Fuzzy Anti-2-Normed Linear Space

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Abstract
In this paper, we study fuzzy anti-2-norm on a linear space and some results are introduced in fuzzy anti-2-norms on a linear space. We shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-2-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-2-normed linear space.

Keywords: Fuzzy norm, Fuzzy anti-norm, Fuzzy 2-norm, Fuzzy anti-2-norm

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1. Introduction
Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by Katsaras in [1984]. Felbin [1992] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [1984]. Cheng and Mordeson [1994] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [1975].

Bag and Samanta in [2003] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [1975]. They also studied some properties of the fuzzy norm in [2005] and [2008]. Bag and Samanta discussed the notions of convergent sequence and Cauchy sequence in fuzzy normed linear space in [2003]. They also made in [2008] a comparative study of the fuzzy norms defined by Katsaras [1984], Felbin [1992], and Bag and Samanta [2003].

In [2010] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [2008] and investigated their important properties.

In this paper, we study fuzzy anti-2-norm on a linear space and some results are introduced in fuzzy anti-2-norms on a linear space. We shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-2-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-2-normed linear space.

In [1992], Felbin introduced the concept of a fuzzy norm based on a Kaleva and Seikkala type [1984] of fuzzy metric using the notion of fuzzy number. Let \( X \) be a vector space over \( R \)(set of real numbers). Let \( \| \bullet \| : X \rightarrow R^+(I) \) be a mapping and let the mappings \( L, U : [0, 1] \times [0, 1] \rightarrow [0, 1] \), be symmetric, non-decreasing in both arguments and satisfying \( L(0, 0) = 0 \) and \( U(1, 1) = 1 \). Write \( \| x \|_\alpha = \sup \{ \| x \|_\alpha(R) \} \) for \( x \in X, 0 < \alpha \leq 1 \) and suppose for all \( x \in X, x \neq 0 \) there exists \( \alpha_0 \in (0, 1] \) independent of \( x \) such that for all \( \alpha \leq \alpha_0 \),

\[
(A) \quad \| x \|_\alpha < \infty \quad \text{and} \quad \alpha > 0
\]

The quadruple \( (X, \| \bullet \|, L, U) \) is called a Felbin-fuzzy normed linear space and \( \| \bullet \| \) is a Felbin-fuzzy norm if

(i) \( \| x \| = 0 \) if and only if \( x = 0 \) (the null vector),

(ii) \( \| r x \| = |r| \| x \|, x \in X, r \in R \),

(iii) For all \( x, y \in X \), (a) Whenever \( s \leq \| x \|_\alpha \), \( t \leq \| y \|_\alpha \) and \( s + t \leq \| x + y \|_\alpha \),

\[
\| x + y \|_\alpha + t \leq \| x + y \|_\alpha \alpha + \| y \|_\alpha \alpha
\]

(b) Whenever \( s \geq \| x \|_\alpha \), \( t \geq \| y \|_\alpha \) and \( s + t \geq \| x + y \|_\alpha \),

\[
\| x + y \|_\alpha + \| x + y \|_\alpha \alpha \leq \| x + y \|_\alpha \alpha + \| y \|_\alpha \alpha
\]

Definition 1.1. Let \( X \) be a vector space over \( R \)(set of real numbers). Let \( \| \bullet, \bullet \| : X \times X \rightarrow R^+(I) \) be a mapping and let the mappings \( L, U : [0, 1] \times [0, 1] \rightarrow [0, 1] \), be symmetric, non-decreasing in both arguments and satisfying \( L(0, 0) = 0 \) and \( U(1, 1) = 1 \). Write \( \| x, z \|_\alpha = \sup \{ \| x, z \|_\alpha(R, x, z) \} \) for \( x, z \in X, 0 < \alpha \leq 1 \) and suppose for all \( x, z \in X, x \neq 0, z \neq 0 \) there exists \( \alpha_0 \in (0, 1] \) independent of \( x, z \) such that for all \( \alpha \leq \alpha_0 \),

\[
(A) \quad \| x, z \|_\alpha < \infty \quad \text{and} \quad \alpha > 0
\]

The quadruple \( (X, \| \bullet, \bullet \|, L, U) \) is called a Felbin-fuzzy 2-normed linear space and \( \| \bullet, \bullet \| \) is a Felbin-fuzzy 2-norm if
Definition 2.1. This section contains a few basic definitions and preliminary results which will be needed in the sequel.

2. Fuzzy 2-Norms on a Linear Space

Let $\mathcal{V}$ be a fuzzy 2-norm on a linear space $X$. Let $(X, \|\cdot\|)$ be a fuzzy 2-normed linear space. Define $N(x, y, t) = \inf \{\|c\| : c \in X, \|c\| \leq t\}$, for every $x, y, t \in X$.

Theorem 2.5. Let $X$ be a normed linear space. If $N(x, y, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to +\infty} N(x, y, t) = 1$.

