{\Delta_{1,2}}(\Delta_{2,3} b_3 + x''_3), \frac{1}{\Delta_{1,2}}(\Delta_{1,3} b_3 + x''_3), -\frac{(1+x''_3)^2}{2x_3^3}), \Delta_{1,2} \neq 0, x_3' \neq 0.$$  

Where $\Delta_{2,3} = (x_2' x_3' - x_1' x_3''), \Delta_{1,3} = (x_1' x_2' - x_3' x_3'')$ and $\Delta_{1,2} = (x_3' - x_2' x_3'' - x_3' x_2'').$

**Proof.** Let $\xi(s) = (x_1(s), x_2(s), x_3(s))$, be the parametric equation of any null curve in $E^3_1$ where the tangent vector $T(s) = (x'_1(s), x'_2(s), x'_3(s))$ and the normal vector $N(s) = T'(s) = (x''_1(s), x''_2(s), x''_3(s))$. To calculate the bi-normal vector of the curve $\xi(s)$, let $B(s) = (b_1, b_2, b_3)$, since $B(s)$ is unique light-like vector, hence $\langle B, B \rangle = 0$ and so,

$$b_1^2 + b_2^2 - b_3^2 = 0.$$  

Also, $g(T, B) = -1$ and so,
Since $B$ is orthogonal to $N$ where $\langle N, B \rangle = 0$ so we get,

$$x''_1 b_1 + x''_2 b_2 - x''_3 b_3 = -1. \quad (2)$$

Multiply equation (2) by $x''_1$ and equation (3) by $x'_1$ and subtracting the product equations so we get,

$$b_2 = \frac{1}{\Delta_{1,2}} \left( \Delta_{1,3} b_3 + x''_1 \right), \Delta_{1,2} \neq 0. \quad (4)$$

Multiply equation 2 by $x''_2$ and equation 3 by $x''_2$ and subtracting the product equations. Then,

$$b_1 = -\frac{1}{\Delta_{1,2}} \left( \Delta_{2,3} b_3 + x''_2 \right), \Delta_{1,2} \neq 0. \quad (5)$$

By substituting equations 4 and 5 in equation 1. Then,

$$(\Delta_{2,3}^2 + \Delta_{1,3}^2 - \Delta_{1,2}^2) b_3^2 + x''_1^2 + x''_2^2 + 2(\Delta_{1,3} x''_1 + \Delta_{2,3} x''_2) b_3 = 0.$$  

But $$(\Delta_{2,3}^2 + \Delta_{1,3}^2 - \Delta_{1,2}^2) = 0 \text{ and so we get,}$$

$$b_3 = \frac{-\left(x''_1^2 + x''_2^2\right)}{2(\Delta_{1,3} x''_1 + \Delta_{2,3} x''_2)}. \quad (6)$$

Also, $b_3$ can be written in the form,

$$b_3 = \frac{-\left(g(N,N) + x''_3^2\right)}{2(0 \cdot g(T,N)^2 - g(N,N) x''_3^2)}. \quad (7)$$

In equation (7), when the parameterization is pseudo-arc so $g(N, N) = 1, g(T, N) = 0$ and we get,

$$b_3 = \frac{-\left(1 + x''_3^2\right)}{2 x''_3^2}, x'_3 \neq 0. \quad (8)$$

Where $\Delta_{2,3} = (x''_2 x''_3^2 - x''_3 x''_2^2), \Delta_{1,3} = (x''_1 x''_3^2 - x''_3 x''_1^2)$ and $\Delta_{1,2} = (x''_1 x''_2^2 - x''_2 x''_1^2)$. Then we get,

$$B(s) = \left( \frac{1}{\Delta_{1,2}} \left( \Delta_{2,3} b_3 + x''_2 \right), \frac{1}{\Delta_{1,2}} \left( \Delta_{1,3} b_3 + x''_1 \right), \frac{-1 + x''_3^2}{2 x''_3^2} \right). \quad (9)$$

Where $\Delta_{1,2} \neq 0, x'_3 \neq 0$ with curvature $k = 1$ and torsion $\tau = -(N', B) = \frac{1}{2} g(\alpha''', \alpha''')$.

