Geometric Properties of Special Spacelike Curves in Three-Dimension Minkowski Space-Time

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Abstract
In this paper, we introduce a special spacelike Smarandache curves \( \mathcal{C} \) reference to the Bishop frame of a regular spacelike curve \( \mathcal{C} \) in Minkowski 3-space \( \mathbb{R}^3 \). From that point, we investigate the Frenet invariants of a special case in \( \mathbb{R}^3 \) and we obtain some properties of these curves when the base curve \( \mathcal{C} \) is contained in a plane. Lastly, we shall give two examples to illustrate these curves.

Keywords: smarandache curve, bishop frame, Minkowski 3-space

AMS Subject Classification (2010):

1. Introduction
When considering the theory of curves in the Euclidean spaces \( \mathbb{R}^3 \) and Minkowski spaces \( \mathbb{R}^3 \), we discovered that the Smarandache curves are this regular curve whose position vector is composed of Frenet frame vectors on other regular curves(M. M. Wageeda, E. M. Solouma, & M. Bary., 2019)(C. Ashbacher., 1997).


In this work, we mention spacelike special curves (Smarandache curves) according to Bishop frame of a spacelike curve \( \mathcal{C} \) in the three-dimension Minkowski space \( \mathbb{R}^3 \). In Section 2, we give the basic conceptions of three-dimension Minkowski 3-space \( \mathbb{R}^3 \) and give of Bishop frame that will be used during this work. In Section 3, we investigate the Bishop special spacelike \( T B_1, T B_2, B_1 B_2 \) and \( T B_1 B_2 \) – curves in terms of the curvature functions \( \kappa_1(\sigma) \) and \( \kappa_2(\sigma) \) of the base curve in \( \mathbb{R}^3 \). On top of that, we obtain some properties on these special curves when the curve \( \mathcal{C} \) is contained in a plane. Finally, in Section 4, we give two examples to clarify these curves.

2. Preliminaries
The Minkowski 3-space \( \mathbb{R}^3 \) is three-dimensional Euclidean space provided with the Lorentzian inner product,

\[ D = -d \xi_1^2 + d \xi_2^2 + d \xi_3^2 \]

where \((\xi_1, \xi_2, \xi_3)\) is a rectangular coordinate system of \( \mathbb{R}^3 \). An arbitrary vector \( u \in \mathbb{R}^3 \) can have one of three characters; it can be spacelike if \( D(u, u) > 0 \) or \( u = 0 \), timelike if \( D(u, u) < 0 \) and null if \( D(u, u) = 0 \) and \( u \neq 0 \). Similarly, an arbitrary curve \( \mathcal{C} = \mathcal{C}(\sigma) \) can be locally spacelike, timelike or null if all of its velocity vectors \( \mathcal{C}' = \mathcal{C}'(\sigma) \) are spacelike, timelike or null, respectively (R. Lopez., 2014).(B. O'Neill., 1983).

Let \( \{T, N, B\} \) denote that Frenet frame, and suggest that \( \{T, N, B\} \) moving along the spacelike special curve \( \mathcal{C} \) with arc-length parameter \( \sigma \). The Frenet trihedron consists of the following: (1) the tangent vector \( \{T\} \), 2. the
principal normal vector \( \{N\} \), 3. the binormal vector \( \{B\} \)). Then this frame (Frenet frame) has the following properties: (B. O'Neill., 1983).

\[
\begin{pmatrix}
N'(\sigma) \\
B'(\sigma)
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa(\sigma) & 0 \\
-\varepsilon\kappa(\sigma) & 0 & \tau(\sigma) \\
0 & \tau(\sigma) & 0
\end{pmatrix}
\begin{pmatrix}
T(\sigma) \\
N(\sigma) \\
B(\sigma)
\end{pmatrix}
\]

(1)

Where \( \varepsilon = \pm 1 \), \( \mathcal{D}(T(\sigma),T(\sigma)) = 1 \), \( \mathcal{D}(N(\sigma),N(\sigma)) = \varepsilon \), \( \mathcal{D}(B(\sigma),B(\sigma)) = -\varepsilon \), \( \mathcal{D}(T(\sigma),B(\sigma)) = \mathcal{D}(N(\sigma),B(\sigma)) = 0 \). If \( \varepsilon = 1 \), then \( \zeta(\sigma) \) is a spacelike curve, and the \( \zeta(\sigma) \) consists of the following: (spacelike principal normal \( \{N\} \) and timelike binormal \( \{B\} \)). Also, if \( \varepsilon = -1 \), then \( \zeta(\sigma) \) is a spacelike curve with timelike principal normal \( \{N\} \) and spacelike binormal \( \{B\} \).

Let \( \zeta = \zeta(\sigma) \) be a regular curve in \( \mathbb{R}^3 \). If the tangent vector field of this curve forms a constant angle with a constant vector field \( U \), then this curve is called a general helix or an inclined curve (M. P. Do Carmo, 1976).

The Lorentzian sphere of radius \( r > 0 \) and with a center in the origin in the space \( \mathbb{R}^3 \) is defined by,

\[
S^2_r = \{ p \in \mathbb{R}^3 : \mathcal{D}(p,p) = r^2 \}.
\]

The parallel transport (or Bishop) frame we can say is an alternative approach to defining a moving frame that is well defined even when the curve has vanished the second derivative (L. R. Bishop, 1975) (B. Bukcu, & M. K. Karacan, 1975).

