Some Properties and Convergence Theorems of Conditional g-expectation

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Abstract

Peng Shige introduced the notions of g-expectation and conditional g-expectation, which is the nonlinear extension of classic mathematical expectation and conditional mathematical expectation. Just as g-expectation preserves most properties of classic mathematical expectation except the linearity, conditional g-expectation preserves most properties of classic conditional mathematical expectation except the linearity. Based on the properties and theorems of classic conditional mathematical expectation, this paper will discuss and sum up some properties and convergence theorems of conditional g-expectation.

Keywords: Backward stochastic differential equation(BSDE), g-expectation, Conditional g-expectation

1. Basic assumptions and notations

Let (Ω, F, P) be a probability space, in which $\{B_t\}_{0\leq t\leq T}$ is a d-dimensional standard Brownian motion. Let $\{F_t\}_{0\leq t\leq T}$ be the filtration generated by this Brownian motion denoted by $F_t = \sigma\{B_s, s \leq t\}$. For $\forall T \in [0, \infty)$, we define the following two spaces first:

$$L^2(0, F_t, P; R) := \{X : X_t \text{ is a } F_t\text{-adapted process and } \|X\|_L^2 := E\int_0^T |X_s|^2 ds < \infty\};$$
$$L^2(\Omega, F_t, P; R) := \{\xi : \xi \text{ is a } F_T\text{-measurable random variable and } E|\xi|^2 ds < \infty\}.$$

We assume the function $g(y, z, t) : R \times R^d \times [0, T] \to R$ satisfies the following assumptions:

(H_1) For $\forall (y, z) \in R \times R^d$, we have $\int_0^T |g(y, z, s)|^2 ds < \infty\}$;

(H_2) g satisfies consistent Lipschitz-conditions: There exists a constant $\mu \geq 0$, for $\forall t \in [0, T], y_1, y_2 \in R, z_1, z_2 \in R^d$, we have

$$|g(y_1, z_1, t) - g(y_2, z_2, t)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|);$$

(H_3) For $\forall t \in [0, T], y \in R$, we have $g(y, 0, t) \equiv 0$;

Pardoux and Peng Shige(Pardoux, 1990, p. 55-61)have proved that the following backward stochastic differential equation(BSDE in short):

$$y_t = \xi + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dB_s, 0 \leq t \leq T \tag{1}$$

exists an unique adapted and square integrable solution denoted by

$$(y, z) \in L^2(0, T, P; R) \times L^2(0, T, P; R^d).$$

Furthermore, Peng Shige introduced the notions of g-expectation and conditional g-expectation by BSDE(Peng, 1997, p. 141-159).

Definition 1.1(g-expectation and conditional g-expectation)

For any given $T \geq 0$, $\xi \in L^2(\Omega, F_T, P; R)$, we assume the generator g satisfis (H_1) and (H_2), $(y_t^{\xi,g,T}, z_t^{\xi,g,T})$ is the adapted solution to BSDE(1), then we can call $\varepsilon_g[\xi] = y_0^{\xi,g,T}$ and $\varepsilon_g[\xi \backslash F_t] = y_t^{\xi,g,T}$ are the mathematical expectation and conditional mathematical expectation generated by function g, denoted by g-expectation and conditional g-expectation shortly.

Since g-expectation and conditional g-expectation can be considered as the extension of classic mathematical expectation and conditional mathematical expectation, they preserve most properties of classic mathematical expectation and conditional mathematical expectation except the linearity. Many scholars have studied all kinds of properties and convergence theorems and Jensen inequality of g-expectation, such as Peng Shige(Peng, 1997, p. 141-159), Cheng Zengjing(Chen, 1999, p. 175-180), Jang Long(Jiang, 2003, p. 13-17)and(Jiang, 2004, p. 401-412), Zhang Hui(Zhang, 2005, p. 29-30), Lin Qun(Lin, 2008, p. 28-30)and so on. This paper will discuss and sum up some properties and convergence theorems of conditional g-expectation basing on the properties and theorems of the classic conditional mathematical expectation given by the previous studies.

2. Properties of conditional g-expectation

Lemma 2.1 Conditional g-expectation preserves most properties of classic conditional mathematical expectation except the linearity((Yan, J. A., 2000),(Yan, J. A., 2004),(Wang, J.G., 2005)).