**Example 3.1.** Let $\alpha(s) = \frac{1}{r^2} (\cosh(rs), rs, \sinh(rs))$ if we calculate $1^{st}$ and $2^{nd}$ order derivatives (with respect to $s$) of $\alpha(s)$ and so $T(s) = \frac{1}{r}(\sinh(rs), 1, \cosh(rs))$. Since $\langle T, T \rangle = 0$ so $\alpha(s)$ is a null curve and $N(s) = T'(s) = (\cosh(rs), 0, \sinh(rs))$ so $\langle N, N \rangle = 1$, since $B(s)$ is unique light like vector such that $g(T, B) = 1$ and it is orthogonal to $T$, by substituting in the equation (9). We get $B(s) = \frac{r}{2} (\sinh(rs), -1, \cosh(rs))$ and so $\langle B, B \rangle = 0, N' = r(\sinh(rs), 0, \cosh(rs))$. The pseudo torsion is $\tau = -(N', B) = \frac{-r^2}{2}$ where $N$ is space like vector. Then $\alpha(s)$ is a null curve with curvature $k = 1$ and the Frenet equations of $\alpha(s)$ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ \tau & 0 & k \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} -\frac{r^2}{2} & 0 & 1 \\ 0 & -\frac{r^2}{2} & 1 \\ \frac{1}{r} (\sinh(rs), 1, \cosh(rs)) \end{pmatrix} \begin{pmatrix} \frac{1}{r} (\cosh(rs), 0, \sinh(rs)) \\ \frac{r}{2} (\sinh(rs), -1, \cosh(rs)) \end{pmatrix}. \quad (10)$$

**Corollary 3.1** Let $\xi(s)$ be a null curve in $E^3_1$ with non-zero curvature and pseudo torsion $\tau$, then the bi-normal vector of $\xi(s)$ can be calculated by the form,

$$B(s) = \left( \frac{1}{k} N'(s) - \frac{k}{\tau} T(s) = \left( \frac{1}{k} \right) \xi'''(s) - \left( \frac{1}{k} \right) \xi''(s). \right.$$  

Such that $\tau = -g(N', B)$ or $\tau = \frac{1}{2} g(\xi''', \xi''').$

**Theorem 3.2.** Let $\xi(s)$ be a null curve in $E^3_1$ with non-zero curvature and pseudo torsion $\tau(s)$. Then $\xi(s)$ satisfies a vector differential fourth order as follow,

$$\frac{d^4 \xi}{ds^4} = 2\tau \left( \frac{d^2 \xi}{ds^2} \right) - \tau' \frac{d \xi}{ds} = 0.$$
Proof. Since $\xi(s)$ be a null curve in $E^3_1$ from the Frenet equation ($\ast$). We get,

$$T''(s) = kN(s), N'(s) = \tau T(s) + kB(s) and B'(s) = \tau N(s),$$

with $k = 1$ and so we have,

$$T''(s) = N'(s) = \tau T + B(s) and T'''(s) = \tau' T + \tau T' + B'(s). Then,$$

$$T'''(s) = \tau' T + 2\tau T' and so T'''(s) = 2\tau T' - \tau' T = 0 denoting T = \frac{d\xi}{ds},$$

$$\frac{d4\xi}{ds^4} - 2\tau \left( \frac{d2\xi}{ds^2} \right) - \tau' \frac{d\xi}{ds} = 0.$$

4. Folding of Null Curves

Theorem 4.1. Let $\xi(s)$ be a null curve in $E^3_1$ with non-zero curvature and $\psi(s) = f(\xi(s))$ be a topological folding of $\xi(s)$ for all $s \in$ Domain ($\psi(s)) = I \subset$ Domain $\xi(s)$ defined by frame vectors. Then $\psi(s) = f(\xi(s))$ is a null curve and the Frenet apparatus of the folded curve $\psi(s)$ can be formed by the Frenet apparatus of $\xi(s)$.

