Conditionally Permutable Subgroup and $p$-supersolubility of Finite Groups

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Abstract

In this paper, we research $p$-supersolubility of finite groups. We determine the structure of some groups by using the conditionally permutable subgroups. We obtain some sufficient or necessary and sufficient conditions of a finite group is $p$-supersolvable.

Keywords: Conditionally permutable, Maximal subgroup, $p$-supersolvable

1. Introduction

All groups considered in this paper are finite. The product $HT$ of subgroups $H$ and $T$ is still a subgroup if and only if $HT = TH$. Thus the permutability plays an important in the study of the structure of finite groups. For example, Ore O., 1939, P.431-460, proved that every permutable subgroups $H$ of a group $G$ is subnormal in $G$. However, for two subgroups $H$ and $T$ of a group $G$, maybe they are not permutable but there exists an element $x \in G$ such that $HTx = Txx$. Guo W.B., Shum K.P., Skiba A.N., 2004, P.128-133, 2005, P.493-510, introduced the concepts of conditionally permutable subgroups and completely conditionally permutable subgroups. With these concepts, some new elegant results, Hu Y.S., Guo X.Y., 2007, P.28-32, Hu Y.S., Wang L.L., 2007, P.1-4, Li C.W., Yu Q., 2007, P.8-10, Zhang X.M., Liu X., 2010, P.51-59, have been obtained. In this paper, we determine the structures of some groups by using the conditionally permutable subgroups. Some new criterions of $p$-supersolubility of some finite groups will be given and some known results are generalized.

We use “c-permutable” to denote “conditionally permutable”. As usual, we denote a maximal subgroup $M$ of $G$ by $M < G$ and a minimal normal subgroup $A$ of $G$ by $A \triangleleft G$. All unexplained notions and terminologies are standard, see Refs. Guo W.B., 2000 and Xu M.Y., 1987.

2. Preliminaries

We cite here some known results which are useful in the later.


(1) $H$ is called c-permutable with $T$ in $G$ if there exists some $x \in G$ such that $HTx = Txx$.
(2) $H$ is called c-permutable in $G$ if for every subgroup $K$ of $G$, there exists some $x \in G$ such that $HKx = Kxx$.

Lemma 2.1 (Guo W.B., 2000; Theorem 1.9.4) The following conditions are equivalent:
(1) $G$ is $p$-supersolvable;
(2) $G$ is $p$-solvable and the index of every maximal subgroup of $G$ either equal to $p$ or be $p'$-number.

Lemma 2.2 (Guo W.B., 2000; Theorem 1.7.7) Let $G$ be $\pi'$-solvable group. Then there at least exists one $\pi'$-Hall subgroup $G_\pi'$ of $G$, and for every $\pi'$-subgroup $A$ of $G$, there exists some $x \in G$ such that $Axx \subseteq G_\pi'$. In particular, any two $\pi'$-Hall subgroups of $G$ conjugated in $G$.

Lemma 2.3 (Guo W.B., 2000; Theorem 1.7.6) Let $G$ be $\pi$-solvable group. Then there at least exists one $\pi$-Hall subgroup $G_\pi$ of $G$, and for every $\pi$-subgroup $A$ of $G$, there exists some $x \in G$ such that $Axx \subseteq G_\pi$. In particular, any two $\pi$-Hall subgroups of $G$ conjugated in $G$.

Lemma 2.4 (Guo W.B., Shum K.P., Skiba A.N., 2004, P.128-133) Let $G$ be a group. Suppose that $N \triangleleft G$ and $H \leq G$. Then
(1) If $N \leq T \leq G$ and $H$ is c-permutable with $T$ in $G$, then $HN/N$ is c-permutable with $T/N$ in $G/N$;
(2) Assume that $N \leq H$ and $T \leq G$, if $H/N$ is c-permutable with $TN/N$ in $G/N$, then $H$ is c-permutable with $T$ in $G$;
(3) Assume that $T \leq G$ and $H$ is c-permutable with $T$ in $G$, then $H^x$ is c-permutable with $T^x$ in $G$ for any $x \in G$. 