Then $N$ is said to be a fuzzy 2-norm on a linear space $X$ and the pair $(X, N)$ is called a fuzzy 2-normed linear space (briefly F-2-NLS).

The following condition of fuzzy 2-norm $N$ will be required later on.

Example 2.3. Let $(X, \|\cdot\|)$ be a 2-normed linear space. Define

$$N(x, y, t) = \begin{cases} \frac{t}{t + \|x, y\|}, & \text{when } t > 0, \ t \in R, \ x, y \in X \\ 0, & \text{when } t \leq 0, \ t \in R, \ x, y \in X. \end{cases}$$

Then $(X, N)$ is an F-2-NLS.

Example 2.4. Let $(X, \|\cdot\|)$ be a 2-normed linear space. Define

$$N(x, y, t) = \begin{cases} 0, & \text{when } t \leq \|x, y\|, \ t \in R, \ x, y \in X \\ 1, & \text{when } t > \|x, y\|, \ t \in R, \ x, y \in X. \end{cases}$$

Then $(X, N)$ is an F-2-NLS.

Theorem 2.5. Let $(X, N)$ be a fuzzy 2-normed linear space. Define $\|x, y\|_\alpha = \inf \{t : N(x, y, t) \geq \alpha\}; \alpha \in (0, 1)$. Then $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2-norms on $X$. These 2-norms are called $\alpha$-2-norms on $X$ corresponding to fuzzy 2-norm on $X$.

Theorem 2.6. Let $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on linear space $X$. Define a function $N' : X \times X \times R \to [0, 1]$ as

$$N'(x, y, t) = \begin{cases} \sup \{\alpha \in (0, 1) : \|x, y\|_\alpha \leq t\}, & \text{when } (x, y, t) \neq 0, \\ 0, & \text{when } (x, y, t) = 0. \end{cases}$$
Then $N'$ is a fuzzy 2-norm on $X$.

If the index set $(0, 1)$ of the family of crisp 2-norms $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ of Theorem 2.6 is extended to $(0, 1]$ then a fuzzy 2-norm $N$ is generated, satisfying an additional property that $N(x, y, t)$ attains the value 1 at some finite value $t$.

**Theorem 2.7.** Let $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on linear space $X$. Define a function $N' : X \times X \times R \rightarrow [0, 1]$ as

$$N'(x, y, t) = \sup[\alpha \in (0, 1) : \|x, y\|_\alpha \leq t], \text{ when } (x, y, t) \neq 0,$$

$$= 0, \text{ when } (x, y, t) = 0.$$

Then (a) $N'$ is a fuzzy 2-norm on $X$.

(b) For each $x, y \in X$, if $t = t(x, y) > 0$ such that $N'(x, y, s) = 1$, for all $s \geq t$.

**Example 2.8.** Let $X = \mathbb{R}^3$ be a linear space over $R$. Define $\|\cdot, \cdot\| : X \times X \times R \rightarrow [0, 1]$ by

$$\|x, y\| = \max\{|x_1 y_2 - x_2 y_1|, |x_2 y_3 - x_3 y_2|, |x_3 y_1 - x_1 y_3|\},$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(y_1, y_2, y_3) \in \mathbb{R}^3$ then $(X, \|\cdot, \cdot\|)$ is a 2-normed linear space.

Define $N : X \times X \times R \rightarrow [0, 1]$ by

$$N(x, y, t) = 1, \text{ if } t > \|x, y\|$$

$$= 0.5, \text{ if } \frac{1}{2} \|x, y\| < t \leq \|x, y\|$$

$$= 0, \text{ if } t \leq \frac{1}{2} \|x, y\|$$

Then $(X, N)$ is a fuzzy 2-normed linear space. Define $\|x, y\|_{\alpha} = \inf \{t : N(x, y, t) \geq \alpha\}, \alpha \in (0, 1)$. Then $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on a linear space $X$. The $\alpha$-2-norms corresponding to fuzzy 2-norm on $X$ are given by

$$\|x, y\|_\alpha = \|x, y\| \text{ if } 1 > \alpha > 0.5$$

$$= \frac{1}{2} \|x, y\| \text{ if } 0 < \alpha \leq 0.5$$

If the index set $(0, 1)$ of the family of crisp 2-norms $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ of Theorem 2.6 is extended to $(0, 1]$ then a fuzzy 2-norm $N$ is generated, satisfying an additional property that $N(x, y, t)$ attains the value 1 at some finite value $t$.

Let $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on linear space $X$. Define a function $N' : X \times X \times R \rightarrow [0, 1]$ as $N'(x, y, t) = \sup[\alpha \in (0, 1) : \|x, y\|_\alpha \leq t], \text{ when } (x, y, t) \neq 0,$

$$= 0, \text{ when } (x, y, t) = 0.$$

Then $N'$ is a fuzzy 2-norm on $X$.

**Definition 2.9.** Let $(X, N)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $X$ then $x_n$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, y, t) = 1$, for all $t > 0$.