Proof. Let $\xi = \xi(s)$ be a null curve in $E^3_1$ with non-zero curvature and $\psi(s) = f(\xi(s)), s \in I \subset$ domain $\xi(s)$ is a topological folding of $\xi(s)$ with curvatures $k_f$ and $\tau_f$ and so,$$

\psi(s) = f(\xi(s)), \psi'(s) = f'(\xi)\psi(s) = f'(\xi) T(s).$$

And we get,

$(\psi', \psi'') = (f'\xi'(s), f''\xi'(s)) = f''(T(s), T'(s)) = 0.$ Since $\xi(s)$ is a null curve with $(T(s), T'(s)) = 0, f'' > 0$ for all $s$. Then $\psi(s)$ is a null curve with curvatures $k_f = k = 1$ and $T_f = f'(s) T(s)$ where,$$

\psi'''(s) = T^3 f'''(\xi) + 3T N f'''(\xi) + f'(\xi)\psi'''(s).$$

By substituting the value of $\xi'''(s)$ from the Frenet apparatus of the curve $\xi(s)$ in corollary 3.1. Then,$$

T_f = T(s) f'(\xi),$$

$N_f = \psi''(s) = N(s) f'(\xi) + T^{2}(s) f''(\xi),$$

$$B_f = \psi'''(s) = f'(\xi) B(s) + T^3 f'''(\xi) + 3T^2 N f'''(\xi), \tau_f = \tau = 0,$$

$$B_f = \psi''' - \tau_f \psi' = (\tau - \tau_f) f'(\xi) T' + f'(\xi) B = T^3 f'''(\xi) + 3T^2 N f'''(\xi),$$

for all $\tau \neq 0$ and $\tau_f \neq 0$.

Corollary 4.1. Let $\xi(s)$ be a null curve in $E^3_1$ and $\psi(s) = f(\xi(s))$ be a topological folding of $\xi(s)$. Then the limit of folding's of $\xi(s)$ is a null point.

Proof. Let $\psi(s) = f(\xi(s))$ be a topological folding of the null curve $\xi(s)$ in $E^3_1$ so $\psi(s)$ be null curve and we have,$$

\psi_1(s): f(\xi(s)) \rightarrow f(\xi(s)), \psi_2(s): \psi_1(f(\xi(s))) \rightarrow \psi_1(f(\xi(s))), \psi_3(s): \psi_2(f(\xi(s))) \rightarrow \psi_2(\psi_1(f(\xi(s)))).$$

$$\psi_n(s): \psi_{n-1}(\psi_{n-2}(\ldots \psi_1(f(\xi)))) \ldots ) \rightarrow \psi_{n-1}(\psi_{n-2}(\ldots \psi_1(f(\xi)) \ldots ))$$

Then $\lim_{n \to \infty} \psi_n = p = (0, 0, 0)$, which is a null point.

Definition 4.1. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))\}$ be a null curve in $E^3_1$. Then $\psi(s)$ be an isometric folding defined as follows,$$

\psi(s): \xi(s) = \{x_1(s), x_2(s), x_3(s)) \rightarrow \xi_f = \left\{ \left\{ \frac{|x_1(s)|}{m}, \frac{|x_2(s)|}{m}, \frac{|x_3(s)|}{m} \right\} \right\} for all s, \ |m| > 1, m \neq 0.$$

Theorem 4.2. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s))$ be a null curve in $E^3_1$ and $\psi(\xi) = \left( \begin{array}{c} \frac{|x_1(s)|}{m} \\ \frac{|x_2(s)|}{m} \\ \frac{|x_3(s)|}{m} \end{array} \right) for all s be an isometric folding of $\xi(s), |m| > 1. Then the folding $\psi(s)$ be a null curve and,$$

\left( \begin{array}{ccc} \frac{\delta}{m} & 0 & 0 \\ 0 & \frac{\delta}{m} & 0 \\ \delta & 0 & 0 \end{array} \right) \begin{pmatrix} T_f \\ N_f \end{pmatrix}, \delta = 1 if x_i(s) > 0 and \delta = -1 if x_i(s) < 0, i \in \{1, 2, 3\}.$$

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Proof. Let $\psi(\xi): \xi(s) = (x_1(s), x_2(s), x_3(s)) \rightarrow \left(\frac{|x_1(s)|}{m}, \frac{|x_2(s)|}{m}, \frac{|x_3(s)|}{m}\right)$, $|m| > 1$, be an isometric folding of the null curve $\xi(s) = (x_1(s), x_2(s), x_3(s))$ in $E_1^3$. If $x_i(s) > 0$, $i \in \{1, 2, 3\}$, then