Suppose that we consider the parallel transport (or Bishop) frame \( \{T(\sigma),B_1(\sigma),B_2(\sigma)\} \) of the special spacelike curve \( \zeta(\sigma) \) such that \( T(\sigma) \) the spacelike unit tangent vector, \( B_1(\sigma) \) is spacelike unit normal vector, and \( B_2(\sigma) \) the timelike unit binormal vector. The Bishop frame \( \{T(\sigma),B_1(\sigma),B_2(\sigma)\} \) is expressed as (B. Bukcu, & M. K. Karacan, 1975) (B. Bukcu, & M. K. Karacan, 2010).

\[
\begin{pmatrix}
T'(\sigma) \\
B_1'(\sigma) \\
B_2'(\sigma)
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa_1(\sigma) & -\kappa_2(\sigma) \\
-\varepsilon\kappa_1(\sigma) & 0 & 0 \\
-\varepsilon\kappa_2(\sigma) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T(\sigma) \\
B_1(\sigma) \\
B_2(\sigma)
\end{pmatrix}
\]

(2)

Where \( \mathcal{D}(T(\sigma),T(\sigma)) = 1 \), \( \mathcal{D}(B_1(\sigma),B_1(\sigma)) = \varepsilon \), \( \mathcal{D}(B_2(\sigma),B_2(\sigma)) = -\varepsilon \), \( \mathcal{D}(T(\sigma),B_1(\sigma)) = \mathcal{D}(B_1(\sigma),B_2(\sigma)) = 0 \). Here, we shall call \( \kappa_1(\sigma) \) and \( \kappa_1(\sigma) \) as Bishop curvatures. The relation matrix may be expressed as,

\[
\begin{pmatrix}
T(\sigma) \\
B_1(\sigma) \\
B_2(\sigma)
\end{pmatrix} =
\begin{pmatrix}
1 & \cosh \theta(\sigma) & 0 \\
0 & \sinh \theta(\sigma) & \kappa_1(\sigma) \\
0 & \sinh \theta(\sigma) & \cosh \theta(\sigma)
\end{pmatrix}
\begin{pmatrix}
T(\sigma) \\
N(\sigma) \\
B(\sigma)
\end{pmatrix}
\]

(3)

Where

\[
\begin{align*}
\theta(\sigma) &= \arctanh \left( \frac{\kappa_2(\sigma)}{\kappa_1(\sigma)} \right); \quad \kappa_1(\sigma) \neq 0, \\
\tau(\sigma) &= -\varepsilon \frac{d\theta(\sigma)}{d\sigma}, \\
\kappa(\sigma) &= \sqrt{\kappa_1^2(\sigma) - \kappa_2^2(\sigma)}.
\end{align*}
\]

(4)

And

\[
\begin{align*}
\kappa_1(\sigma) &= \kappa(\sigma) \cosh \theta(\sigma), \\
\kappa_1(\sigma) &= \kappa(\sigma) \sinh \theta(\sigma).
\end{align*}
\]

Let \( \zeta = \zeta(\sigma) \) be a regular non-null curve parametrized by arc-length in three-dimension Minkowski space \( \mathbb{R}^3 \) with its Bishop frame \( \{T(\sigma),B_1(\sigma),B_2(\sigma)\} \). Then \( \{TB_1, TB_2, B_1 B_2 \} \) \( - \text{ curves} \) of \( \zeta \) are defined, respectively as follows:

\[
\begin{align*}
\varphi(\sigma) &= \varphi(\sigma^*), \\
\varphi(\sigma) &= \varphi(\sigma^*), \\
\varphi(\sigma) &= \varphi(\sigma^*), \\
\varphi(\sigma) &= \varphi(\sigma^*).
\end{align*}
\]
3. Main Results

In this section, we introduce the special spacelike curves reference to the parallel transport frame in three-dimension Minkowski space $\mathbb{R}^3_1$. by the same token, we obtain the natural curvature functions of these curves and studying some properties on it when the base curve $\zeta = \zeta(\sigma)$ especially is contained in a plane.

3.1 Spacelike TB1-Special Curves

**Definition 3.1.** Let $\zeta = \zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}^3_1$ reference to the parallel transport frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then the special spacelike $T B_1 - curves$ (Smarandache curve) are defined by,

$$\varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}} \left( T(\sigma) + B_1(\sigma) \right).$$

(5)

**Theorem 3.1.** Let $\zeta = \zeta(\sigma)$ be a spacelike curve in $\mathbb{R}^3_1$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. If the base curve $\zeta$ is contained in a plane, then the spacelike $T B_1 - curve$ (Smarandache curve) is a circular helix with $\varepsilon k_2^2(\sigma) \neq (1 + \varepsilon)k_2^2(\sigma)$ and its natural curvature functions satisfying the following equations;

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2} \left( (1 + \varepsilon)k_1^2 \right)^2 - \varepsilon k_1^2 k_2^2}{\left( (1 + \varepsilon)k_1^2 \right)^2},$$

$$\tau_\varphi(\sigma^*) = \frac{\sqrt{2} \left( (1 + \varepsilon)k_2^2 \right)^2 - \varepsilon k_2^1 k_2^2}{\left( (1 + \varepsilon)k_2^2 \right)^2}.$$  

(6)