**Definition 2.10.** Let $(X, N)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $X$ then $x_n$ is said to be a Cauchy sequence if $\lim_{n \to \infty} N(x_{n+p} - x_n, y, t) = 1$, for all $t > 0$ and $p = 1, 2, 3, \ldots$.

**Definition 2.11.** A subset $B$ of a fuzzy 2-normed linear space $(X, N)$ is said to be bounded if and only if there exists $t > 0$ and $0 < r < 1$ such that $N(x, y, t) > 1 - r$ for all $x, y \in B$.

**Definition 2.12.** A subset $B$ of a fuzzy 2-normed linear space $(X, N)$ is said to be compact if any sequence $\{x_n\}$ in $B$ has a subsequence converging to an element of $B$.

3. Fuzzy Anti-2-Norms on a Linear Space

In this section, we introduce the notion of fuzzy anti-2-normed linear space and investigate their important properties.

**Definition 3.1.** Let $U$ be a linear space over a real field $F$. A fuzzy subset $N^*$ of $U \times U \times R$ such that for all $x, y, u \in U$

$(2 - N^*1)$: For all $t \in R$ with $t \leq 0$, $N^*(x, y, t) = 1,$

$(2 - N^*2)$: For all $t \in R$ with $t > 0$, $N^*(x, y, t) = 0$ if and only if $x, y$ are linearly dependent

$(2 - N^*3)$: $N^*(x, y, t)$ is invariant under any permutation of $x, y$

$(2 - N^*4)$: For all $t \in R$ with $t > 0$, $N^*(x, cy, t) = N^*(x, y, \frac{1}{|c|}t)$ if $c \neq 0, c \in F$
Thus \( N^* \) is said to be a fuzzy anti-2-norm on a linear space \( U \) and the pair \((U, N^*)\) is called a fuzzy anti-2-normed linear space (briefly Fa-2-NLS).

The following condition of fuzzy anti-2-norm \( N^* \) will be required later on.

(2 \( \rightarrow \) \( N^* \)): For all \( t \in R \) with \( t > 0 \), \( N^*(x, y, t) < 1 \) implies that \( x, y \) are linearly dependent.

**Example 3.2.** Let \((U, \| \cdot \|)\) be a 2-normed linear space. Define

\[
N^*(x, y, t) = \frac{\|x, y\|}{t + \|x, y\|}, \quad \text{when } t > 0, \, t \in R, \, x, y \in U
\]

\[
= 1, \quad \text{when } t \leq 0, \, t \in R, \, x, y \in U.
\]

Then \((U, N^*)\) is an Fa-2-NLS.

**Proof.** Now we have to show that \( N^*(x, y, t) \) is a fuzzy anti-2-norm in \( U \).

(2 \( \rightarrow \) \( N^* \)): For all \( t \in R \) with \( t \leq 0 \), we have by definition \( N^*(x, y, t) = 1 \).

(2 \( \rightarrow \) \( N^* \)): For all \( t \in R \) with \( t > 0 \),

\[
N^*(x, y, t) = 0 \Leftrightarrow \frac{\|x, y\|}{t + \|x, y\|} = 0 \Leftrightarrow \|x, y\| = 0 \Leftrightarrow x, y \text{ are linearly dependent}
\]

(2 \( \rightarrow \) \( N^* \)): As \( \|x, y\| \) is invariant under any permutation of \( x, y \), it follows that \( N^*(x, y, t) \) is invariant under any permutation of \( x, y \).

(2 \( \rightarrow \) \( N^* \)): For all \( t \in R \) with \( t > 0 \) and \( c \neq 0, \, c \in F \), we get

\[
N^*(x, cy, t) = \frac{\|x, cy\|}{t + \|x, cy\|} = \frac{|c| \|x, y\|}{t + |c| \|x, y\|} = \frac{\|x, y\|}{t + \|x, y\|} = N^*(x, y, t/c).
\]

(2 \( \rightarrow \) \( N^* \)): For all \( s, t \in R \) and \( x, y, u \in U \). We have to show that \( N^*(x, y + u, s + t) \leq \max\{N^*(x, y, s), N^*(x, u, t)\} \). If (a) \( s + t < 0 \) (b) \( s = t = 0 \) (c) \( s + t > 0 \); \( s > 0, \, t < 0; \, s < 0, \, t > 0 \), then in these cases the relation is obvious. If (d) \( s > 0, \, t > 0, \, s + t > 0 \), then assume that

\[
N^*(x, y, s) \leq N^*(x, u, t) \Rightarrow \frac{\|x, y\|}{s + \|x, y\|} \leq \frac{\|x, u\|}{s + \|x, u\|} \Rightarrow \|x, y\|(s + \|x, y\|) \leq \|x, u\|(s + \|x, y\|)
\]

\[
\Rightarrow t\|x, y\| \leq s\|x, u\|\]  \hspace{1cm} (1)