$$\psi'(\xi) = \frac{dx}{ds} = \frac{1}{m} (x_1'(s), x_2'(s), x_3'(s)),$$

since $\xi(s)$ be a null curve where $(T(T(s), T'(s)) = 0$ and $(T'(T(s), T'(s)) = 0$. for the folded curve $\xi_f(s) = (x_1(s)/m, x_2(s)/m, x_3(s)/m)$ since $(T_f(s), T'_f(s)) = \frac{1}{m^2} (T(s), T'(s)) = 0$ and $(T'_f(s), T''_f(s)) = \frac{1}{m} T''(s), T'(s)) = 0$, then the folded curve $\xi_f(s)$ is a null curve. Since $B(s)$ is unique light like vector, also $g(T, B) = -1$ and $B$ is orthogonal to $N$. Then,

$$T_f(s) = \psi'(s) = \frac{1}{m} T(s), N_f(s) = T' = \frac{1}{m} N(s)$$

and from theorem(1), we get,

$$B_f(s) = m B(s).$$

If $x_i(s) < 0$, $i \in \{1, 2, 3\}$ and $\xi_f(s) = \left(-\frac{x_1(s)}{m}, -\frac{x_2(s)}{m}, -\frac{x_3(s)}{m}\right)$, so $T_f(s) = \frac{1}{m} T(s)$, $N_f(s) = \frac{1}{m} N(s)$ and $B_f(s) = -m B(s)$. Then the Frenet apparatus of the folding $\psi(\xi)$ can be formed by the Frenet apparatus of $\xi(s)$.

Now we introduce a type of folding which make the null curves to be space like curves and time like curves and the converse as follows,

5. Conditional Fractal Folding of Null Curves

Definition 5.1 Let $\xi(s)$ be any curve in $E_1^n$ the map which is defined as $\xi_f: (x_1(s), x_2(s), ..., x_1(s), ..., x_n(s)) \rightarrow (x_1(s), x_2(s), ..., ex_1(s), ..., x_n(s))$ for $\varepsilon \leq 1, \varepsilon \neq 0$ is called conditional fractal folding of the coordinates $x_1, x$ depends on the type of curve $\xi_f$ (space like, time like and null curve) ( M. EL-Ghoul & A. M. Soliman. 2002).

Theorem 5.1. Let $\xi(s)$ be a null curve in $E_1^3$. Under the conditional fractal folding $\psi(s): \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \rightarrow \xi_f = (x_1(s), x_2(s), ex_3(s), \varepsilon \neq 0$ for all $s$, then $\xi_f$ is space like curve if $|\varepsilon| < 1$, $\xi_f$ is null curve if $\varepsilon = 1$ and $\xi_f$ is time like curve if $|\varepsilon| > 1$.

Proof. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s)\}$ be a null curve in $E_1^3$, $(T, T') = 0$, so $x_1^2 + x_2^2 = x_3^2$ and $\psi(s)$ be conditional folding defined as $\psi(s): \xi(s) \rightarrow \xi_f$, if $\xi_f = (x_1(s), (x_2(s), ex_3(s), \varepsilon \neq 0$, so $(T_f, T'_f) = x_1^2 + x_2^2 - \varepsilon^2 x_3^2$ and then let $g(s) = (T_f, T'_f)$, then we have $g'(s) = 2(T_f, T'_f) = 2(T_f, k_f N_f)$ where $k_f \neq 0$ is constant, so $g'(s) = 0$ and $g(s) = c_1, c_1$ is constant.

If $c_1 > 0, (T_f, T'_f) > 0$ and $x_1^2 + x_2^2 - \varepsilon^2 x_3^2 > 0$ so $x_3^2 (1 - \varepsilon^2) > 0, \varepsilon < 1$, then $\xi_f$ is space- like if $|\varepsilon| < 1$.

If $c_1 < 0$ we have $\langle T_f, T'_f \rangle < 0$ and $\varepsilon^2 > 1$, then $\xi_f$ is time like curve if $|\varepsilon| > 1$.

If $c_1 = 0$, $\langle T_f, T'_f \rangle = 0$ and so $\varepsilon^2 = 1$, then $\xi_f$ is null curve if $\varepsilon = \pm 1$.

Corollary 5.1. Let $\xi(s)$ be a null curve in $E_1^3$. Under the conditional fractal folding which is defined as,

$\psi(s): \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \rightarrow \xi_f = (x_1(s), x_2(s), ex_3(s), \varepsilon)$ for all $\varepsilon \leq 1, \varepsilon \neq 0$.

