Strong Convergence Theorem According to Hybrid Methods for Mapping Asymptotically Quasi-Nonexpansive Types

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Abstract
The purpose of this article is to prove strong convergence theorems for mapping of asymptotically quasi-nonexpansive types in a Hilbert space according to hybrid methods. The results obtained in this paper extend and improve upon those recently announced by Qin, X., Su, Y. and Shang, M. (Qin, X. et al., 2008), and many others.

Keywords: Asymptotically quasi-nonexpansive type, Metric projection, Uniformly L-Lipschitzian

1. Introduction
Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Further, for a mapping $T : C \to C$, let $\emptyset \neq F(T)$ be the set of all fixed points of $T$. A mapping $T : C \to C$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ of real number with $k_n \geq 1$ and $\lim_{n} k_n = 1$ such that for all $x \in C$, $q \in F(T)$

$$\|T^n x - q\| \leq k_n \|x - q\|, \text{ for all } n \geq 1. \tag{1}$$

$T$ is called asymptotically quasi-nonexpansive type (Sahu, 2003) provided $T$ is uniformly continuous, and

$$\lim sup_n \sup_{x \in C} (\|T^n x - q\| - \|x - q\|) \leq 0, \text{ for all } q \in F(T). \tag{2}$$

$T$ is called uniformly L-Lipschitzian if there exists a positive constant $L$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \text{ for all } x, y \in C \text{ and all } n \geq 1. \tag{3}$$

Fixed point iteration processes for asymptotically quasi-nonexpansive type mappings in Banach spaces have been studied extensively by many authors, (Chang, 2004; Li, 2007; Puturong, 2008 & 2009; Quan, 2006; Sahu, 2003; Saluja, 2007; Tian, 2007). The other nonlinear mappings, which are all special cases of asymptotically quasi-nonexpansive type mappings, have been also studied both in Banach spaces and Hilbert spaces. Those nonlinear mappings are nonexpansive mappings, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. However, In recently years, the hybrid iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors as follows:

In 2003, Nakajo, K. and Takahashi, W. introduced an iterative scheme for a single nonexpansive mapping $T$ in a Hilbert space $H$

$$x_0 \in C \text{ chosen arbitrarily,}$$
$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$
$$C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},$$
$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n}(x_0), \tag{4}$$

where $C$ is a closed convex subset of $H$, $P_K(x_0)$ denotes the metric projection from $H$ onto a closed convex subset $K$ of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (4) converges strongly to $P_{F(T)}(x_0)$. Where $F(T)$ denote the fixed points set of $T$.

In 2006, Kim, T. H. and Xu, H. K introduced an iterative scheme for asymptotically nonexpansive mapping $T$ in a Hilbert...
space $H$:

\[ x_0 \in C \text{ chosen arbitrarily}, \]
\[ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \]
\[ C_n = \{ z \in C : ||y_n - z||^2 \leq ||x_n - z||^2 + \theta_n \}, \]
\[ Q_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \geq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}(x_0), \]

where $C$ is a closed convex subset and

\[ \theta_n = (1 - \alpha_n)(\frac{k_n}{n} - 1)(\text{diam} \ C)^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \]

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (5) converges strongly to $P_{F(T)}(x_0)$.

They also introduced an iterative scheme for asymptotically nonexpansive semigroup $\mathcal{S}$ in a Hilbert space $H$:

\[ x_0 \in C \text{ chosen arbitrarily}, \]
\[ y_n = \alpha_n x_n + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \]
\[ C_n = \{ z \in C : ||y_n - z||^2 \leq ||x_n - z||^2 + \bar{\theta}_n \}, \]
\[ Q_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \geq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}(x_0), \]

where $C$ is a closed convex subset and

\[ \bar{\theta}_n = (1 - \alpha_n)(\frac{1}{t_n} \int_0^{t_n} L(u)du)^2 - 1)(\text{diam} \ C)^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \]

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (6) converges strongly to $P_{F(\mathcal{S})}(x_0)$. Where $F(\mathcal{S})$ denote the common fixed points set of $\mathcal{S}$.