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Lemma 2.1. For any subgroup $(ii)$ Every maximal subgroup of $G$ with the index of $p^\alpha$ is $c$-permutable in $G$, where $\alpha$ is an integer;

$(iii)$ Every maximal subgroup of $G$ with the index of $p^\alpha$ is $c$-permutable with every maximal subgroup of Sylow $p$-subgroup of $G$ in $G$;

$(iv)$ Every maximal subgroup of $G$ is $c$-permutable with every maximal subgroup of Sylow $p$-subgroup of $G$ in $G$;

Proof: $(i) \implies (ii)$ Let $G$ be a $p$-solvable group and $M$ be a maximal subgroup of $G$, $G/M = p^\beta$. By Lemma 2.1, for any subgroup $K$ of $G$, $K = K_pK'$, and $M = M_pM'$, $K_p \in \text{Syl}_p(K)$, $M_p \in \text{Syl}(M)$, $K' \subseteq \text{Hall}_{p'}(K)$, $M' \subseteq \text{Hall}(M)$ and $G' \subseteq \text{Hall}(G)$. By Lemma 2.2, there exists some $x \in G$ such that $K^x_p \subseteq G_p \subseteq M$. If $K^x_p \subseteq M$, then $M_p = M'$. If $K^x_p \nsubseteq M$, then $G = M^x_p = K^xM = K^xM$. All imply that $M$ is $c$-permutable in $G$.

$(ii) \implies (iii)$ It is concluded from the definition of $c$-permutable subgroups.

$(iii) \implies (iv)$ Let $G$ be a $p$-solvable group and every maximal subgroup of $G$ with the index of $p^\alpha$ is $c$-permutable with every maximal subgroup of Sylow $p$-subgroup of $G$ in $G$.

For any maximal subgroup $M$ of $G$, $\lvert G : M \rvert = p^\beta$ or $\lvert G : M \rvert$ is a $p^\beta$-number, where $\beta$ is an integer. Set $P \in \text{Syl}(G)$ and $P_1 < P$. If $\lvert G : M \rvert = p^\beta$, then $M$ is $c$-permutable with $P_1$ in $G$ by the hypothesis. If $\lvert G : M \rvert$ is a $p^\beta$-number, then $M = M_pM_p' = G_pM_p'$, where $G_p \in \text{Syl}(M)$ and $M_p \in \text{Syl}(G)$ and $M_p' \in \text{Hall}(M)$. By Lemma 2.3, there exists some $y \in (M, P_1)$, $G_y$ such that $P_y \subseteq G_p \subseteq M$. Hence $MP_y' = M = P_yM$. All imply that $M$ is $c$-permutable with $P_1$ in $G$.

$(iv) \implies (i)$ Let $G$ be a $p$-solvable group and every maximal subgroup of $G$ is $c$-permutable with every maximal subgroup of Sylow $p$-subgroup of $G$ in $G$.

Assume that the proposition $(i)$ is false and let $G$ be a counterexample of a minimal order. Let $H \triangleleft G$, $M/H \lhd G/H$, $P/H \in \text{Syl}(G/H)$ and $P_1/H < P/H$. If $P_0 \in \text{Syl}(P)$ and $P_2 \in \text{Syl}(P_1)$, then $M < G$, $P_0 \in \text{Syl}(G)$ and $P_2 < P_0$. Hence by the hypothesis $M$ is $c$-permutable with $P_2$ in $G$. Clearly $P_2H/H = P_1/H$ and $P_0H/H = P/H$ by Lemma 2.4. Then $P_1/H$ is $c$-permutable with $M/H$ in $G/H$. This shows that the hypothesis holds on $G/H$.

Since $G$ is $p$-solvable and outer $p$-supersolvable subgroup, by Lemma 2.5, $G = AN$ and $A \cap N = 1$, where $A < G$, $N \lhd G$ and $\lvert N \rvert = p^\alpha$, $\alpha > 1$.