The Frenet equations of the folded curve $\xi_f$ depends on $\varepsilon$.

Corollary 5.2. Let $\xi(s)$ be a null curve in $E_1^3$ and $\psi(t)$ be conditional fractal folding defined as $\psi(s): \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \rightarrow \xi_f$ and $\xi_f = (ex_1(s), ex_2(s), x_3(s)), \varepsilon \neq 0, \varepsilon \neq 1$ for all $s$. Then $\xi_f$ is space like curve if $|\varepsilon| > 1, \xi_f$ is time like curve if $|\varepsilon| < 1$, and $\xi_f$ is null curve if $\varepsilon = \pm 1$.

Corollary 5.3. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s)\}$ be any curve in $E_1^3$ under the conditional fractal folding $\psi(s): \xi(s) \rightarrow \xi_f, \xi_f = (ex_1(s), x_2(s), x_3(s))$ or $\xi_f = (x_1(s), ex_2(s), x_3(s)) \varepsilon \neq 0, |\varepsilon| < 1$, for all $s$. Then the limit of a sequence of foldings of $\xi(s)$ is never being null curve.

Proof. Let the limit of a sequence of foldings of any curve $\xi(s)$ in $E_1^3$ be a null curve with $\xi_f = (0, x_2(s), x_3(s)), \varepsilon \neq 1, \xi_f = (x_1(s), 0, x_3(s))$ and $\xi_f = (x_1(s), x_2(s), 0)$, then from theorem 3.1, the bi-normal vector of the folded curve $B_f$ undefined, also $N_f = \tau T_f - k B_f$ undefined. The Frenet equations of $\xi_f$ cannot appoints and so this contradict with $\xi_f$ be null curve. Then $\xi_f$ never being null curve.

Theorem 5.2. Let $\xi(s) = \{x_1(s), x_2(s), x_3(s)\}$ be a null curve in $E_1^3$. Then the conditional folding $\xi_f = (ex_1(s), ex_2(s), ex_3(s)), |\varepsilon| < 1$, of $\xi(s)$ be null curve. And the Frenet equations of the folded curve $\xi_f$ can be formed by the Frenet equations of $\xi(s)$.
Proof. Let \( \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \) be a null curve in \( E^3_1 \) and \( \xi_f = (\varepsilon x_1(s), \varepsilon x_2(s), \varepsilon x_3(s)) \), \( |\varepsilon| < 1 \) be a conditional fractal folding of \( \xi(s) \) and so \( \langle T_f, T_f \rangle = \varepsilon^2 \langle T(s), T(s) \rangle = 0 \). Then the folded curve \( \xi_f \) is a null curve, with curvature \( k_f = k = 1 \) and torsion \( \tau_f = \tau \), by using the form of Frenet equations in theorem 1. Then we have,

\[
\begin{align*}
T_f(s) &= \varepsilon T(s) \\
N_f(s) &= \varepsilon N(s) \\
B_f(s) &= \left( \frac{1}{\varepsilon} \right) B(s).
\end{align*}
\]

Corollary 5.4. Let \( \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \) be a null curve in \( E^3_1 \). Then the conditional fractal folding \( \xi_f = (\varepsilon x_1(s), \varepsilon x_2(s), \varepsilon x_3(s)) \), \( \varepsilon \in \mathbb{N}, |\varepsilon| < 1 \), \( \varepsilon_i \neq 0 \) be a null curve and the limit of a sequence of foldings of a null curve \( \xi(s) \) be a null point.

Proof. Let \( \xi(s) = \{x_1(s), x_2(s), x_3(s)\} \) be a null curve in \( E^3_1 \). So \( \langle T(s), T(s) \rangle \), since \( \langle T_f(s), T_f(s) \rangle = \varepsilon_i^2 \langle T(s), T(s) \rangle = 0, \varepsilon_i \neq 0 \), then \( \xi_f \) is a null curve.