In 2006, Carlos Martinez-Yanes and Hong-Kun Xu introduced an iterative scheme for nonexpansive mapping $T$ in a Hilbert space $H$:

\[ x_0 \in C \text{ chosen arbitrarily}, \]
\[ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \]
\[ z_n = \beta_n x_n + (1 - \beta_n) T x_n, \]
\[ C_n = \{ z \in C : ||y_n - z||^2 \leq ||x_n - z||^2 \]
\[ + (1 - \alpha_n)(||z_n||^2 - ||x_n||^2 + 2\langle x_n - z_n, z \rangle) \}, \]
\[ Q_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \geq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}(x_0), \]

where $C$ is a closed convex subset of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \rightarrow 0$, then the sequence $\{x_n\}$ generated by (7) converges strongly to $P_{F(T)}(x_0)$.

They also introduced an iterative scheme for nonexpansive mapping $T$ in a Hilbert space $H$:

\[ x_0 \in C \text{ chosen arbitrarily}, \]
\[ y_n = \alpha_n x_0 + (1 - \alpha_n) T x_n, \]
\[ C_n = \{ z \in C : ||y_n - z||^2 \leq ||x_n - z||^2 \]
\[ + \alpha_n(\langle x_0 \rangle^2 + 2\langle x_n - x_0, z \rangle) \}, \]
\[ Q_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \geq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}(x_0), \]

where $C$ is a closed convex subset of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\alpha_n \rightarrow 0$, then the sequence $\{x_n\}$ generated by (8) converges strongly to $P_{F(T)}(x_0)$. 

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In 2007, Su, Y., Wang, D. and Shang, M. introduced an iterative scheme for quasi-nonexpansive mapping \( T \) in a Hilbert space \( H \):

\[
\begin{align*}
x_0 & \in C \text{ chosen arbitrarily,} \\
y_n & = \alpha_n x_0 + (1 - \alpha_n) T^n x_n, \\
C_n & = \{ z \in C_{n-1} \cap Q_{n-1} : \| z - y_n \| \leq \| z - x_n \| \}, \\
C_0 & = \{ z \in C : \| z - y_0 \| \leq \| z - x_0 \| \}, \\
Q_n & = \{ z \in C_{n-1} \cap Q_{n-1} : (x_n - z, x_0 - x_n) \geq 0 \}, \\
Q_0 & = C, \\
x_{n+1} & = P_{C \cap Q_n}(x_0),
\end{align*}
\]

where \( C \) is a closed convex subset of \( H \). They proved that if \( \{\alpha_n\} \) is a sequence in \([0, 1]\) such that \( \limsup_{n \to \infty} \alpha_n < 1 \), then the sequence \( \{x_n\} \) generated by (9) converges strongly to \( P_{F(T)}(x_0) \).

In 2008, Inchan, I. and Plubtieng, S. introduced an iterative scheme for two asymptotically nonexpansive mappings \( S \) and \( T \) in a Hilbert space \( H \):

\[
\begin{align*}
y_n & = \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\
z_n & = \beta_n x_n + (1 - \beta_n) S^n z_n, \\
C_{n+1} & = \{ z \in C_n : \| y_n - z \| \leq \| y_n - \theta \| \} (10) \leq \| x_n - \theta \| + \theta_n, \\
x_{n+1} & = P_{C_{n+1}}(x_0), \quad n \in \mathbb{N}
\end{align*}
\]

They also introduced an iterative scheme for two asymptotically nonexpansive semigroups \( S(t) : 0 \leq t \leq \infty \) and \( T(t) : 0 \leq t \leq \infty \) in a Hilbert space \( H \):

\[
\begin{align*}
y_n & = \alpha_n x_n + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(t) z_n dt, \\
z_n & = \beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^{s_n} S(t) z_n dt, \\
C_{n+1} & = \{ z \in C_n : \| y_n - z \| \leq \| y_n - \theta \| \} + \beta_n \\
x_{n+1} & = P_{C_{n+1}}(x_0), \quad n \in \mathbb{N},
\end{align*}
\]

They proved that the sequence \( \{x_n\} \) generated by (11) converges strongly to \( P_{F(S \cap T)}(x_0) \).