**Proof.** Let $\varphi = \varphi(\sigma^*)$ be a spacelike $T B_1 - Smarandache curve$ with base curve $\zeta = \zeta(\sigma)$. From Eq. (5) and using Eq. (2), we get

$$\varphi'(\sigma^*) = \frac{d\varphi}{d\sigma} \frac{d\sigma^*}{d\sigma} = \frac{1}{\sqrt{2}} \left( -\varepsilon k_1 T(\sigma) + k_1 B_1(\sigma) - k_2 B_2(\sigma) \right).$$

(7)

Hence

$$T_\varphi(\sigma^*) = \frac{1}{\sqrt{2} \left( (1 + \varepsilon)k_1^2 \right)^2 - \varepsilon k_1^2 k_2^2} \left( \xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3 B_2(\sigma) \right).$$

(8)

Where

$$\frac{d\sigma^*}{d\sigma} = \frac{\sqrt{2} \left( (1 + \varepsilon)k_1^2 \right)^2 - \varepsilon k_1^2 k_2^2}{\sqrt{2}}.$$  

(9)

Differentiating Eq. (8) with respect to $\sigma$, we have

$$T_\varphi'(\sigma^*) = \frac{\sqrt{2}}{(1 + \varepsilon)k_1^2 - \varepsilon k_2^2} \left( \xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3 B_2(\sigma) \right).$$

Where

$$\begin{align*}
\xi_1 &= (k_1^2 - k_2^2)(k_1^2 \varepsilon + 1) k_1^2 - k_1^2 + k_1 k_1^2 - k_2 k_2^2, \\
\xi_2 &= (k_1^2 - k_2^2)(k_1^2 - 1) - \varepsilon k_1^2(k_1^2 + k_1 k_1^2 - k_2 k_2^2), \\
\xi_3 &= (k_1 - k_2)(k_1^2 - k_2^2) + \varepsilon k_1^2(k_1 k_1^1 - k_2 k_2^1) + (1 + \varepsilon) k_1 k_1^1 k_2^2.
\end{align*}$$

Then, the curvature and the principal normal vector field of $\varphi$ are respectively,

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2} \left( \xi_1^2 + (1 + \varepsilon)k_1^2 \right)^2}{(1 + \varepsilon)k_1^2 - \varepsilon k_2^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3 B_2(\sigma)}{\sqrt{\xi_1^2 + \varepsilon (\xi_2^2 - \xi_3^2)}}.$$

Also, the binormal vector of $\varphi$ is

$$B_\varphi(\sigma^*) = \frac{\left[ \{\xi_3 - \xi_2 k_2 \} T(\sigma) + \{e \xi_3 k_1 - \xi_1 k_2 \} B_1(\sigma) - \xi_1 \xi_2 \right]}{\sqrt{(1 + \varepsilon)k_1^2 - \varepsilon k_2^2}}.$$

Now, from Eq. (7) we have,
\[ \varphi''(\sigma^*) = \frac{1}{\sqrt{2}} [\varepsilon [\kappa_1^2 - \kappa_2^2 - \kappa_1^2]T(\sigma) + [\kappa_1' - \varepsilon \kappa_1^2] B_1(\sigma) + [\kappa_1 \kappa_2 - \kappa_2^2] B_2(\sigma)], \]

and

\[ \varphi'''(\sigma^*) = \frac{1}{\sqrt{2}} (\alpha_1 T(\sigma) + \alpha_2 B_1(\sigma) + \alpha_3 B_2(\sigma)), \]

where

\[ \begin{align*}
\alpha_1 &= \kappa_1 (\kappa_1^2 - \kappa_2^2) + \varepsilon (\kappa_2^2 + 2 \kappa_2 \kappa_1^2 - 3 \kappa_1 \kappa_1' - \kappa_1''), \\
\alpha_2 &= \kappa_1^2 + \varepsilon \kappa_1 (\kappa_2^2 - \kappa_1^2 - 3 \kappa_1'), \\
\alpha_3 &= -\varepsilon \kappa_1 (\kappa_2^2 - \kappa_1^2 - 3 \kappa_1') - \kappa_1''.
\end{align*} \]

Then the torsion of \( \varphi \) is given by formulae

\[ \tau_{\varphi} = \sqrt{2} \left( \frac{\kappa_1^2 (\alpha_3 \kappa_1 + \alpha_2 \kappa_2) - \varepsilon \kappa_1 (\alpha_3 \kappa_1^2 + \alpha_2 \kappa_1' + \alpha_1 \kappa_1 (\kappa_1' - \varepsilon \kappa_1^2))}{[\varepsilon \kappa_1 (\kappa_2^2 - \kappa_1^2) - \kappa_1'']^2 + \varepsilon \kappa_2 (2 \kappa_1^2 - \kappa_2^2 - \kappa_1') - \kappa_1'']^2 - \varepsilon \kappa_1^2 (\kappa_2^2 - \kappa_1^2 - \kappa_1') - \kappa_1''). \]

So, if \( \zeta(\sigma) \) is contained in a plane, then \( \kappa_{\varphi} \) and \( \tau_{\varphi} \) are constants which implies that the spacelike \( T B_1 - Smarandache curve \) is a circular helix and Eq. (6) holds which complete the proof.