Let $N \in \text{Syl}(G)$ and $N_1 \lhd N$. By the hypothesis, $A$ is $c$-permutable with $N_1$ in $G$. Hence By Lemma 2.4, there exists some $z \in (A, N_1)$ such that $D = N_1^Az = A^zN_1$. If $D = G$, then $\lvert G : A^z \rvert = \lvert N_1 \rvert = \lvert G : A \rvert = \lvert N \rvert$, this is a contradiction since $N_1 \lhd N$. So $D \neq G$, and $N_1A^z = A^z$ since $A^z < G$. Then $N_1^z \subseteq A \cap N = 1$ and $\lvert N_1 \rvert = 1, \lvert N \rvert = p$, this is a contradiction. This induces that $N$ is not a Sylow $p$-subgroup of $G$.

Let $A_1 \in \text{Syl}(A)$, by Lemma 2.3, there exists some subgroup $P \in \text{Syl}(G)$ such that $A_1 < P$. And there exists some subgroup $P_1$ of $P$ such that $P_1 < P$ and $A_1 \nsubseteq P_1$. By the hypothesis, $A$ is $c$-permutable with $P_1$ in $G$. So by Lemma 2.3, there exists some $w \in (A, P_1)$ such that $B = P_1A^w = A^wP_1$. Since $G = AN$, then there exists some $a \in A$ and $n \in N \subseteq P$ such that $w = an$. Hence $B = P_1A^n$ and $A^n_1 \nsubseteq P_1$, $P_1$ $\triangleleft P$. If $B = G$, then $P = P \cap P_1A^n = P_1(P \cap A^n) = P_1A^n_1 = P_1$, this is a contradiction. This implies that $B \neq G$. Thus $A^n < G$ and $B = A^n, P_1 \subseteq A^n$. $\lvert G : A^n \rvert = \lvert G : A \rvert = p = \lvert N \rvert$. This
Theorem 2. Let $G$ be a $p$-solvable group, $G = AB$ and $A \in Syl_p(G)$, $B \in Hall_p(G)$. If $B$ is $c$-permutable in $G$, then $G$ is $p$-supersolvable.

Proof: Assume that the assertion is false and $G$ be a counterexample of a minimal order. Let $H \cdot <G$. Then $G/H$ is $p$-solvable group and $G/H = AH/H \cdot BH/H$ which $AH/H \in Syl(G/H)$ and $BH/H \in Hall_p(G/H)$. By the hypothesis and Lemma 2.4, $BH/H$ is $c$-permutable in $G/H$. This shows that the hypothesis holds on $G/H$.

Since $G$ is $p$-solvable and outer $p$-supersolvable group, $G = MN$ and $M \cap N = 1$ by Lemma 2.5, where $M < G$, $N < G$ and $|N| = p^\alpha, \alpha > 1$. Hence $N \leq A$ and $A = A \cap M = A \cap N = N(A \cap M)$. If $A \cap M = A$, then $N \leq A \subseteq M$, this is a contradiction. Since $A \cap M \neq A$ and there exists subgroup $T$ of $G$ such that $T < A$ and $A \cap M \subseteq T$. By the hypothesis, $B$ is $c$-permutable in $G$. So there exists some $x \in G$ such that $BT^x = T^xB$. Hence $G = AB = N(A \cap M)B = (NT)B = (NT)^x B = NB^x$. This implies that either $BT^x = G$ or $BT^x$ is a supplement of $N$ in $G$. If $BT^x = G$, then $G = BT^x = BT$ by Lemma 2.6 and $A = A \cap BT = T(A \cap B) = T$. If $BT^x \cap N = 1$, then $T^x \cap N = 1$ and $N = |A : T| = p$ since $A = NT$. This contradiction completes the proof.