In 2008, Qin, X., Su, Y. and Shang, M. introduced an iterative scheme for asymptotically nonexpansive mapping \( T \) in a Hilbert space \( H \):

\[
\begin{align*}
x_0 & \in C \text{ chosen arbitrarily,} \\
z_n & = \beta_n x_n + (1 - \beta_n) T^n z_n, \\
y_n & = \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\
C_n & = \{ v \in C : \| y_n - v \| \leq \| y_n - \theta \|^\frac{2}{2} \\
&\quad + (1 - \alpha_n) \| k_n^2 \| \| z_n \|^2 - \| x_n \|^2 \} (12) + \| x_n - k_n^2 z_n, v \| \}, \\
Q_n & = \{ v \in C : (x_n - x_n, x_n - v) \geq 0 \}, \\
x_{n+1} & = P_{C \cap Q_n}(x_0),
\end{align*}
\]

where \( M \) is a appropriate constant such that \( M > \| v \|^2 \) for each \( v \in C_n \). They proved that if \( \{k_n\} \) is a sequence such that \( k_n \to 1 \) as \( n \to \infty \) and \( \{\alpha_n\} \) is a sequence in \((0, 1)\) such that \( \alpha_n \geq 1 - \delta \) for all \( n \) and for some \( \delta \in (0, 1) \) and \( \beta_n \to 1 \), then the sequence \( \{x_n\} \) generated by (12) converges strongly to \( P_{F(T)}(x_0) \).
They also introduced an iterative scheme for asymptotically nonexpansive mapping $T$ in a Hilbert space $H$:

$$x_0 \in C \text{ chosen arbitrarily,}$$

$$y_n = \alpha_n x_0 + (1 - \alpha_n)T^n x_n,$$

$$C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (k_n^2 - 1 - \alpha_n k_n^2)\|x_n - v\|^2$$

$$+ \alpha_n \|x_0\|^2 - 2(\alpha_n x_0 + (k_n^2 - 1 - \alpha_n k_n^2)x_n, v) + (1 - \alpha_n)(k_n^2 - 1)M\},$$

$$Q_n = \{v \in C : \langle x_0 - x_n, x_n - v \rangle \geq 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n}(x_0),$$

where $M$ is a suitable constant such that $M > \|v\|^2$ for each $v \in C_n$. They proved that if $\{k_n\}$ is a sequence such that $k_n \to 1$ as $n \to \infty$ and $\{|\alpha_n|\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ generated by (13) converges strongly to $P_{F(T)}(x_0)$.

The purpose of this paper is to prove strong convergence theorems for mapping of asymptotically quasi-nonexpansive types in Hilbert space. The results obtained in this paper extend and improve upon those recently announced by X. Qin, Y. Su, and M. Shang and many others.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; let $C$ be a nonempty closed convex subset of $H$. In a real Hilbert space $H$, we have

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle,$$

$$\langle \alpha x + (1 - \alpha) y, z \rangle = \alpha \langle x, z \rangle + (1 - \alpha) \langle y, z \rangle,$$

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2,$$

$$\|x - y\|^2 = \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2,$$

for all $x, y, z \in H$.

Let $\{x_n\}$ be a sequence of $H$ and let $x \in H$. Then, $\{x_n\}$ is said to converge weakly to $x$, denoted by $x_n \rightharpoonup x$, if for any $y \in H$, $\langle x_n, y \rangle \to \langle x, y \rangle$.

For any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for any $y \in C$. Such a $P_C$ is called the metric projection of $H$ onto $C$.

Recalling a well-known concept, and the following essential lemma, in order to prove our main results:

**Lemma 2.1** There holds the identity in a Hilbert space $H$:

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

**Lemma 2.2** [Qin, 2008] Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $P_C$ be the metric projection from $H$ onto $C$. Given $x \in H$ and $z \in C$. Then $z = P_C(x)$ if and only if there holds the relations:

$$\langle x - z, y - z \rangle \leq 0 \quad \text{for all } y \in C.$$

**Lemma 2.3** [Takahashi, 2009] Let $H$ be a real Hilbert space and let $\{x_n\}$ be a bounded sequence of $H$ such that $x_n \to x$. Then the following inequality hold:

$$\|x\| \leq \liminf_{n \to \infty} \|x_n\|.$$

**Lemma 2.4** [Takahashi, 2009] Let $H$ be a real Hilbert space and let $\{x_n\}$ be a sequence of $H$. If $x_n \to x$, then $x_n \to x$.