3.2 Spacelike \( T B_2 \) - special curves

**Definition 3.2.** Let \( \zeta = \zeta(\sigma) \) be a regular spacelike curve in \( \mathbb{R}^3 \) reference to moving Bishop frame \( \{T(\sigma), B_1(\sigma), B_2(\sigma)\} \). Then the special spacelike \( T B_2 - Smarandache curves \) are defined by,

\[ \varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{2}} (T(\sigma) + B_2(\sigma)). \] (10)

**Theorem 3.2.** Let \( \zeta = \zeta(\sigma) \) be a spacelike curve in \( \mathbb{R}^3 \) with the moving Bishop frame \( \{T(\sigma), B_1(\sigma), B_2(\sigma)\} \). If the base curve \( \zeta \) is contained in a plane, then the spacelike \( T B_2 - Smarandache curve \) is a circular helix with \( \varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma) \neq 0 \) and its curvature functions are satisfying the following equations;

\[ \kappa_{\varphi}(\sigma^*) = \frac{\sqrt{2(\kappa_1^2 - \kappa_2^2)(\kappa_1^2 - \kappa_2^2 + 1)}}{\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)}, \]

\[ \tau_{\varphi}(\sigma^*) = \frac{\sqrt{2} \kappa_2 (\kappa_2^2 + \varepsilon \kappa_1) (\kappa_1^2 - \kappa_2^2)}{[\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)]^2}. \] (11)

**Proof.** Let \( \varphi = \varphi(\sigma^*) \) be a spacelike \( T B_2 - Smarandache curve \) curves according to the base curve \( \zeta = \zeta(\sigma) \). From Eq. (10), we get

\[ \varphi'(\sigma^*) = \frac{d \varphi}{d \sigma^*} \frac{d \sigma^*}{d \sigma} = \frac{1}{\sqrt{2}} (-\varepsilon \kappa_2 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2 B_2(\sigma)). \] (12)

Hence

\[ T_{\varphi}(\sigma^*) = \frac{1}{\sqrt{2}} \frac{\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)}{\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)} (-\varepsilon \kappa_2 T(\sigma) + \kappa_1 B_1(\sigma) - \kappa_2 B_2(\sigma)). \] (13)

Where

\[ \frac{d \sigma^*}{d \sigma} = \frac{\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)}{\varepsilon \kappa_1^2 (\sigma) + (1 - \varepsilon) \kappa_2^2 (\sigma)}. \] (14)

Then

\[ T_{\varphi}'(\sigma^*) = \sqrt{2} \frac{\lambda_1 T(\sigma) + \lambda_2 B_1(\sigma) + \lambda_3 B_2(\sigma)}{(\lambda_1 T(\sigma) + \lambda_2 B_1(\sigma) + \lambda_3 B_2(\sigma))}. \]

Where

\[ \begin{align*}
\lambda_1 &= (\kappa_2^2 - \kappa_1^2 - \kappa_2^2)[\kappa_1^2 + \varepsilon (1 - \varepsilon) \kappa_2^2] + \kappa_2 [\kappa_1 \kappa_2^2 + \varepsilon (1 - \varepsilon) \kappa_2 \kappa_2^2], \\
\lambda_2 &= (\kappa_1' - \varepsilon \kappa_1 \kappa_2^2) (\kappa_2^2 + (1 - \varepsilon) \kappa_2^2) - \kappa_4 [\varepsilon \kappa_1 \kappa_2^2 + (1 - \varepsilon) \kappa_2 \kappa_2^2], \\
\lambda_3 &= (\varepsilon \kappa_2^2 - \kappa_2^2)(\kappa_1^2 + (1 - \varepsilon) \kappa_2^2) + \kappa_2 [\varepsilon \kappa_1 \kappa_2^2 + (1 - \varepsilon) \kappa_2 \kappa_2^2].
\end{align*} \]

So,
$\kappa_{\sigma}(\sigma^*) = \frac{\sqrt{2} \sqrt{\lambda_1^2 + \epsilon (\lambda_2^2 - \beta_3^2)}}{[\epsilon \kappa_1^2 (\sigma) + (1 - \epsilon) \kappa_2^2 (\sigma)]^2}$,

and

$N_{\sigma}(\sigma^*) = \frac{\lambda_1 T(\sigma) + \lambda_2 B_1(\sigma) + \lambda_3 B_2(\sigma)}{\sqrt{\lambda_1^2 + \epsilon (\lambda_2^2 - \beta_3^2)}}$.

Also,

$B_{\sigma}(\sigma^*) = -\frac{[\lambda_3 \kappa_1 + \lambda_2 \kappa_2] T(\sigma) + (\epsilon - 1) \lambda_1 \kappa_2 B_1(\sigma) - [\lambda_1 \kappa_1 + \epsilon \lambda_2 \kappa_2] B_2(\sigma)}{\sqrt{\epsilon \kappa_1^2 (\sigma) + (1 - \epsilon) \kappa_2^2 (\sigma) \sqrt{\lambda_1^2 + \epsilon (\lambda_2^2 - \beta_3^2)}}}$.

