Solution of Time-Independent Schrödinger Equation for a Two-Dimensional Quantum Harmonic Oscillator Using He's Homotopy Perturbation Method

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Abstract

In this paper, time-independent Schrödinger equation for a charged particle, in the presence of electric potential and vector potential, has been solved using He's Homotopy Perturbation Method (HPM). HPM is one of the newest analytical methods to solve linear and nonlinear differential equations. In contrast to the traditional perturbation methods, the Homotopy method does not require a small parameter in the equation. In this method, according to the homotopy technique, a Homotopy with an embedding parameter $\delta \in [0, 1]$ is constructed, and the embedding parameter is considered as a small parameter. Using cylindrical coordinates, it has been found that the z-equation of the charged particle is a one-dimensional harmonic oscillator and the r equation is actually a two-dimensional harmonic oscillator. The obtained results show the evidence of simplicity, usefulness, and effectiveness of the HPM for obtaining approximate analytical solutions for the time-independent Schrödinger equation for a charged particle in parallel electric and magnetic fields.

Keywords: Homotopy Perturbation Method, Scalar Potential, Vector Potential, Two-Dimensional Harmonic Oscillator, Bessel functions.

1. Introduction

The dynamics of charged particles in electric and magnetic fields is of both academic and practical interest in physics and engineering. The areas where this problem finds applications include the development of cyclotron accelerators, free electron lasers, plasma physics, cathode-ray, and X-ray tubes. Classically, a charged particle in a time-independent homogeneous magnetic field executes a circular motion in the plane perpendicular to the direction of the field. The period of this motion is the inverse of the cyclotron frequency $\omega_c = qB/m$, where q is the charge of the particle, m is the mass of the particle, and B is the strength of the magnetic field. A charged quantum particle in a time-independent homogeneous magnetic field also executes this circular motion (Dirac, 1958). In addition, the probability distribution oscillates harmonically with time. Jesus et al (1999) have studied the classical and the quantum dynamics of a charged particle in oscillating magnetic and electric fields, which are related through the Faraday law. The equations of motion show two resonance frequencies, one at the Larmor frequency (ω_L) and another at the cyclotron frequency (ω_c). When the field frequency equals ω_c, the particle is confined to a simple closed trajectory, but when $\omega_L = \omega_c$, it drifts away, the same happening to off resonance particles whose frequencies are very close to ω_c. In addition, the particle eigenstates and eigenvalues are calculated.

The harmonic oscillator (HO) is one of the most discussed problems in physics. There is a large number of quantum systems which can be approximated, at least in the limit of small amplitudes, by the HO equations. On the other hand, there are "quasi-classical" states for the quantum HO (coherent states) which illustrates the relation between quantum and classical mechanics when limit $\hbar \to 0$ is studied (semi-classical limit). Thus, the separability of the HO problem in different coordinate systems as well as the corresponding eigenstates and eigenvalues are point of considerable interest (Fendrik and Bernath, 1989). Apart from being one of the few exactly solvable quantum mechanical problems, the HO physical relevance reaches far beyond the most obvious interpretation of the oscillator as an analogue of the classical spring force problem. It can be applied rather directly to the explanation of the vibration spectra of diatomic molecules (Robert and Thomas, 1985). Furthermore, it is the foundation for the understanding of complex modes of vibration in larger molecules, the motion of atoms in a solid lattice, the theory of heat capacity, and the electromagnetic fields. Any potential, of arbitrarily complicated form, which possesses a minimum or equilibrium, can, to lowest non-trivial order, be treated as a harmonic oscillator. Higher order terms in the expansion of the physical potential can then be added as perturbations (Shankar, 1994).