Now

\[
\frac{\|x, y + u\|}{s + t + \|x, y + u\|} - \frac{\|x, u\|}{t + \|x, u\|} \leq \frac{\|x, y\| + \|x, u\|}{s + t + \|x, y\| + \|x, u\|} - \frac{\|x, u\|}{t + \|x, u\|} = \frac{t\|x, y\| - s\|x, u\|}{(s + t + \|x, y\| + \|x, u\|)(t + \|x, u\|)}.
\]

By using equation (1), we get

\[
\frac{\|x, y + u\|}{s + t + \|x, y + u\|} \leq \frac{\|x, u\|}{t + \|x, u\|} \leq \frac{\|x, y\|}{s + \|x, y\|}.
\]

Hence \( N^*(x, y + u, s + t) \leq \max\{N^*(x, y, s), N^*(x, u, t)\} \).

(2 \( \rightarrow \) \( N^* \)): If \( t_1 < t_2 \leq 0 \), then we have \( N^*(x, y, t_1) = N^*(x, y, t_2) = 1 \). If \( 0 < t_1 < t_2 \), then

\[
\frac{\|x, y\|}{t_1 + \|x, y\|} - \frac{\|x, y\|}{t_2 + \|x, y\|} = \frac{\|x, y\|(t_2 - t_1)}{(t_1 + \|x, y\|)(t_2 + \|x, y\|)} > 0 \Rightarrow N^*(x, y, t_1) \geq N^*(x, y, t_2).
\]

Thus \( N^*(x, y, t) \) is a non-increasing function of \( t \in R \). Again

\[
\lim_{t \to -\infty} N^*(x, y, t) = \lim_{t \to -\infty} \frac{\|x, y\|}{t + \|x, y\|} = 0, \text{ for all } x, y \in U. \text{ Hence } (U, N^*) \text{ is an Fa-2-NLS.}
\]

**Example 3.3.** Let \((U, \| \cdot \|)\) be a 2-normed linear space. Define \( N^* : U \times U \times R \to [0, 1] \) by

\[
N^*(x, y, t) = 0, \text{ when } t > \|x, y\|, \, t \in R, \, x, y \in U
\]

\[
= 1, \text{ when } t \leq \|x, y\|, \, t \in R, \, x, y \in U.
\]

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Then \((U, N^*)\) is an Fa-2-NLS.

**Proof.** It can be easily verified that \((U, N^*)\) is an Fa-2-NLS.

**Remark 3.4.** \(N^*\) is a fuzzy anti-2-norm on \(U \ni 1 - N^*\) is a fuzzy 2-norm on \(U\).

**Lemma 3.5.** Let \((U, N^*)\) be an Fa-2-NLS. Then \(N^*(x, y - u, t) = N^*(x, u - y, t)\) for all \(x, y, u \in U\) and \(t \in (0, \infty)\).

**Proof.** For \(x, y, u \in U\) and \(t \in (0, \infty)\), \(N^*(x, y - u, t) = N^*(x, -(u - y), t) = N^*(x, u - y, t)\).

**Definition 3.6.** Let \(N^*\) be a fuzzy anti-2-norm on \(U\) satisfying \((2 - N^*)\). Define \(\|x, y\|_{N^*} = \inf\{t > 0 : N^*(x, y, t) < \alpha, \alpha \in (0, 1]\}\).

**Lemma 3.7.** Let \((U, N^*)\) be a Fa-2-NLS. For each \(\alpha \in (0, 1]\) and \(x, y, u \in U\) then we have

(i) \(\|x, y\|_{N^*} \geq \|x, y\|_{c, 0}^*\) for \(0 < \alpha_1 < \alpha_2 \leq 1\).

(ii) \(\|x, cy\|_{N^*} = |c| \cdot \|x, y\|_{N^*}\) for any scalar \(c\).

(iii) \(\|x, y + u\|_{N^*} \leq \|x, y\|_{N^*} + \|x, u\|_{N^*}\).

**Proof.** (i) For \(0 < \alpha_1 < \alpha_2 \leq 1\), we note that

\[ \inf\{t > 0 : N^*(x, y, t) < \alpha_1\} \geq \inf\{t > 0 : N^*(x, y, t) < \alpha_2\} \Rightarrow \|x, y\|_{\alpha_2}^* \geq \|x, y\|_{\alpha_1}^* \]

(ii) For any scalar \(c\) and for all \(\alpha \in (0, 1]\),

\[ \|x, cy\|_{N^*} = \inf\{t > 0 : N^*(x, cy, t) < \alpha, \alpha \in (0, 1]\} = \inf\{t > 0 : N^*(x, y, \frac{t}{|c|}) < \alpha, \alpha \in (0, 1]\} = |c| \cdot \|x, y\|_{N^*}\]

(iii) For any \(\alpha \in (0, 1]\),

\[ \|x, y\|_{N^*} + \|x, u\|_{N^*} = \inf\{t > 0 : N^*(x, y, t) < \alpha\} + \inf\{s > 0 : N^*(x, u, s) < \alpha\} \]

\[ \geq \inf\{t + s > 0 : N^*(x, y, t) < \alpha, N^*(x, u, s) < \alpha\} = \|x, y + u\|_{N^*}\]

**Theorem 3.8.** Let \((U, N^*)\) be a Fa-2-NLS. Then \(\|\alpha \cdot x, y\|_{\alpha}^* : \alpha \in (0, 1]\) is a decreasing family of 2-norms on a linear space \(U\).