Now, from Eq. (12) we have

$\phi''(\sigma^*) = \frac{1}{\sqrt{2}} \left[ \epsilon [\kappa_2^2 - \kappa_2' - \kappa_2''] T(\sigma) + [\kappa_1' - \epsilon \kappa_1 \kappa_2] B_1(\sigma) + [\epsilon \kappa_2^2 - \kappa_2'] B_2(\sigma) \right]$,

and

$\phi'''(\sigma^*) = \frac{1}{\sqrt{2}} \left[ \beta_1 T(\sigma) + \beta_2 B_1(\sigma) + \beta_3 B_2(\sigma) \right]$,

where

$\begin{align*}
\beta_1 &= \epsilon \left[ 3 \kappa_3 \kappa_2' + \epsilon \kappa_3 \kappa_2^2 - \kappa_2' \kappa_2' - 2 \kappa_1 \kappa_1' - \kappa_1' \right], \\
\beta_2 &= \kappa_1' - \epsilon \kappa_1 \kappa_2 - \kappa_2' + \kappa_1 \kappa_2', \\
\beta_3 &= \epsilon \kappa_2 \kappa_1^2 + \kappa_2^2 - \kappa_2' + \kappa_1 \kappa_2' - \kappa_2'.
\end{align*}$

Then,

$\tau_{\phi} = \sqrt{2} \left( \frac{\epsilon \kappa_2^2 - \kappa_2'}{\kappa_1 (\kappa_2 - \kappa_2') + \epsilon \kappa_2 (\kappa_2' - \kappa_2') \kappa_2' + \epsilon \kappa_2^2 [\kappa_2^2 - (1 + \epsilon) \kappa_2^2]^2 - \epsilon [\kappa_2' - \kappa_2 + \kappa_1 (\kappa_2^2 - \kappa_2') - \kappa_2']^2 \right)$.

Now, if the base curve $\zeta(\sigma)$ is contained in a plane, then the spacelike $T_{B_2} - \text{Smarandache curve}$ is a circular helix and Eqs. (11) holds which complete the proof.

### 3.3 Spacelike $B_1 B_2 - \text{special curves}$

**Definition 3.3.** Let $\zeta = \zeta(\sigma)$ be a regular spacelike curve in $\mathbb{R}^3_1$ reference to moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$. Then the special spacelike $B_1 B_2 - \text{Smarandache curves}$ are defined by,

$\phi(\sigma) = \phi(\sigma^*) = \frac{1}{\sqrt{2}} (B_1(\sigma) + B_2(\sigma))$.  \hspace{1cm} (15)

**Theorem 3.3.** Let $\zeta = \zeta(\sigma)$ be a spacelike curve in $\mathbb{R}^3_1$ with the moving Bishop frame $\{T(\sigma), B_1(\sigma), B_2(\sigma)\}$.

If the base curve $\zeta$ is contained in a plane, then the spacelike $B_1 B_2 - \text{Smarandache curve}$ is also contained in a plane with $\kappa_1(\sigma) + \kappa_2(\sigma) \neq 0$ and its curvature satisfying the following equation;

$\kappa_{\sigma}(\sigma^*) = \frac{\sqrt{2} (\epsilon \kappa_1 - \kappa_2)}{\kappa_1 + \kappa_2}$.  \hspace{1cm} (16)

**Proof:** Let $\phi = \phi(\sigma^*)$ be a spacelike $B_1 B_2 - \text{Smarandache curve}$ according to the base curve $\zeta = \zeta(\sigma)$.

From Eq. (15), we get

$\phi''(\sigma^*) = \frac{d \phi}{d \sigma^*} \frac{d \sigma^*}{d \sigma} = -\frac{\epsilon}{\sqrt{2}} (\kappa_1 + \kappa_2) T(\sigma)$.  \hspace{1cm} (17)

Hence

$T_{\phi}(\sigma^*) = -\epsilon T(\sigma)$.

Where

$\frac{d \sigma^*}{d \sigma} = \frac{\kappa_1 + \kappa_2}{\sqrt{2}}$.  \hspace{1cm} (19)

Then

$T_{\phi}'(\sigma^*) = \frac{\sqrt{2}}{\kappa_1 + \kappa_2} (-\epsilon \kappa_1 B_1(\sigma) + \epsilon \kappa_2 B_2(\sigma))$.  \hspace{1cm} (18)
So,
\[ \kappa_\varphi (\sigma^*) = \frac{\sqrt{2} \varepsilon (\kappa_1 - \kappa_2)}{\kappa_1 + \kappa_2}. \]

and
\[ N_\varphi (\sigma^*) = -\frac{\varepsilon \kappa_1 B_1(\sigma) + \varepsilon \kappa_2 B_2(\sigma)}{\sqrt{\varepsilon (\kappa_1^2 - \kappa_2^2)}}. \]

Also,
\[ B_\varphi (\sigma^*) = \frac{1}{\sqrt{\varepsilon (\kappa_1^2 - \kappa_2^2)}} (\kappa_2 B_1(\sigma) + \kappa_1 B_2(\sigma)). \]

From Eq. (17) we have,
\[ \varphi''(\sigma^*) = -\frac{\varepsilon}{\sqrt{2}} \{ (k_1' + k_2')T(\sigma) + \kappa_1 (k_1 + k_2)B_1(\sigma) - k_2 (k_1 + k_2)B_2(\sigma) \}. \]

And
\[ \varphi'''(\sigma^*) = \frac{1}{\sqrt{2}} \{ \mu_1 T(\sigma) + \mu_2 B_1(\sigma) + \mu_3 B_2(\sigma) \}. \]

Where
\[ \begin{align*}
\mu_1 &= (k_1 + k_2)(k_1^2 - k_2^2) - \varepsilon (k_1' + k_2'), \\
\mu_2 &= -\varepsilon [k_1' (k_1 + k_2) + 2 k_1 (k_1' + k_2)], \\
\mu_3 &= \varepsilon [k_2' (k_1 + k_2) + 2 k_2 (k_1' + k_2)].
\end{align*} \]