**Lemma 2.5** [Takahashi, 2009] Let $H$ be a real Hilbert space and suppose $x_n \to x$. Then $\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$ for all $y \in H$ with $x \neq y$.

3. Main Results

In this section, we provide proof strong convergence theorems for a mapping of asymptotically quasi-nonexpansive type in a Hilbert space by hybrid methods.

**Lemma 3.1** Let $C$ be a nonempty closed convex subset of Hilbert space $H$, and $T : C \to C$ be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set $F(T)$ and $T$ is uniformly L-Lipschitzian. Put

$$G_n = \max_{x \in C} \sup_{x \in C} \|T^n x - p\| - \|x - p\|$$

where $s_n = \frac{1}{\|x_n - p\|}$.
for all \( n \geq 1 \), for all \( p \in F(T) \) so that \( \sum_{n=1}^{\infty} G_n < \infty \). Then \( F(T) \) is a closed and convex.

**Proof** We first show that \( F(T) \) is closed. To see this, let \( \{x_n\} \) be a sequence in \( F(T) \) with \( x_n \to x \), we shall prove that \( x \in F(T) \). Then we have

\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} T x_n = T \lim_{n \to \infty} x_n = T x
\]

and hence \( x \in F(T) \). This implies that \( F(T) \) is closed. Next, we show that \( F(T) \) is convex. Let \( x, y \in F(T) \) and \( \lambda \in (0, 1) \). We show \( z = \lambda x + (1 - \lambda)y \in F(T) \). From Lemma 2.1, we have

\[
||z - T^z||^2 = \lambda||x - (1 - \lambda)y - AT^z - (1 - \lambda)T^z||^2
\]

\[
= \lambda||x - T^z||^2 + (1 - \lambda)||y - T^z||^2 - \lambda(1 - \lambda)||x - y||^2
\]

\[
= \lambda \left( \left( ||T^zx - x|| - ||z - x|| \right) + ||z - x|| \right)^2
\]

\[
+ (1 - \lambda) \left( \left( ||T^zy - y|| - ||z - y|| \right) + ||z - y|| \right)^2
\]

\[
= \lambda \left( \left( ||T^z|| - ||z - x|| \right) + \left( ||z - x|| \right) \right)^2 + 2 \left( ||T^z|| - ||z - y|| \right)
\]

\[
\left( ||z - x|| + ||z - y|| \right)^2 + (1 - \lambda) \left( \left( ||T^z|| - ||z - y|| \right) + \left( ||z - y|| \right) \right)^2
\]

\[
\leq \lambda \left( \sup_{q \in C} (||T^q|| - ||z - q||) \right)^2 + 2 \lambda \left( \sup_{q \in C} (||T^z|| - ||q - y||) \right) \left( ||z - x|| \right)
\]

\[
+ \lambda ||z - x||^2 + (1 - \lambda) \left( \sup_{q \in C} (||T^z|| - ||q - y||) \right)^2
\]

\[
+ 2(1 - \lambda) \left( \sup_{q \in C} (||T^z|| - ||q - y||) \right) \left( ||z - y|| \right) + (1 - \lambda) ||z - y||^2 - \lambda(1 - \lambda)||x - y||^2
\]

\[
\leq \lambda G_n^2 + 2 (1 - \lambda) G_n \left( ||z - x|| \right) + (1 - \lambda) G_n \left( ||z - y|| \right) + ||z - x|| + (1 - \lambda) \left( ||z - y|| \right)^2
\]

\[
= G_n^2 + 2 (1 - \lambda) G_n \left( ||z - y|| \right) + (1 - \lambda) \left( ||z - y|| \right)^2.
\]

Which implies that \( ||z - T^z|| \to 0 \). We obtain,

\[
||z - T^z|| \leq ||z - T^{z+1}|| + ||T^{z+1} - T^z|| \leq ||z - T^{z+1}|| + L||T^z - z|| \to 0,
\]

which implies that \( Tz = z \). So, we get \( z \in F(T) \). This implies that \( F(T) \) is convex.