**Proof.** By lemma 3.7 it can be easily verified.

**Theorem 3.9.** Let \(\|\alpha \cdot x, y\|_{\alpha}^* : \alpha \in (0, 1]\) be a decreasing family of 2-norms on a linear space \(U\). Now define a function \(N^*_1 : U \times U \times R \rightarrow [0, 1]\) as

\[ N^*_1(x, y, t) = \begin{cases} \inf\{\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq t\}, & \text{when } (x, y, t) \neq 0, \\ 1, & \text{when } (x, y, t) = 0. \end{cases} \]

Then (a) \(N^*_1\) is a fuzzy anti-2-norm on \(U\).

(b) For each \(x, y \in U\), \(\exists r = r(x, y) > 0\) such that \(N^*_1(x, y, t) = 1\).

**Proof.** (a) Now we have to show that \(N^*_1\) is a fuzzy anti-2-norm on \(U\).

\((2 - N^*1)\): (i) For all \(t \in R\) with \(t < 0\), \([\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq t\} = \Phi\), \(\forall x, y \in U\), we have

\[ N^*_1(x, y, t) = \inf\{\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq t\} = 1. \]

(ii) For \(t = 0\) and \(x \neq 0, y \neq 0\), \([\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq 0\} = \Phi\), \(\forall x, y \in U\), we have \(N^*_1(x, y, t) = 1\).

(iii) For \(t = 0\) and \(x \neq 0, y \neq 0\), then from the definition \(N^*_1(x, y, t) = 1\).

Thus for all \(t \in R\) with \(t \leq 0\), \(N^*_1(x, y, t) = 1, \forall x, y \in U\).

\((2 - N^*2)\): For all \(t \in R\) with \(t > 0\), \(N^*_1(x, y, t) = 0\). Choose any \(\epsilon \in (0, 1]\). Then for any \(t > 0\), \(\exists \alpha_1 \in (\epsilon, 1]\) such that \(\|x, y\|_{\alpha_1}^{\epsilon} \leq \epsilon\) and \(\|x, y\|_{\alpha_1}^{\epsilon} \leq t\). Since \(t > 0\) is arbitrary, this implies that \(\|x, y\|_{\alpha_1}^{\epsilon} = 0\) then \(x, y\) are linearly dependent.

If \(x, y\) are linearly dependent then for \(t > 0\), \(N^*_1(x, y, t) = \inf\{\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq t\} = 0\). Thus for all \(t \in R\) with \(t > 0\), \(N^*_1(x, y, t) = 0\) if and only if \(x, y\) are linearly dependent.

\((2 - N^*3)\): As \(\|x, y\|_{\alpha}^*\) is invariant under any permutation of \(x, y\), it follows that \(N^*_1(x, y, t)\) is invariant under any permutation of \(x, y\).

\((2 - N^*4)\): For all \(t \in R\) with \(t > 0\), and \(c \neq 0, c \in F\), we have \(N^*_1(x, cy, t) = \inf\{\alpha \in (0, 1] : \|x, cy\|_{\alpha}^* \leq t\} = \inf\{\alpha \in (0, 1] : |c| \cdot \|x, y\|_{\alpha}^* \leq t\} = \inf\{\alpha \in (0, 1] : \|x, y\|_{\alpha}^* \leq \frac{t}{|c|}\} \forall x, y \in U\).

\((2 - N^*5)\): We have to show that \(\forall s, t \in R\) and \(\forall x, y, u \in U\), \(N^*_1(x, y + u, s + t) \leq \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}\).

Suppose that \(\forall s, t \in R\) and \(\forall x, y, u \in U\), \(N^*_1(x, y + u, s + t) > k > \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}\). Choose \(k\) such that \(N^*_1(x, y + u, s + t) > k > \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}\).
Now $N'_1(x,y+u,s+t) > k \Rightarrow \inf\{\alpha \in (0,1] : \|x,y+u\|_{1}^\alpha + s+t) > k \Rightarrow \|x,y+u\|_{1}^{\alpha} + s+t > s+t. Again k > \max[N'_1(x,y,s),N'_1(x,u,t)] \Rightarrow k > N'_1(x,y,s) \text{ and } k > N'_1(x,u,t) \Rightarrow \|x,y\|_1^\alpha \leq s \text{ and } \|x,u\|_1^\alpha \leq t \Rightarrow \|x,y\|_1^\alpha + \|x,u\|_1^\alpha \leq s + t. Thus s + t < \|x,y\|_1^\alpha + \|x,u\|_1^\alpha \leq s + t, which is a contradiction. Hence $N'_1(x,y+u,s+t) \leq \max[N'_1(x,y,s),N'_1(x,u,t)].$