Then,
\[ \tau_\varphi = \left\{ \begin{array}{l}
\sqrt{2} \varepsilon (\mu_2 k_2 + \mu_3 k_1), \\
(\kappa_1^2 - \kappa_2^2)(\kappa_1 + \kappa_2)^3.
\end{array} \right\}. \]

Hence
\[ \tau_\varphi = \left\{ \begin{array}{l}
\sqrt{2} \varepsilon (\mu_2 k_2 + \mu_3 k_1), \\
(\kappa_1^2 - \kappa_2^2)(\kappa_1 + \kappa_2)^3.
\end{array} \right\}. \]

So, if the base curve \[ \zeta(\sigma) \] is contained in a plane, then the spacelike \[ T B_2 - Smarandache curve \] is also contained in a plane and the Eq. (16) holds, this completes the proof.

### 3.4 Spacelike \[ T B_1 B_2 - special curves \]

**Definition 3.4.** Let \[ \zeta = \zeta(\sigma) \] be a regular spacelike curve in \[ \mathbb{R}^3 \] reference to moving Bishop frame \{\[ T(\sigma), B_1(\sigma), B_2(\sigma) \]}. Then the special spacelike \[ T B_1 B_2 - Smarandache curves \] are defined by,
\[ \varphi(\sigma) = \varphi(\sigma^*) = \frac{1}{\sqrt{3}} (T(\sigma) + B_1(\sigma) + B_2(\sigma)). \] (20)

**Theorem 3.4.** Let \[ \zeta = \zeta(\sigma) \] be a spacelike curve in \[ \mathbb{R}^3 \] with the moving Bishop frame \{\[ T(\sigma), B_1(\sigma), B_2(\sigma) \]}. If the base curve \[ \zeta \] is contained in a plane, then the spacelike \[ T B_1 B_2 - Smarandache curve \] is also contained in a plane with \[ \kappa_1(\sigma), \kappa_2(\sigma) \neq 0 \] and its natural curvature satisfying the following equation;
\[ \kappa_\varphi(\sigma^*) = \frac{\sqrt{3} \varepsilon (k_1 + k_2)}{(1 + \varepsilon) k_1^2 + (1 - \varepsilon) k_2^2 + 2 k_1 k_2}. \] (21)

**Proof.** Let \[ \varphi = \varphi(\sigma^*) \] be a spacelike \[ T B_1 B_2 - Smarandache curve \] curves according to the base curve \[ \zeta = \zeta(\sigma) \]. From Eq. (20), we get
\[ \varphi'(\sigma^*) = \frac{d \varphi}{d \sigma} \frac{d \sigma^*}{d \sigma} = \frac{1}{\sqrt{3}} (-\varepsilon (k_1 + k_2) T(\sigma) + k_1 B_1(\sigma) - k_2 B_2(\sigma)). \] (22)

Hence
\[ T_\varphi(\sigma^*) = \frac{-\varepsilon (k_1 + k_2) T(\sigma) + k_1 B_1(\sigma) - k_2 B_2(\sigma)}{(1 + \varepsilon) k_1^2 + (1 - \varepsilon) k_2^2 + 2 k_1 k_2}. \] (23)

Where
\[ \frac{d \sigma^*}{d \sigma} = \frac{\sqrt{(1 + \varepsilon) k_1^2 + (1 - \varepsilon) k_2^2 + 2 k_1 k_2}}{\sqrt{3}}. \] (24)

Then, from Eq. (23) we get
\[ T_\varphi'(\sigma^*) = \frac{\sqrt{3} (y_1 T(\sigma) + y_2 B_1(\sigma) + y_3 B_2(\sigma))}{(1 + \varepsilon) k_1^2 + (1 - \varepsilon) k_2^2 + 2 k_1 k_2} \frac{1}{2}. \]
Where

\[
\begin{align*}
\gamma_1 &= \varepsilon [\kappa_1^2 - (k_2^2 + \kappa_1 + \kappa_2)][(1 + \varepsilon)\kappa_1^2 + (1 - \varepsilon)\kappa_2^2 + 2 \kappa_1 \kappa_2] \\
+ 2\varepsilon (k_1 + k_2) [(1 + \varepsilon)\kappa_1 + (1 - \varepsilon)\kappa_2 k_2' + (\kappa_1 k_2)',] \\
\gamma_2 &= [(1 + \varepsilon)\kappa_1^2 + (1 - \varepsilon)\kappa_2^2 + 2 \kappa_1 \kappa_2][\kappa_1' - \varepsilon \kappa_2 (k_1 + k_2)] \\
- 2\kappa_1 [(1 + \varepsilon)\kappa_1 k_1' + (1 - \varepsilon)\kappa_2 k_2' + (k_1 k_2)',] \\
\gamma_3 &= [(1 + \varepsilon)\kappa_1^2 + (1 - \varepsilon)\kappa_2^2 + 2 \kappa_1 \kappa_2][\kappa_1' - \varepsilon \kappa_2 (k_1 + k_2)] \\
+ 2\kappa_1 [(1 + \varepsilon)\kappa_1 k_1' + (1 - \varepsilon)\kappa_2 k_2' + (k_1 k_2)',]
\end{align*}
\]