(i) (Monotonicity) If $\xi_1 \geq \xi_1 \quad a.s.$, then we have $\varepsilon_g[\xi_1 \setminus F_t] \geq \varepsilon_g[\xi_2 \setminus F_t] \quad a.s.$;

Furthermore, if $P(\xi_1 > \xi_2) > 0$, then we have $P(\varepsilon_g[\xi_1 \setminus F_t] \geq \varepsilon_g[\xi_2 \setminus F_t]) > 0$;

(ii) (Smoothness) For $\forall t, r \in [0, T]$, we have $\varepsilon_g[\varepsilon_g[\xi \setminus F_t] \setminus F_r] = \varepsilon_g[\xi \setminus F_{t\wedge r}]$;

(iii) If ξ is F_t-measurable, then we have $\varepsilon_g[\xi \setminus F_t] \equiv \xi$;

Especially, $\varepsilon_g[0 \setminus F_t] \equiv 0$, $\varepsilon_g[1 \setminus F_t] \equiv 1$;

(iv) For $\forall t \in [0, T]$, we have $\varepsilon_g[\varepsilon_g[\xi \setminus F_t]] = \varepsilon_g[\xi]$;

(v) For $\forall B \in F_t$, we have $\varepsilon_g[\xi 1_B \setminus F_t] = 1_B \varepsilon_g[\xi \setminus F_t]$;

(vi) If g is not depend on y, that is $g(y, z, t) : \Omega \times R^d \times [0, T] \to R$, then for $\forall t \in [0, T]$ we have

$$\varepsilon_g[\xi + \eta \setminus F_t] = \varepsilon_g[\xi \setminus F_t] + \eta, \forall \eta \in L^2(\Omega, F_t, P), \xi \in L^2(\Omega, F_T, P);$$

(vii) If $g(y, z, t)$ is determined, and ξ is independent of F_t, then we have $\varepsilon_g[\xi \setminus F_t] = \varepsilon_g[\xi]$.

3. Convergence theorems for conditional g-expectation

As we all know, for classic conditional mathematical expectation we have Monotone convergence theorem(Levi lemma), Fatou lemma, Lebesgue dominated convergence theorem and Jensen inequality(Wang, J.G., 2005). Similarly, for conditional g-expectation we also have some corresponding convergence theorems and inequality which are not involved with linearity. To proof the theorems easily, we need to introduce a lemma first:

Lemma 3.1 (Comparison theorem(Pardoux, 1990, p. 55-61))

Let $\xi, \overline{\xi} \in L^2(\Omega, F_T, P)$, $g, \overline{g}$ satisfy the assumptions $(H_1), (H_2)$, (y_t, z_t) and $(\overline{y_t}, \overline{z_t})$ be the solutions to BSDE(1) and the following BSDE:

$$\overline{y_t} = \overline{\xi} + \int_t^T g(\overline{y_s}, \overline{z_s}, s)ds - \int_t^T \overline{z_s} dB_s, 0 \leq t \leq T. \tag{2}$$

We have the following conclusions:

(i) If $\xi \geq \overline{\xi}$, $g(\overline{y_s}, \overline{z_s}, s) \geq \overline{g}(\overline{y_s}, \overline{z_s}, s) \quad a.s.a.e.$, then we have $y_t \geq \overline{y_t} \quad a.s.a.e.$;

(ii) Furthermore, if $P(\xi - \overline{\xi} > 0) > 0$, then we have $P(y_t - \overline{y_t} > 0) > 0$. Especially, $y_0 > \overline{y_0}$.

Under the above assumptions(g satisfies the assumptions $(H_1), (H_2), (H_3)$) and the definition of g-expectation, let $\xi_n, n \geq 1$ be a random variable sequence and $\xi, \eta \in L^2(\Omega, F_T, P)$, we can have the following theorems:

Theorem 3.2 (Monotone convergence theorem of conditional g-expectation)

(i) If $\eta \leq \xi_n \uparrow \xi \quad a.s.$, then we have $\lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t] = \varepsilon_g[\xi \setminus F_t] \quad a.s.a.e.$;

(ii) If $\eta \geq \xi_n \downarrow \xi \quad a.s.$, then we have $\lim_{n\to\infty} \varepsilon_g[-\xi_n \setminus F_t] = \varepsilon_g[-\xi \setminus F_t] \quad a.s.a.e.$.

Proof: Here we just proof the situation of(i), the situation of(ii)can be proved similarly.

From the monotonicity of conditional g-expectation we know that $\varepsilon_g[\xi_n \backslash F_t]$ progressively increase with n. Let $\varepsilon_g[\xi_n \backslash F_t] \uparrow \zeta$, then $\zeta \in F_t$.