$(2 - N^6)$: Let $x, y \in U, \alpha \in (0,1)$. Now $t > \|x,y\|_{1}^\alpha \Rightarrow N'_1(x,y,t) = \inf\{\beta \in (0,1] : \|x,y\|_{1}^\beta \leq t \} \leq \alpha$. So $\lim_{t \rightarrow \infty} N'_1(x,y,t)$ is

Next we verify that $N'_1(x,y,t)$ is a non-increasing function of $t \in R$. If $t_1 < t_2 \leq 0$, then $N'_1(x,y,t_1) = N'_1(x,y,t_2) = 1, \forall x, y \in U$. If $0 < t_1 < t_2$ then $[\{x,y\}]^\alpha \times \{1 \} \subseteq [\{0 \}]^{\alpha} \times \{1 \} \Rightarrow \inf\{\alpha \in (0,1] : \|x,y\|_1^\alpha \leq t \} \geq \inf\{\alpha \in (0,1] : \|x,y\|_1^\alpha \leq t \} \geq N'_1(x,y,t_1) \geq N'_1(x,y,t_2). Thus N'_1(x,y,t) is a non-increasing function of $t \in R$ and $N'_1$ is a fuzzy anti-2-norm on $U$.

(b) For each $x \neq 0, y \neq 0, \|x,y\|_{1}^\alpha > 0$. Thus $\exists r = r(x,y) > 0$ such that $\|x,y\|_{1}^\alpha > r(x,y), \forall \alpha \in (0,1] \Rightarrow \inf\{\alpha \in (0,1] : \|x,y\|_1^\alpha \leq t \} = 1 \Rightarrow N'_1(x,y,t) = 1.$

**Definition 3.10.** Let $(U, N^*)$ be a Fa-2-NLS. A sequence $\{x_n\}$ in $U$ is said to be convergent to $x \in U$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N'_{\infty}(x_n-x,y,t) < r$, for all $n \geq n_0$.

**Theorem 3.11.** In a Fa-2-NLS $(U, N^*)$, a sequence $\{x_n\}$ converges to $x \in U$ if and only if $\lim_{n \rightarrow \infty} N'_{\infty}(x_n-x,y,t) = 0, \forall t > 0$.

**Proof.** Fix $t > 0$. Suppose $\{x_n\}$ converges to $x \in U$. Then for a given $r, 0 < r < 1$ there exists an integer $n_0 \in N$ such that $N'_{\infty}(x_n-x,y,t) < r$, for all $n \geq n_0$, and hence $N'_{\infty}(x_n-x,y,t) \rightarrow 0$, as $n \rightarrow \infty$. Conversely, if for each $t > 0$, $N'_{\infty}(x_n-x,y,t) \rightarrow 0$, as $n \rightarrow \infty$, then for every $r, 0 < r < 1$, there exists an integer $n_0$ such that $N'_{\infty}(x_n-x,y,t) < r$, for all $n \geq n_0$. Hence $\{x_n\}$ converges to $x$ in $U$.

**Definition 3.12.** Let $(U, N^*)$ be a Fa-2-NLS. A sequence $\{x_n\}$ in $U$ is said to be a Cauchy sequence if $\lim_{n \rightarrow \infty} N'_{\infty}(x_{n+p}-x_n,y,t) = 0$, for all $n \geq n_0, p = 1,2,3,\ldots$.

**Theorem 3.13.** In a Fa-2-NLS $(U, N^*)$, a sequence $\{x_n\}$ is a Cauchy sequence in $U$ if and only if $\lim_{n \rightarrow \infty} N'_{\infty}(x_{n+p}-x_n,y,t) = 0$, for $p = 1,2,3,\ldots$ and $t > 0$.

**Proof.** Fix $t > 0$. Suppose $\{x_n\}$ is a Cauchy sequence in $U$. Then for a given $r, 0 < r < 1$ and $p = 1,2,3,\ldots$ there exists an integer $n_0 \in N$ such that $N'_{\infty}(x_{n+p}-x_n,y,t) < r$, for all $n \geq n_0$, and hence $N'_{\infty}(x_{n+p}-x_n,y,t) \rightarrow 0$, as $n \rightarrow \infty$. Conversely, if for each $t > 0$, and $p = 1,2,3,\ldots, N'_{\infty}(x_{n+p}-x_n,y,t) \rightarrow 0$, as $n \rightarrow \infty$, then for every $r, 0 < r < 1$, there exists an integer $n_0$ such that $N'_{\infty}(x_{n+p}-x_n,y,t) < r$, for all $n \geq n_0$. Hence $\{x_n\}$ is a Cauchy sequence in $U$.