Then

\[
\kappa_\varphi (\sigma^*) = \frac{\sqrt{3} y_1^2 + \varepsilon(y_2^2 - y_3^2)}{[(1 + \varepsilon)\kappa_1^2 + (1 - \varepsilon)\kappa_2^2 + 2 \kappa_1 \kappa_2]^2},
\]

and

\[
N_\varphi (\sigma^*) = \frac{\gamma_1 T(\sigma) + \gamma_2 B_1 (\sigma) + \gamma_3 B_2 (\sigma)}{\sqrt{y_1^2 + \varepsilon(y_2^2 - y_3^2)}}.
\]

Also, the binormal vector of \( \varphi \) is

\[
B_\varphi (\sigma^*) = \frac{m_1 T(\sigma) + m_2 B_1 (\sigma) + m_3 B_2 (\sigma)}{\sqrt{(1 + \varepsilon)\kappa_1^2 + (1 - \varepsilon)\kappa_2^2 + 2 \kappa_1 \kappa_2 \sqrt{y_1^2 + \varepsilon(y_2^2 - y_3^2)}}}. 
\]

Where

\[
\begin{align*}
m_1 &= -\gamma_3 \kappa_1 - \gamma_2 \kappa_2, \\
m_2 &= \varepsilon \gamma_3 (\kappa_1 + \kappa_2) - \gamma_2 \kappa_2, \\
m_3 &= -\gamma_1 \kappa_1 - \gamma_2 (\kappa_1 + \kappa_2).
\end{align*}
\]

Now, from Eq. (21) we have

\[
\varphi'' (\sigma^*) = \frac{1}{\sqrt{3}} [\varepsilon (k_1^2 - (k_1^2 + \kappa_1 + \kappa_2)]T(\sigma) + [\kappa_1' - \varepsilon \kappa_2 (k_1 + k_2)]B_1 (\sigma) + [\varepsilon \kappa_2 (k_1 + k_2) + k_1'] B_2 (\sigma)].
\]

And

\[
\varphi''' (\sigma^*) = \frac{1}{\sqrt{3}} \left[ \delta_1 T(\sigma) + \delta_2 B_1 (\sigma) + \delta_3 B_2 (\sigma) \right],
\]

where

\[
\begin{align*}
\delta_1 &= (\kappa_1 + \kappa_2) (k_1^2 - k_2^2)^2 + \varepsilon [3(\kappa_1 k_1' + \kappa_2 k_2') - \kappa_1'' - k_2''], \\
\delta_2 &= k_1' - \varepsilon (k_1 + k_2) [k_1' + k_1(k_1 - k_2)], \\
\delta_3 &= \varepsilon (k_1 + k_2) [k_2' + k_2(k_1 - k_2)] - k_1''.
\end{align*}
\]

Then,

\[
\tau_\varphi = \sqrt{3} \left[ \frac{[\kappa_1' - \varepsilon \kappa_2 (k_1 + k_2)][\delta_1 k_2 + \varepsilon \delta_3 (k_1 + k_2)]}{[\kappa_1' - \varepsilon \kappa_2 (k_1 + k_2)][\delta_1 k_2 + \varepsilon \delta_3 (k_1 + k_2)]} \\
+ [\varepsilon \kappa_2 (k_1 + k_2) - k_2''] \left[ \delta_1 k_1 - \varepsilon \delta_2 k_1 + k_1'' \right] \\
- \varepsilon [k_2' - k_2 k_1'] [\delta_1 k_1 + \varepsilon \delta_2 (k_1 + k_2)] \\
+ \varepsilon \kappa_2 (k_1 + k_2) [k_2' + \kappa_1(k_2 - k_1)] - k_1'' \\
- \varepsilon \kappa_2 (k_1 + k_2) [(1 - \varepsilon)k_2^2 + (1 + \varepsilon)k_1 k_2 - k_1''] + (\kappa_1 k_2)'.
\right]
\]

Now, if the base curve \( \zeta (\sigma) \) is contained in a plane, then the spacelike \( TB_1 B_2 - Smarandache \, curve \) is also contained in a plane and the Eq. (21) holds. This completes the proof.

4. Examples

In this section, we construct two examples of the spacelike Smarandache curves in \( \mathbb{R}^2 \) with the moving Bishop frame \( \{ T(\sigma), B_1 (\sigma), B_2 (\sigma) \} \) of the base curve \( \zeta (\sigma) \). The first example corresponds with the case \( \varepsilon = 1 \). In the second example, we assume \( \varepsilon = -1 \).

**Example 4.1.** Case \( \varepsilon = 1 \). Let \( \zeta (\sigma) = (3 \sinh (\sigma), 3 \cosh (\sigma), \frac{5 \sigma}{4}) \) be a spacelike curve parametrized by arc-length with timelike binormal vector (see Figure 1). Then
\[ T(\sigma) = \left( \frac{3}{4} \cosh \left( \frac{\sigma}{4} \right), \frac{3}{4} \sinh \left( \frac{\sigma}{4} \right), \frac{5}{4} \right). \]

This vector is spacelike and future-directed, we have \( \kappa = \frac{3}{16} \neq 0 \). Hence
\[ N(\sigma) = (\sinh \left( \frac{\sigma}{4} \right), \cosh \left( \frac{\sigma}{4} \right), 0), \]
\[ T(\sigma) = \left( \frac{5}{4} \cosh \left( \frac{\sigma}{4} \right), \frac{5}{4} \sinh \left( \frac{\sigma}{4} \right) \right). \]