**Theorem 3.1** Let \( C \) be a nonempty bounded closed convex subset of Hilbert space \( H \) and let \( T : C \to C \) be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set \( F(T) \) and \( T \) is uniformly \( L \)-Lipschitzian. Assume that \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \((0, 1)\) such that \( \alpha_n \leq 1 - \delta \) for all \( n \) and for some \( \delta \in (0, 1) \) and \( \beta_n \to 1 \). Define a sequence \( \{x_n\} \) in \( C \) by the following algorithm:

\[
x_0 \in C \text{ chosen arbitrarily,}
\]

\[
z_n = \beta_n x_n + (1 - \beta_n) T^{x_n},
\]

\[
y_n = \alpha_n x_n + (1 - \alpha_n) T^{z_n},
\]

\[
C_n = \{ v \in C : ||y_n - v||^2 \leq \alpha_n \|z_n\|^2 + (1 - \alpha_n) G_n^2 + MG_n + (1 - \alpha_n) \|z_n\|^2 \}
\]

\[
- 2(\alpha_n x_n + (1 - \alpha_n) z_n, v),
\]

\[
Q_n = \{ v \in C : \langle x_0 - x_n, v \rangle \geq 0 \},
\]

\[
x_{n+1} = P_{C_n \cap Q_n}(x_n),
\]

where \( M = 2(1 - \alpha_n)(diam C) \) and \( G_n = \max\{0, \sup_{x \in C, y \in C} \|\|T^x - p\| - \|x - p\|\|\} \) for all \( n \geq 1 \), for all \( p \in F(T) \) so that \( \sum_{n=1}^{\infty} G_n < \infty \). Then \( \{x_n\} \) converges to \( P_{F(T)}(x_0) \).

**Proof** It is obvious that for \( n \in \mathbb{N} \cup \{0\} \), \( Q_n \) is closed and convex. We show that \( C_n \) is closed and convex for all \( n \in \mathbb{N} \cup \{0\} \). Let \( \{y_m\}_{m=1}^{\infty} \subseteq C \) with \( y_m \to v \) as \( m \to \infty \). Since \( C \) is closed and \( y_m \in C_n \), we have \( v \in C \) and

\[
||y_n - v||^2 \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n) G_n^2 + MG_n + (1 - \alpha_n) \|z_n\|^2 - 2(\alpha_n x_n + (1 - \alpha_n) z_n, v),
\]

\[
+ ||v_m||^2.
\]
Then
\[ \|y_n - v\|^2 = \|y_n - v_m + v_m - v\|^2 = \|y_n - v_m\|^2 + \|v_m - v\|^2 + 2\langle y_n - v_m, v_m - v \rangle \]
\[ \leq \|y_n - v_m\|^2 + \|v_m - v\|^2 + 2\|y_n - v_m\||v_m - v| \]
\[ \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, v_m) \]
\[ + \|v_m\|^2 + \|v_m - v\|^2 + 2\|y_n - v_m\||v_m - v|. \]

Taking \( m \to \infty \)
\[ \|y_n - v\|^2 \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, v) + \|v\|^2. \]

Then \( v \in C_n \) and hence \( C_n \) is closed.

Let \( x, y \in C_n \subset C \) with \( z = \lambda x + (1 - \lambda)y \) where \( \lambda \in (0, 1) \). Since \( C \) is convex, \( z \in C \). Thus, we have
\[ \|y_n - x\|^2 \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, x) + \|x\|^2. \]

and
\[ \|y_n - y\|^2 \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, y) + \|y\|^2. \]