Ciann-Dong (2006) has demonstrated how quantum harmonic oscillator can be analyzed classically in complex domain. It has been found that the motion in abstract eigenstate is characterized by eigen-trajectories constructed from the eigenfunctions of Schrödinger equation, along which a particle's position, velocity, and acceleration in the related eigenstate can be identified. Granados and Aquino (1999) have studied the correspondence between the states of a two-dimensional isotropic harmonic oscillator and the states of a Morse oscillator (MO). They proved that the states of a well of the MO are mapped in a degenerated multiplet of the harmonic oscillator. Gao-Feng et al (2008) have studied the isotropic charged HO in uniform magnetic field in the non-commutative phase space. The corresponding exact energy is obtained, and the analytic eigenstates is presented in terms of the confluent hypergeometric function. It has been also shown that in the non-commutative space, the isotropic charged HO in uniform magnetic field has the similar behaviors to the Landau problem. Fendrik and Bernath (1989) has solved the Schrödinger equation for the two-dimensional simple harmonic oscillator using elliptic coordinates where it is separable. It has been shown that the separability of the HO problem in such coordinates is independent of the selection of the focal distance.

In this paper, the eigenstates and the corresponding eigenvalues of a charged particle, in the presence of electric potential and vector potential, are obtained by solving the time-independent Schrödinger equation in cylindrical coordinates. The separability of Schrödinger equation in cylindrical coordinates in turns leads to one-dimensional and two-dimensional HO problems. The solution of the One-dimensional HO can be read off directly in terms of Hermit polynomials. On the other hand, the solution of the two-dimensional HO will be carried out using He's Homotopy Perturbation Method (HPM).

HPM was proposed first by He (1999). The HPM is designed for solving differential and integral equations, linear and nonlinear, and has been the subject of extensive analytical and numerical studies. The method, which is a coupling of a homotopy technique and a perturbation technique, deforms continuously to a simple problem which is easily solved. This method, which does not require a small parameter in an equation, in contrast to the traditional perturbation methods, has a significant advantage in that it provides an analytical approximate solution to a wide range of linear and nonlinear problems in applied sciences. This method doesn't need linearization, perturbation or un-justified assumptions. The HPM yields the solution in terms of a rapid convergent series with easily computable components (He, 2003).

In the last two decades with the rapid development of differential equations science, there has appeared ever-increasing interest in the analytical techniques for linear and nonlinear problems. The widely applied techniques are perturbation methods. Latif (2005) applied the HPM to search for exact analytical solutions of linear differential equations with constant coefficients. In addition, based on the precise integration method, a coupling technique of the variational iteration method (VIM) and HPM is proposed to solve nonlinear matrix differential equations. Rezania et al (2009) used HPM and VIM to solve the heat equations which are functions on time and space. This type of equation governs numerous scientific and engineering experimentations. Zhang et al (2006) obtained an explicit analytical solution for nonlinear Poisson-Boltzmann equation by the HPM. Wang et al (2007) applied HPM to solve reaction-diffusion equations which is governed by the nonlinear ordinary differential equation. Furthermore, HPM is also applied to solve the Helmholtz equation, and the results reveal that this method is very effective and simple (Bizzar et al, 2008). Recently, Mahasneh et al (2010) have solved the heat conduction equation for a homogenous solid metallic sphere using HPM.

HPM has been also used to formulate a new analytical solution for free-particle radial dependent Schrödinger equation (Mahasneh et al., 2010).

This paper has been divided into six parts. The first part discusses importance of HO in physics, and a brief overview of the recent studies of the HO problems and the wide use of HPM has also been discussed. For the general Knowledge, part two highlights on the quantum physics of one-dimensional and three-dimensional harmonic oscillators. Part three lays out the separability on time independent Schrödinger equation in cylindrical coordinates for a charged particle in the presence of electric potential and vector potential. The formulation of He's HPM is given in part four. In part five, the solution of two-dimensional harmonic oscillator using HPM will be solved and discussed in details. The main conclusions are summarized in the last part.