Let \( f: \xi \to \xi_f \) be a conditional fractal folding of the null curve \( \xi \) such that \( \forall x, y \in \xi, d(x, y) \geq d(f(x), f(y)) \) where \( \xi(s) \) be a null curve. By successive steps of conditional fractal folding's we get,

\[
\begin{align*}
f_1: \xi &\to \varepsilon_1 \xi, |\varepsilon_1| < 1, \\
f_2: \varepsilon_1 \xi &\to \varepsilon_2 \xi, \varepsilon_2 < \varepsilon_1, \\
f_3: \varepsilon_2 \xi &\to \varepsilon_3 \xi, \varepsilon_3 < \varepsilon_2 \cdots, \\
f_n: \varepsilon_{n-1} \xi &\to \varepsilon_n \xi, \varepsilon_n < \varepsilon_{(n-1)} \ll 1,
\end{align*}
\]

\[
\lim_{n \to \infty} f_n(\xi) = p \quad \text{where} \quad p = (0, 0, 0) \quad \text{is a null point}.
\]

Theorem 5.5. If \( \xi(s) \) and \( \xi_f(s) \) are null curves with non-zero curvature in \( E^3_1 \) and \( F_i: \xi \to \xi_f \) is an isotorsion folding, then the torsion of \( \xi \) identically zero if and only if \( \xi_f \) is a part of the null cubic.

Proof. Let \( \xi \) be a null curve in \( E^3_1 \) has torsion identically zero. Since \( F \xi \) is an isotorsion folding from \( \xi \) into \( \xi_f \). Then the torsion of \( \xi_f \) is zero and the Maclaurin series can be written as,

\[
\xi(s) = \xi(0) + \xi'(0)s + \xi''(0)\frac{s^2}{2} + \xi'''(0)\frac{s^3}{6}.
\]

Since \( B(s) = -\xi'''(s) \) when \( \tau = 0 \). So we get,

\[
\xi(s) = \xi(0) + T(0)s + N(0)\frac{s^2}{2} - B(0)\frac{s^3}{6}. \quad \text{With Frenet frame } \{T, N, B\} \text{ of } \xi(s) \text{ in this case } g(T,T) = g(B,B) = 0, \ g(T,B) = g(N,N) = 1. \quad \text{Without loss of generality,}
\]

assume that \( T(0) = \frac{1}{\sqrt{2}}(1,0,1), N(0) = (0,1,0) \) and \( B(0) = \frac{1}{\sqrt{2}}(1,0,-1) \) so we get,

\[
\xi(s) = \frac{1}{6\sqrt{2}}(6s - s^3, 3\sqrt{2}s^2, 6s + s^3). \quad \text{Then } \xi(s) \text{ is a part of null cubic. Conversely let the curve } \xi(s) \text{ be a part of the null cubic, then the torsion of } \xi(s) \text{ identically zero. Since } F \xi \text{ is an isotorsion folding and } \xi_f \text{ has torsion identically zero.}
\]

6. Conditional Deformations of Null Curves in \( E^3_1 \)

Theorem 6.1. Let \( \xi(s) \) be a null curve in \( E^3_1 \) and \( F(x) = Mx + c, c \in \mathbb{R}, M \neq 0 \) be a conditional deformation of \( \xi(s) \) defined as \( F(s) = (Mx_1(s) + c, Mx_2(s) + c, Mx_3(s) + c) \). Then the deformation \( F(s) \) be a null curve and,

\[
\begin{pmatrix}
T_F \\
N_F \\
B_F
\end{pmatrix} =
\begin{pmatrix}
M & 0 & 0 \\
0 & M & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
T \\
N \\
B
\end{pmatrix}.
\]

Proof. Let \( \xi(s) \) be a null curve in \( E^3_1 \) and \( F(s) \) be a conditional deformation of \( \xi(s) \) defined as \( F(s) = M\xi(s) + c \), since \( \xi(s) \) is a null curve so \( \langle \xi', \xi' \rangle = 0, M \neq 0 \) we get,

\[
F'(s) = M\xi'(s), \langle F', F' \rangle = M^2 \langle \xi', \xi' \rangle = 0. \quad \text{Then } F(s) \text{ is a null curve with } k = k_d = 1 \quad \text{and we get,}
\]

\[
T_F(s) = F'(s) = M\xi'(s) = MT(s),
\]

\[
N_F(s) = T_F(s) = MT'(s) = MN(s),
\]

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Theorem 6.2. Let \( F(s) = \frac{1}{M}B \), where the torsion be \( \tau_F = -\langle N_F', B_F \rangle = \tau \). Then the Frenet apparatus of \( F(s) \) can be formed by the Frenet apparatus of \( \xi(s) \).