For $\forall B \in F_t$, from properties(v)and(ii)we have:

$$
\begin{aligned}
\varepsilon_g[1_B\zeta \setminus F_t] &= \varepsilon_g[1_B \lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t] \setminus F_t] \\
&= 1_B\varepsilon_g[\lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t] \setminus F_t] \\
&= 1_B\varepsilon_g[\lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t]] \\
&= \varepsilon_g[1_B \lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t]] \\
&= \varepsilon_g[\varepsilon_g[1_B \lim_{n\to\infty} \xi_n \setminus F_t]] \\
&= \varepsilon_g[1_B\varepsilon_g[\xi \setminus F_t]]
\end{aligned}
$$

Then we have $\zeta = \varepsilon_g[\xi \setminus F_t]$, that is $\lim_{n\to\infty} \varepsilon_g[\xi_n \setminus F_t] = \varepsilon_g[\xi \setminus F_t]$, so the theorem is proved.

Theorem 3.3 (Fatou lemma of conditional g-expectation)

(i) If $\xi_n \geq \eta$, then we have $\varepsilon_g[\underline{\lim}_n\xi_n \setminus F_t] \leq \underline{\lim}_n\varepsilon_g[\xi_n \setminus F_t]$;

(ii) If $\xi_n \leq \eta$, then we have $\varepsilon_g[\overline{\lim}_n\xi_n \setminus F_t] \geq \overline{\lim}_n\varepsilon_g[\xi_n \setminus F_t]$.

Proof: Here we also just proof the situation of(i), the situation of(ii)can be proved similarly.

Let $\eta_n = \inf_{k\geq n} \xi_k$, then we have $\eta \leq \eta_n \uparrow \underline{\lim}_n\xi_n \quad a.s.$. Besides, from lemma 3.1 we have

$$
\begin{aligned}
\varepsilon_g[\underline{\lim}_n\xi_n \setminus F_t] &= \lim_{n\to\infty} varepsilon_g[\eta_n \setminus F_t] \\
&\leq \underline{\lim}_n\varepsilon_g[\xi_n \setminus F_t]
\end{aligned}
$$

Then the inequality is proved.

Theorem 3.4 (Lebesgue dominated convergence theorem of conditional g-expectation)

If $|\xi_n| \leq \eta$ and $\xi_n \to \xi \quad a.s.$ when $n \to \infty$, then we have that $\lim_n \varepsilon_g[\xi_n \setminus F_t] = \varepsilon_g[\xi \setminus F_t] \quad a.s.a.e.$.

Proof: From theorem 3.3 we have

$$
\begin{aligned}
\varepsilon_g[\xi \setminus F_t] &= \varepsilon_g[\lim_n \xi_n \setminus F_t] \\
&\leq \underline{\lim}_n\varepsilon_g[\xi_n \setminus F_t] \\
&\leq \overline{\lim}_n\varepsilon_g[\xi_n \setminus F_t] \\
&\leq \varepsilon_g[\lim_n \xi_n \setminus F_t] \\
&= \varepsilon_g[\xi \setminus F_t]
\end{aligned}
$$

Then the theorem is proved.

For classic conditional mathematical expectation we have the following Jensen inequality(Wang, J. G, 2005):

If ξ and $\varphi(\xi)$ are integrable random variables and $\varphi(\xi)$ is a convex function in R, then we have

$$\varphi(\varepsilon_g[\xi \setminus F_t] \leq \varepsilon_g[\varphi(\xi) \setminus F_t]) \quad a.s..$$

To proof it, we need an important property of classic conditional expectation:

$$E[XY \setminus F_t] = YE[X \setminus F_t], \text{ for } \forall Y \in F_t.$$

However, for general g, conditional g-expectation doesn't have the property:

$$\varepsilon_g[\xi\eta \setminus F_t] = \eta\varepsilon_g[\xi \setminus F_t], \text{ for } \forall \eta \in F_t$$

But if g and φ are specific, we can still discuss the Jensen inequality of conditional g-expectation using comparison theorem.

Theorem 3.5 (Jensen inequality of conditional g-expectation)

If function φ is monotonous nondecreasing, and for $\forall\xi \in L^2(\Omega, F_T, P)$ we have $\varphi(\xi) \in L^2(\Omega, F_T, P)$, then we have the following conclusion:

(i) when $g \geq 0$ and φ is a limited convex function, we have $\varepsilon_g[\varphi(\xi) \setminus F_t] \geq \varphi(\varepsilon_g[\xi \setminus F_t])$;

(ii) when $g \leq 0$ and φ is a limited concave function, we have $\varepsilon_g[\varphi(\xi) \setminus F_t] \leq \varphi(\varepsilon_g[\xi \setminus F_t])$.

Referring the proof of Jensen inequality of classic conditional mathematical expectation(Yan, J. A., 2004), it's not difficult to proof this theorem. Here we omit it.

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