Theorem 3. Let $G$ be a $p$-solvable group. $G = AB$ and $A$ and $B$ are $p$-supersolvable groups and $(|A|, |B|) = 1$. If $A$ is $c$-permutable with every maximal subgroup of $B$ in $G$, and $B$ is $c$-permutable with every maximal subgroup of $A$ in $G$, then $G$ is $p$-supersolvable group.

Proof: Suppose that the theorem is false and let $G$ be a counterexample of minimal order.

Let $H < G$. Obviously, $G/H$ is a $p$-solvable group and $G/H = AH/H \cdot BH/H$, where $AH/H$ and $BH/H$ are $p$-supersolvable groups. Since $(|A|, |B|) = 1$, $(|AH/H|, |BH/H|) = (|A|/|A \cap H|, |B|/|B \cap H|) = 1$.

Let $T/H < AH/H$. Then there exists subgroup $A_0$ of $G$ such that $A_0 < A$ and $A_0H/H = T/H$. By the hypothesis, $B$ is $c$-permutable with $A_0$ in $G$. By Lemma 2.4, $BH/H$ is $c$-permutable with every maximal subgroup of $B$ in $G$. Similarly, it can be proved that $AH/H$ is $c$-permutable with every maximal subgroup of $B$ in $G$. Thus $G/H$ satisfies the hypothesis and $G/H$ is $p$-supersolvable.

Since $G$ is a $p$-solvable and outer $p$-supersolvable group. By Lemma 2.5, $G = MN$ and $|N| = p^\alpha, \alpha > 1$, where $N < G$ and $M < G$. Since $(|A|, |B|) = 1$, without loss of generality, we may assume that $N \subseteq A$ and $B \subseteq M$. Then $A = A \subseteq M = A \cap M$ and $N \leq A \subseteq M$. This is a contradiction. Since $A \cap M \neq A$ and there exists subgroup $T$ of $G$ such that $T < A$ and $A \cap M \subseteq T$. By the hypothesis, $B$ is $c$-permutable with $T$ in $G$ and there exists some $x \in G$ such that $BT^x = T^xB$. Hence $G = AB = N(A \cap M)B = (NT)B = NB^x$. Then $BT^x \cap N = 1$ since $N < G$ and $N$ is a abelian group. So $T^x \cap N = 1$ and $T \cap N = 1$. Then $|N| = |A : T| = p$ since $A = NT$, this is a contradiction. This implies that $G$ is $p$-supersolvable group.

Corollary 4. Let $G$ be $p$-solvable group. $G = AB$ which $A$ and $B$ are $p$-nilpotent groups and $(|A|, |B|) = 1$. If $A$ is $c$-permutable with every maximal subgroup of $B$ in $G$ and $B$ is $c$-permutable with every maximal subgroup of $A$ in $G$, then $G$ is $p$-supersolvable group.

Corollary 5 (Liu X., Li B.J., Yi X.L., 2008, P.79-86; Theorem 3.1) A group $G$ is supersolvable if and only if $G = AB$ is the product of two supersoluble subgroups $A$ and $B$ of coprime orders such that $A$ permutes with every maximal subgroup of $B$ and $B$ permutes with every maximal subgroup of $A$.

Proof: We only need to prove the sufficiency part as the necessity part is trivial. It is easy to see that a supersoluble group is also a $p$-supersolvable group and a permutable subgroup is also a $c$-permutable subgroup. Hence, we know that the corollary holds by our Theorem 2 and Lemma 2.7.

Corollary 6 (Liu X., Li B.J., Yi X.L., 2008, P.79-86; Corollary 3.3) A group $G$ is supersolvable if and only if $G = AB$ is the product of two supersoluble subgroups $A$ and $B$ of coprime orders such that every Sylow subgroup of $B$ is permutable with every maximal subgroup of $A$ and every Sylow subgroup of $A$ is permutable with every maximal subgroup of $B$.

Proof: Clearly, by our Theorem 3 and Lemma 2.7, the corollary holds.

References