Hence
\[ \|y_n - z\|^2 = \|y_n - (\lambda x + (1 - \lambda)y)\|^2 \]
\[ = \|\lambda(y_n - x) + (1 - \lambda)(y_n - y)\|^2 \]
\[ = \lambda \|y_n - x\|^2 + (1 - \lambda)\|y_n - y\|^2 - \lambda(1 - \lambda)\|y - x\|^2 \]
\[ \leq \alpha_n \|x_n\|^2 + \lambda(1 - \alpha_n)G_n^2 + \lambda MG_n + (1 - \alpha_n)|z_n|^2 - 2\lambda(\alpha_n x_n + (1 - \alpha_n)z_n, x) \]
\[ + (1 - \alpha_n)\|x_n\|^2 + (1 - \lambda)(\alpha_n\|x_n\|^2 + (1 - \lambda)G_n^2 + (1 - \alpha_n)MG_n + (1 - \alpha_n)(1 - \alpha_n)|z_n|^2 \]
\[ - 2(1 - \lambda)(\alpha_n x_n + (1 - \alpha_n)z_n, y) + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|y - x\|^2 \]
\[ = \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, \lambda x + (1 - \lambda)y) \]
\[ + \|\lambda x + (1 - \lambda)y\|^2 \]
\[ = \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)|z_n|^2 - 2(\alpha_n x_n + (1 - \alpha_n)z_n, z) + \|z\|^2. \]

It follows that \( z \in C_n \) and hence \( C_n \) is closed and convex. Then \( C_n \cap Q_n \) is closed and convex.

Next, we show that \( F(T) \subset C_n \) for all \( n \in \mathbb{N} \cup \{0\} \). Indeed, let \( p \in F(T) \), we have
\[ \|y_n - p\|^2 = \|\alpha_n(x_n - p) + (1 - \alpha_n)(T^n x_n - p)\|^2 \]
\[ = \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|T^n x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T^n x_n\|^2 \]
\[ \leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|T^n x_n - p\|^2 \]
\[ = \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)(\|T^n x_n - p\| - \|z_n - p\|) + \|z_n - p\|^2 \]
\[ \leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)(\|T^n x_n - p\| - \|z_n - p\|) \]
\[ + 2(1 - \alpha_n)(\|T^n x_n - p\| - \|z_n - p\|) + (1 - \alpha_n)\|z_n - p\|^2 \]
\[ \leq \alpha_n\|x_n - p\|^2 - 2(\alpha_n x_n, p) + \alpha_n\|p\|^2 + (1 - \alpha_n)G_n^2 \]
\[ + 2(1 - \alpha_n)G_n(diam C) + (1 - \alpha_n)\|z_n - p\|^2. \]
\[ = \alpha_n\|x_n\|^2 - 2(\alpha_n x_n, p) + \alpha_n\|p\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)\|z_n\|^2 - 2\langle (1 - \alpha_n)z_n, p \rangle \]
\[ +\|p\|^2 - \alpha_n\|p\|^2 \]
\[ = \alpha_n\|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)\|z_n\|^2 - 2\langle (1 - \alpha_n)z_n, p \rangle + \|p\|^2. \quad (14) \]
that \( F(T) \subset C_k \cap Q_k \) for some \( k \in \mathbb{N} \). Since \( C_k \cap Q_k \) is closed and convex, we can define \( x_{k+1} = P_{C_k \cap Q_k}(x_0) \). From \( x_{k+1} = P_{C_k \cap Q_k}(x_0) \), by Lemma 2.2 we have

\[
\langle x_0 - x_{k+1}, x_{k+1} - z \rangle \geq 0 \quad \text{for all } z \in C_k \cap Q_k.
\]

Since \( F(T) \subset C_k \cap Q_k \), we also have

\[
\langle x_0 - x_{k+1}, x_{k+1} - u \rangle \geq 0 \quad \text{for all } u \in F(T).
\]

So, we get \( F(T) \subset Q_{k+1} \). Then we obtain \( F(T) \subset C_n \cap Q_{k+1} \). Next, let us show that \( \{x_n\} \) is bounded. Since \( F(T) \) is a closed and convex. Put \( z_0 = P_{F(T)}(x_0) \). From \( x_{k+1} = P_{C_k \cap Q_k}(x_0) \), we get

\[
\|x_{n+1} - x_0\| \leq \|z - x_0\| \quad \text{for all } z \in C_n \cap Q_n.
\]