**Theorem 3.14.** If a sequence $\{x_n\}$ in a Fa-2-NLS $(U, N^*)$ is convergent then its limit is unique.

**Proof.** Let $\{x_n\}$ converges to $x$ and $z$. Also let $s, t \in R^+$ then $\lim_{n \rightarrow \infty} N'_{\infty}(x_n-x,y,t) = 0$ and $\lim_{n \rightarrow \infty} N'_{\infty}(x_n-z,y,s) = 0$. Now $N'_{\infty}(x-z,y,t+s) = N'_{\infty}(x-x_n+x_n-z,y,t+s) \leq \max[N'_{\infty}(x-x_n,y,t),N'_{\infty}(x_n-z,y,s)] = \max[N'_{\infty}(x-x_n,y,t),N'_{\infty}(x_n-z,y,s)].$

Taking limit, we have $N'_{\infty}(x-z,y,t+s) \leq \max[\lim_{n \rightarrow \infty} N'_{\infty}(x_n-x,y,t), \lim_{n \rightarrow \infty} N'_{\infty}(x_n-z,y,s)] \Rightarrow N'_{\infty}(x-z,y,t+s) = 0, \forall s, t \in R^+ \Rightarrow x-z = 0 \Rightarrow x = z.$

**Theorem 3.15.** In a Fa-2-NLS $(U, N^*)$, every subsequence of a convergent sequence converges to the limit of a sequence.

**Proof.** The proof is obvious.

**Theorem 3.16.** Let $L$ be a linear space, $N^*$ be a fuzzy anti-2-norm on $L$ and $\widehat{N} = (1 - N^*)$ be a fuzzy 2-norm on $L$. Then (a) $D_n$ is a convergent sequence in $(L, N^*)$ if and only if $\{x_n\}$ is a convergent sequence in $(L, \widehat{N})$.

(b) $\{x_n\}$ is a Cauchy sequence in $(L, N^*)$ if and only if $\{x_n\}$ is a Cauchy sequence in $(L, \widehat{N})$.

**Proof.** (a) Let $\{x_n\}$ be a convergent sequence in $(L, N^*) \Leftrightarrow \lim_{n \rightarrow \infty} N^*(x_n-x,y,t) = 0, \forall t > 0 \Leftrightarrow \lim_{n \rightarrow \infty} \widehat{N}(x_n-x,y,t) = 1$ for all $t > 0 \Leftrightarrow \{x_n\}$ is a convergent sequence in $(L, \widehat{N})$.

(b) Let $\{x_n\}$ be a Cauchy sequence in $(L, N^*) \Leftrightarrow \lim_{n \rightarrow \infty} N^*(x_{n+p}-x_n,y,t) = 0, p = 1,2,3,\ldots$ for all $t > 0 \Leftrightarrow \lim_{n \rightarrow \infty} \widehat{N}(x_{n+p}-x_n,y,t) = 1, p = 1,2,3,\ldots$, for all $t > 0 \Leftrightarrow \{x_n\}$ is a Cauchy sequence in $(L, \widehat{N})$.

**Theorem 3.17.** In a Fa-2-NLS $(U, N^*)$, every convergent sequence is a Cauchy sequence.

**Proof.** Let $\{x_n\}$ be a convergent sequence in a Fa-2-NLS $(U, N^*)$ then $\lim_{n \rightarrow \infty} N^*(x_n-x,y,t) = 0$, for all $t > 0$. Let $s, t \in R^+$ and $p = 1,2,3,\ldots$, we have $N^*(x_{n+p}-x_n,y,s+t) = N^*(x_{n+p}-x-x_n,y,s+t) \leq \max[N^*(x_{n+p}-x,y,s), N^*(x-x_n,y,t)] = \max[N^*(x_{n+p}-x,y,s), N^*(x-x_n,y,t)].$

Taking limit, we have $\lim_{n \rightarrow \infty} N^*(x_{n+p}-x_n,y,s+t) \leq \max[\lim_{n \rightarrow \infty} N^*(x_{n+p}-x_n,y,s), \lim_{n \rightarrow \infty} N^*(x-x_n,y,t)] = 0$.
exists a sequence \( \{x_n\} \) be a Cauchy sequence in \( (X, \|\cdot\|) \)

\[
 N^*(x, y, t) = \frac{\|x, y\|}{t + \|x, y\|} \quad \text{when } t > 0, \ x, y \in X \\
 = 1, \quad \text{when } t \leq 0, \ x, y \in X.
\]

Then \( (X, N^*) \) is an \( F_2 \)-NLS. Let \( \{x_n\} \) be a sequence in \( X \), then
(a) \( \{x_n\} \) is a Cauchy sequence in \( (X, \|\cdot\|) \) if and only if \( \{x_n\} \) is a Cauchy sequence in \( (X, N^*) \).
(b) \( \{x_n\} \) is a convergent sequence in \( (X, \|\cdot\|) \) if and only if \( \{x_n\} \) is a convergent sequence in \( (X, N^*) \).