Corollary 6.1. A null curve \( \xi(s) \) in \( E_{1}^{3} \) under the conditional deformation \( F(s) = M\xi(s) + c \) of \( \xi(s), M \neq 0 \) has the first curve identity zero if and only if \( F(\xi) \) be a part of a straight line.

Proof. Assume that the conditional deformation \( F(\xi) \) of the null curve \( \xi(s) \) be \( F(s) = (Mx_1(s) + c_1, Mx_2(s) + c_2, Mx_3(s) + c_3) \) where \( M \neq 0 \) such that \( \dim F(\xi) = \dim \xi(s) \) and from the Frenet equations with first curvature \( k = 0 \), then \( F''(\xi) = MN(s) = 0 \) and this implies that \( F(\xi) \) is a straight line, where \( k_F = ||T'_F(s)||, N_F(s) = T'_F(s) \). Conversely, let \( F(\xi) \) be a straight line then \( F''(\xi) = MN(s) = 0 \) and \( F(\xi) \) has the curvature \( k_F \) which is identically zero.

Remark 6.1. If \( \alpha(s) \) be a light-like curve in \( E_{1}^{3} \) with standard flat metric \( g = -dx^2 + dy^2 + dz^2 \), Under the conditional deformation, \( \xi(s) = (x(s), y(s), z(s)) \rightarrow D(\xi) = (\bar{x}(s), \bar{y}(s), \bar{z}(s)) = (z(s), y(s), -x(s)) \) which rotation the coordinates \( x \) and \( z \) in x-z plane with rotation angle \( \theta = \frac{\pi}{2} \), \( n \in \mathbb{R}, n \) is odd integer. Then \( D(\xi) \) be a null curve with standard flat metric \( g = -dx^2 + dy^2 + dz^2 \).

Proof. Let \( \xi(s) \) be a null curve in \( E_{1}^{3} \) with standard flat metric \( g = dx^2 + dy^2 - dz^2 \). Under the Frenet equations, \( D(\xi) = (\bar{x}(s), \bar{y}(s), \bar{z}(s)) = (x(s), y(s), -x(s)) \) also, \( g(D', D'') = -dx^2 + dy^2 + dz^2 \), \( \langle D', D'' \rangle = -dx^2 + dy^2 + dz^2 = dx^2 + dy^2 - dz^2 = \langle \xi', \xi'' \rangle = 0 \). Then the conditional deformation \( D(\xi) \) be a null curve with standard flat metric \( g = -dx^2 + dy^2 + dz^2 \).

Theorem 6.3. Let \( \xi(s) \) be a null curve in \( E_{1}^{3} \) with the standard flat metric given by \( g = -dx_1^2 + dx_2^2 + dx_3^2 \). Then the bi-normal vector can be calculated by,

\[
B(s) = \left( \frac{1}{\Delta_{12}}(x_2'' - \Delta_{23} b_3), \frac{1}{\Delta_{12}}(\Delta_{13} b_3 - x_1'''), \frac{-1}{\Delta_{12}}(x_3'') \right), \Delta_{12} \neq 0, x_3'' \neq 0.
\]

Where \( \Delta_{23} = (x_2' x_3' - x_2 x_3'') + \Delta_{13} = (x_1' x_3' - x_1 x_3'') \) and \( \Delta_{12} = (x_1' x_2' - x_1 x_2'') \).

Proof. Let \( \xi(s) = (x_1(s), x_2(s), x_3(s)) \) be a null curve in \( E_{1}^{3} \) with tangent vector \( T(s) = (x_1'(s), x_2'(s), x_3'(s)) \) and the normal vector \( N(s) = T'(s) = (x_1''(s), x_2''(s), x_3''(s)) \), to calculate the bi-normal vector of the curve \( \xi(s) \). Let \( B(s) = (b_1, b_2, b_3) \), since \( B(s) \) is unique light-like vector. Then, \( \langle B, N \rangle = 0 \). By solving these equations as theorem1, we get the bi-normal vector be.
\[ B(s) = \left( \frac{1}{\Delta_{1,2}} \left( \Delta_{2,3} b_3 + x_3'' \right), \frac{1}{\Delta_{1,2}} \left( \Delta_{1,3} b_3 + x_3'' \right) \right) = \frac{1}{\Delta_{1,2}} \left( \left( \Delta_{1,3} b_3 + x_3'' \right), \left( \Delta_{2,3} b_3 + x_3'' \right) \right), \Delta_{1,2} \neq 0, x_3' \neq 0. \tag{14} \]