From \( z_0 \in F(T) \subset C_n \cap Q_n \), we also have

\[
\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\},
\]

and hence \( \{x_n\} \) is bounded. Since \( x_{n+1} \in C_k \cap Q_n \subset C_n \) and from the definition of \( Q_n \) we have \( x_n = P_{Q_n}(x_0) \), we get \( \|x_n - x_0\| \leq \|x_{n+1} - x_0\| \). From boundedness of \( \{x_n\} \), we get that \( \lim_{n \to \infty} \|x_n - x_0\| \) exists. So, we obtain \( \|x_{n+1} - x_0\| \to 0 \). On the other hand, from \( x_{n+1} \in Q_n \), we have

\[
\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0.
\]

So, for all \( n \in \mathbb{N} \cup \{0\} \) we get

\[
\|x_n - x_{n+1}\|^2 = \|(x_n - x_0) - (x_{n+1} - x_0)\|^2
\]

\[
= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+1} - x_0 \rangle + \|x_{n+1} - x_0\|^2
\]

\[
= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_{n+1}, x_0 - x_n \rangle
\]

\[
= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_n - x_{n+1} \rangle
\]

\[
\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\]

This implies

\[
\|x_{n+1} - x_0\| \to 0. \tag{15}
\]

However, since \( \lim_{n \to \infty} \beta_n = 1 \) and \( \{x_n\} \) is bounded, we obtain

\[
\|z_n - x_n\| = \|\beta_n x_n - x_n\| + (1 - \beta_n)\|T^n x_n - x_n\| \to 0. \tag{16}
\]

Since \( \|z_n - x_{n+1}\| \leq \|z_n - x_n\| + \|x_n - x_{n+1}\| \). It follows from (15) and (16) that

\[
\|z_n - x_{n+1}\| \to 0. \tag{17}
\]

On the other hand, it follows from \( x_{n+1} \in C_n \) that

\[
\|y_n - x_{n+1}\|^2 \leq \alpha_n \|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)\|z_n\|^2 - 2\alpha_n x_n + (1 - \alpha_n)\|x_n\|^2 + \|x_{n+1}\|^2
\]

\[
= \alpha_n \|x_n\|^2 + (1 - \alpha_n)\|x_n\|^2 + \|x_{n+1}\|^2 - 2\alpha_n(x_n, x_{n+1})
\]

\[
+ 2\alpha_n(z_n, x_{n+1}) + (1 - \alpha_n)G_n^2 + MG_n
\]

\[
= \alpha_n \|x_n\|^2 + \|z_n\|^2 - \alpha_n \|x_n\|^2 - \|x_{n+1}\|^2 - 2\alpha_n(x_n, x_{n+1}) - 2\langle z_n, x_{n+1} \rangle
\]

\[
+ 2\alpha_n(z_n, x_{n+1}) + (1 - \alpha_n)G_n^2 + MG_n
\]

\[
= \|z_n - x_{n+1}\|^2 - 2\langle z_n, x_{n+1} \rangle + \alpha_n \|x_n\|^2 + \alpha_n \|x_n\|^2 - 2\alpha_n(x_n, x_{n+1}) + \alpha_n \|x_{n+1}\|^2
\]

\[
- \alpha_n \|z_n\|^2 + 2\alpha_n(z_n, x_{n+1}) - \alpha_n \|x_{n+1}\|^2 + (1 - \alpha_n)G_n^2 + MG_n
\]

\[
= \|z_n - x_{n+1}\|^2 + \alpha_n \|x_n - x_{n+1}\|^2 - \alpha_n \|z_n - x_{n+1}\|^2 + (1 - \alpha_n)G_n^2 + MG_n.
\]

It follows from (15), (17) and \( \sum_{n=1}^{\infty} G_n < \infty \) that

\[
\|y_n - x_{n+1}\| \to 0 \tag{18}
\]