**Proof.** (a) Let \( \{x_n\} \) be a Cauchy sequence in \( (X, \|\cdot\|) \)

\[
 \lim_{n \to \infty} \{x_{n+p} - x_n, y, t\} = 0, \quad \forall \ a, t \in R^+ \quad \text{and } \quad p = 1, 2, 3, \ldots
\]

\[
 N^*(x, y, t) = \lim_{n \to \infty} \frac{\|x_n - y\|}{t + \|x_n - y\|} = 0, \quad \forall \ a, t \in R^+ \quad \text{and } \quad p = 1, 2, 3, \ldots
\]

(b) Let \( \{x_n\} \) be a convergent sequence in \( (X, \|\cdot\|) \)

\[
 \lim_{n \to \infty} N^*(x_n - x, y, t) = \lim_{n \to \infty} \frac{\|x_n - x\|}{t + \|x_n - x\|} = 0, \quad \forall \ a, t \in R^+ \quad \text{and } \quad p = 1, 2, 3, \ldots
\]

**Remark 3.19.** If there exist a 2-normed linear space \( (X, \|\cdot\|_0) \) which is not complete, then the fuzzy anti-2-norm induced by such a crisp 2-norm \( \|\cdot\|_0 \) on an incomplete linear space \( X \), is an incomplete \( F_2 \)-NLS.

**Definition 3.20.** Let \( (U, N^*) \) be a \( F_2 \)-NLS. A subset \( B \) of \( U \) is said to be closed if for any sequence \( \{x_n\} \) in \( B \) converges to \( x \in B \), that is \( \lim_{n \to \infty} N^*(x_n - x, y, t) = 0, \forall \ t \in R^+ \) implies that \( x \in B \).

**Definition 3.21.** Let \( (U, N^*) \) be a \( F_2 \)-NLS. A subset \( W \) of \( U \) is said to be the closure of \( B \subseteq W \) if for any \( w \in W \), there exists a sequence \( \{x_n\} \) in \( B \) such that \( \lim_{n \to \infty} N^*(x_n - x, y, t) = 0, \forall \ t \in R^+ \), we denote the set \( W \) by \( \bar{B} \).

**Definition 3.22.** A subset \( B \) of a \( F_2 \)-NLS \( (U, N^*) \) is said to be bounded if and only if there exists \( t > 0 \) and \( 0 < r < 1 \) such that \( N^*(x, y, t) < r, \forall \ x, y \in B \).

**Definition 3.23.** A subset \( B \) of a \( F_2 \)-NLS \( (U, N^*) \) is said to be compact if any sequence \( \{x_n\} \) in \( B \) has a subsequence converging to an element of \( B \).

**Theorem 3.24.** Let \( (U, N^*) \) be a \( F_2 \)-NLS then every Cauchy sequence in \( (U, N^*) \) is bounded.

**Proof.** Let \( \{x_n\} \) be a Cauchy sequence in \( (U, N^*) \). Then \( \lim_{n \to \infty} N^*(x_{n+p} - x_n, y, t) = 0, \quad \forall \ p = 1, 2, 3, \ldots \)

Choose a fixed \( a_0, 0 < a_0 < 1 \). Then we have \( \lim_{n \to \infty} N^*(x_n - x_{n+p}, y, t) = 0 < 1 - a_0, \forall \ t \in R^+ \quad \text{and } \quad p = 1, 2, 3, \ldots \) \( \Rightarrow \) For \( t' > 0 \), \( \exists \ n_0' = n_0'(t') \) such that \( \lim_{n \to \infty} N^*(x_n - x_{n+p}, y, t') = 1 - a_0, \forall \ n \geq n_0', \ t > 0, \ p = 1, 2, 3, \ldots \) Since \( \lim_{n \to \infty} N^*(x, y, t) = 0 \), we have for each \( x_n, \exists \ t'' > 0 \) such that \( N^*(x_n, y, t'') < 1 - a_0, \forall \ t' > t'' \), \( t = 1, 2, 3, \ldots \). Let \( t_0'' = t' + \max(t_1'', t_2'', \ldots, t_n'') \). Then \( N^*(x_n, y, t_0'') \leq N^*(x_n, t', t_0'') = N^*(x_n - x_{n+p} + x_{n+p}, y, t' + t_0'') \leq \max(N^*(x_n - x_{n+p}, y, t'), N^*(x_{n+p}, y, t_0'')) = (1 - a_0), \forall \ n \geq n_0'' \), i.e., \( N^*(x_n, y, t_0'') \leq (1 - a_0), \forall \ n \geq n_0'' \). Therefore \( \{x_n\} \) is bounded in \( (U, N^*) \).

**Conclusion**
One can introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-n-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-n-normed linear space.

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**References**