The torsion is \( \tau = \frac{5}{16} \neq 0 \) and \( (\sigma) = - \int_0^\sigma \frac{5}{16} \, ds = -\frac{5}{16} \). From Eq. (4), we get \( \kappa_1 = \frac{3}{16} \cosh \left( \frac{5\sigma}{16} \right), \)
\[ \kappa_2 = \frac{3}{16} \sinh \left( \frac{5\sigma}{16} \right). \]

Also from Eq. (2), we get \( B_1(\sigma) = - \int \kappa_1(\sigma) T(\sigma) \, d\sigma, B_2(\sigma) = - \int \kappa_2(\sigma) T(\sigma) \, d\sigma \), the we have
\[ B_1(\sigma) = \left( -\frac{1}{8} \sinh \left( \frac{9\sigma}{16} \right) - \frac{9}{8} \sinh \left( \frac{\sigma}{16} \right), -\frac{1}{8} \cosh \left( \frac{9\sigma}{16} \right) + \frac{9}{8} \cosh \left( \frac{\sigma}{16} \right), -\frac{3}{4} \sinh \left( \frac{5\sigma}{16} \right) \right), \]
\[ B_2(\sigma) = \left( \frac{1}{8} \cosh \left( \frac{9\sigma}{16} \right) + \frac{9}{8} \cosh \left( \frac{\sigma}{16} \right), -\frac{1}{8} \sinh \left( \frac{9\sigma}{16} \right) - \frac{9}{8} \sinh \left( \frac{\sigma}{16} \right), \frac{3}{4} \cosh \left( \frac{5\sigma}{16} \right) \right). \]

Figure 1. The spacelike \( T \, B_1 - Smarandache curve \) with base curve \( \zeta(\sigma) \) on \( S_1^2 \).

Figure 2. The spacelike \( T \, B_2 - Smarandache curve \) with base curve \( \zeta(\sigma) \) on \( S_1^2 \).
Example 4.2. Case $\varepsilon = -1$. Let now $\omega(\sigma) = \frac{1}{\sqrt{2}}(\cosh(\sigma), \sinh(\sigma), \sigma)$ be a spacelike curve parametrized by arc-length with timelike principal normal vector (see Figure 5). Then it is easy to show that $T(\sigma) = \frac{1}{\sqrt{2}}(\sinh(\sigma), \cosh(\sigma), 1)$, $\kappa = \frac{1}{\sqrt{2}} \neq 0$, $\tau = \frac{1}{\sqrt{2}} \neq 0$ and $\theta(\sigma) = \int_{0}^{\sigma} \frac{1}{\sqrt{2}} \, dt = \frac{\sigma}{\sqrt{2}}$. From Eq. (4), we get $\kappa_{1} = \frac{1}{\sqrt{2}} \cosh \left( \frac{\sigma}{\sqrt{2}} \right)$, $\kappa_{2} = \frac{1}{\sqrt{2}} \sinh \left( \frac{\sigma}{\sqrt{2}} \right)$.

From Eq. (2), we get $B_{1}(\sigma) = \int \kappa_{1}(\sigma) T(\sigma) \, d\sigma$, $B_{2}(\sigma) = \int \kappa_{2}(\sigma) T(\sigma) \, d\sigma$. Then we have,

$$B_{1}(\sigma) = \left( -\frac{\sqrt{2}}{4(\sqrt{2} + 1)} \cosh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) - \frac{\sqrt{2}}{4(\sqrt{2} + 1)} \cosh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) \right) \cdot \sqrt{2} \sinh \left( \frac{\sqrt{2}}{2} \cosh \left( \frac{\sigma}{\sqrt{2}} \right) \right)$$

$$+ \frac{\sqrt{2}}{4(\sqrt{2} + 1)} \sinh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) \cdot \frac{\sqrt{2}}{2} \sinh \left( \frac{\sigma}{\sqrt{2}} \right) \right).$$

$$B_{2}(\sigma) = \left( -\frac{\sqrt{2}}{4(\sqrt{2} + 1)} \sinh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) + \frac{\sqrt{2}}{4(\sqrt{2} + 1)} \sinh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) \right) \cdot \sqrt{2} \cosh \left( \frac{\sqrt{2}}{2} \cosh \left( \frac{\sigma}{\sqrt{2}} \right) \right)$$

$$- \frac{\sqrt{2}}{4(\sqrt{2} + 1)} \cosh \left( \frac{\sqrt{2} + 1}{\sqrt{2}} \sigma \right) \cdot - \frac{\sqrt{2}}{2} \cosh \left( \frac{\sigma}{\sqrt{2}} \right) \right).$$
Figure 5. The spacelike curve $\omega = \omega(\sigma)$ on $S^2_1$.

Figure 6. The spacelike $T B_1 - Smarandache curve$ with base curve $\zeta(\sigma)$ on $S^2_1$.

Figure 7. The spacelike $T B_2 - Smarandache curve$ with base curve $\zeta(\sigma)$ on $S^2_1$. 
Figure 8. The spacelike \( B_1 B_2 \) — Smarandache curve with base curve \( \zeta(\sigma) \) on \( S_2^2 \).

Figure 9. The spacelike \( T B_1 B_2 \) — Smarandache curve with base curve \( \zeta(\sigma) \) on \( S_2^2 \).

References


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