It follows from (15) and (18) that

\[
\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{19}
\]
Again, noticing that $T^n z_n = y_n + \alpha_n ||T^n z_n - x_n||$, we have

$$ ||T^n z_n - x_n|| = ||y_n - x_n|| + \alpha_n ||T^n z_n - x_n||. $$

It follows that $||T^n z_n - x_n|| = \frac{1}{1-\alpha_n} ||y_n - x_n||$. Since $\alpha_n \leq 1 - \delta$ and $T$ is uniformly $L$-Lipschitzian, we have

$$ ||T^n x_n - x_n|| \leq ||T^n x_n - T^n z_n|| + ||T^n z_n - x_n|| \leq L||x_n - z_n|| + \frac{1}{\delta} ||y_n - x_n||. $$

Therefore, it follows from (16) and (19) that

$$ ||T^n x_n - x_n|| \rightarrow 0. $$

We obtain

$$ ||T x_n - x_n|| \leq ||T x_n - T_{n+1} x_n|| + ||T_{n+1} x_n - x_{n+1}|| + ||x_{n+1} - x_n|| \leq L||x_n - T x_n|| + (L + 1)||x_n - x_{n+1}|| + ||T^n x_n - x_{n+1}||, $$

which implies that

$$ ||T x_n - x_n|| \rightarrow 0. $$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow q$. Assume $q \neq Tq$. From Opial’s theorem (Lemma 2.5), we have

$$ \lim inf_{i \rightarrow \infty} ||x_{n_i} - q|| < \lim inf_{i \rightarrow \infty} ||x_{n_i} - T q|| $$

$$ \leq \lim inf_{i \rightarrow \infty} (||x_{n_i} - T x_n|| + ||T x_n - T q||) $$

$$ \leq \lim inf_{i \rightarrow \infty} (L||x_{n_i} - q||) $$

$$ = (L)\lim inf_{i \rightarrow \infty} ||x_{n_i} - q||. $$

This is a contradiction. So, we have $q = T q$, and hence $q \in F(T)$. From $z_0 = P_{F(T)} x_0$, Lemma 2.3, 2.4 and $||x_{n+1} - x_n|| \leq ||x_n - z_n||$, we have

$$ ||x_0 - z_0|| \leq ||x_0 - q|| \leq \lim inf_{i \rightarrow \infty} ||x_0 - x_n|| \leq \lim sup_{i \rightarrow \infty} ||x_0 - x_n|| \leq ||x_0 - z_0||. $$

So, we get

$$ \lim_{i \rightarrow \infty} ||x_0 - x_{n_i}|| = ||x_0 - q|| = ||x_0 - z_0||. $$

It follows that $x_0 = x_0 \rightarrow x_0 \rightarrow z_0$; hence, $x_{n_i} \rightarrow z_0$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \rightarrow z_0$. This complete the proof.

**Theorem 3.2** Let $C$ be a nonempty bounded closed convex subset of Hilbert space $H$ and let $T : C \rightarrow C$ be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set $F(T)$ and $T$ is uniformly $L$-Lipschitzian. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. Define a sequence $\{x_n\}$ in $C$ by the following algorithm:

$$ x_0 \in C \text{ chosen arbitrarily,} $$

$$ y_n = \alpha_n x_0 + (1 - \alpha_n) T^n x_n, $$

$$ C_n = \{v \in C : ||y_n - v||^2 \leq \alpha_n ||x_0||^2 + (1 - \alpha_n) ||G_n^2 + MG_n + (1 - \alpha_n)||x_0||^2 $$

$$ - 2(\alpha_n x_0 + (1 - \alpha_n) x_n, v) + ||v||^2), $$

$$ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v \rangle \geq 0\}, $$

$$ x_{n+1} = P_{C_n \cap Q_n}(x_0), $$

where $M = 2(1 - \alpha_n)(\text{diam } C)$ and $G_n = \max[0, \sup_{x \in C} ||T^n x - p|| - ||x - p||]$ for all $n \geq 1, \text{ for all } p \in F(T)$ so that $\sum_{n=1}^{\infty} G_n < \infty$.

Then $\{x_n\}$ converges to $P_{F(T)} x_0$.

**Proof** Similarly as in the proof of Theorem 3.1, we can get the proof is completed.

**References**