Where \( \Delta_{2,3} = (x_2' x_3'' - x_3' x_2''), \Delta_{1,3} = (x_1' x_3'' - x_3' x_1'') \) and \( \Delta_{1,2} = (x_1' x_2'' - x_2' x_1'') \) and so,

\[ b_1 = \frac{1}{\Delta_{1,2}} \left( \Delta_{2,3} b_3 + x_3'' \right), b_2 = \frac{1}{\Delta_{1,2}} \left( \Delta_{1,3} b_3 + x_3'' \right) \text{ and } b_3 = \frac{(x_2'' - x_1'')}{2(\Delta_{1,3} x_3'' - \Delta_{2,3} x_3'')} \]

Also, \( b_3 \) can be written in the form,

\[ b_3 = \frac{[g(N,N) - x_3'']}{2[g(N,N)x_3' - g(T,N)x_3']}. \tag{15} \]

In equation (15), when the parameterization is pseudo-arc so \( g(N,N) = 1, g(T,N) = 0 \). Then,

\[ b_3 = \frac{(1 - x_3'')}{2x_3'}, x_3' \neq 0. \tag{16} \]

**Example 6.1.** Let \( \alpha(s) = \frac{1}{2} (\cos(rs), rs, \sinh(rs)) \) be a null curve in \( E^3_1 \) with standard flat metric \( g = dx^2 + dy^2 - dz^2 \) and \( \alpha_0(s) = \frac{1}{2} (\sinh(rs), rs, \cos(rs)) \) be deformation of the null curve \( \alpha(s) \) by rotation coordinates \( x \) and \( z \) with rotation angle \( \theta = \frac{\pi n}{2} \), \( n \) is odd integer with standard flat metric \( g = -d\hat{s}^2 + d\hat{y}^2 + d\hat{z}^2 \). If we calculate 1st and 2nd order derivatives (with respect to \( s \)) of \( \alpha_0(s) \) and so \( T(s) = (\cos(rs), 1, \sinh(rs)) \), since \( (T,T) = 0 \) so \( \alpha(s) \) is null a curve and \( N(s) = T'(s) = (\sinh(rs), 0, \cos(rs)) \), so \( (N,N) = 1 \), since \( B(s) \) is unique light like vector such that \( g(T,B) = 1 \) and it is orthogonal to \( T \) by substituting in the equation (13). Then \( B(s) = \frac{r}{2} (\cos(rs), -1, \sinh(rs)) \), so \( (B,B) = 0, N' = r(\cos(rs), 0, \sinh(rs)) \), the pseudo torsion is \( \tau = -N'(B) = \frac{1}{2} g(\alpha'', \alpha''') = \frac{r}{2} \), \( N \) is space like vector. Then \( \alpha(s) \) is a null curve with curvature \( k = 1 \) and the Frenet equations of \( \alpha(s) \) are given by

\[
\begin{pmatrix}
T' \\
N'
\end{pmatrix}
= \begin{pmatrix}
0 & k & 0 \\
\tau & 0 & -k
\end{pmatrix}
\begin{pmatrix}
T \\
N
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 \\
-r^2 & 0 & -1 \\
0 & r^2 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} (\cos(rs), 1, \sinh(rs)) \\
\tau (\sinh(rs), 0, \cos(rs)) \\
\frac{r}{2} (\cos(rs), -1, \sinh(rs))
\end{pmatrix}
\]

**Corollary 6.2.** Under the conditional deformation which is defined by,

\[ D: \xi(s) = (x(s), y(s), z(s)) \rightarrow D(\xi) = (z(s), y(s), x(s)) \], the Frenet equations of \( D(\xi) \) are invariant.

**Proof.** The proof is clear from theorem 6.3, the Frenet equations of \( D(\xi) \) calculates from equation (10).

**References**


R. Lopez. (2008). Differential geometry of curves and surfaces in Lorentz –Minkowski space, Instituto de Matematica Estatistica, University of Sao Paulo, Brazil.
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