2. Physics of Quantum Harmonic Oscillators

In the one-dimensional HO problem, a particle of mass M is subject to a potential $V(z)$ given by:

$$V(z) = \frac{1}{2} M\omega^2 z^2 \tag{1}$$

where ω is the angular frequency of the oscillator. The Hamiltonian of the particle is

$$H = \left[\underbrace{\frac{-\hbar^2}{2\,M} \frac{d^2}{dz^2}}_{1} + \underbrace{\frac{1}{2} M\,\omega^2\,z^2}_{2} \right] \tag{2}$$

The first term in the Hamiltonian represents the kinetic energy of the particle, and the second term represents the potential energy in which it resides. The quantity $\hbar$ is Planck's reduced constant. In order to find the eigenvalues and the corresponding eigenstates, we must solve the time independent Schrödinger equation

$$\hat{H}\,\Psi = E\,\Psi \quad \Leftrightarrow \quad \frac{-\hbar^2}{2\,M} \frac{d^2\Psi(z)}{dz^2} + \frac{1}{2} M\,\omega^2\,z^2\,\Psi(z) \;=\; E\,\Psi(z) \tag{3}$$

The solution of this second order differential equation in the coordinate basis turns out that there is a family of solutions, which in the position basis given by (Griffiths, 1995)

$$\Psi_n(z) = \left[\sqrt{\frac{1}{2^n\,n!}} \left(\frac{M\omega}{\pi\,\hbar}\right)^{1/4} \times \exp\left[\frac{-M\omega\,z^2}{2\,\hbar}\right] \times H_n\left(\sqrt{\frac{M\omega}{\hbar}}\,z\right)\right], \quad n = 0, 1, 2, ... \tag{4}$$

The functions H_n are the Hermite polynomials, which are given by the generating function

$$H_n(z) \;=\; (-1)^n \,\exp(z^2) \times \frac{d^n}{d^n z}\left(\exp(-z^2)\right) \tag{5}$$

The corresponding eigenvalues of HO are labeled by a single quantum number n,

$$E_n \;=\; \left(n + \tfrac{1}{2}\right) \hbar\,\omega \tag{6}$$

The eigenvalues of the first six eigenstates are shown in Fig.(1). The energy spectrum shown in Fig.(1) is noteworthy for three reasons. Firstly, the energies are "quantized", and may only take the discrete half-integer multiples of $\hbar\,\omega$. This is a feature of many quantum mechanical systems. Secondly, the lowest achievable energy is not zero, but $\hbar\,\omega/2$, which is called the "ground state energy". In the ground state, according to quantum mechanics, an oscillator performs null oscillations and its average kinetic energy is positive. It is not obvious that this is significant, because normally the zero of energy is not a physically meaningful quantity, only differences in energies. The final reason is that the energy levels are equally spaced, unlike the Bohr model or the particle in a box (Schiff, 1968).

In three dimensional HO problem, the potential $V(x, y, z)$ is given by

$$V(x, y, z) \;=\; \tfrac{1}{2}\,M\,(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{7}$$

and the time-independent Schrödinger equation

$$\frac{-\hbar^2}{2\,M}\,\nabla^2\Psi(x, y, z) + \left(\tfrac{1}{2}\,M\,\omega_x^2 x^2 + \tfrac{1}{2}\,M\omega_y^2 y^2 + \tfrac{1}{2}\,M\omega_z^2 z^2\right)\Psi(x, y, z) \;=\; E\,\Psi(x, y, z) \tag{8}$$

Let $\Psi(x, y, z) = \; X(x)\; Y(y)\, Z(z)$ and substituting back in Eq. (8) to get

$$\begin{aligned} &\left(\underbrace{\frac{-\hbar^2}{2\,M}\,\frac{1}{X(x)}\,\frac{d^2X(x)}{dx^2} + \frac{1}{2}\,M\;\omega_x^2 x^2}_{1}\right) + \left(\underbrace{\frac{-\hbar^2}{2\,M}\,\frac{1}{Y(y)}\,\frac{d^2Y(y)}{dy^2} + \frac{1}{2}\,M\;\omega_y^2 y^2}_{2}\right) \\ &\quad + \left(\underbrace{\frac{-\hbar^2}{2\,M}\,\frac{1}{Z(z)}\,\frac{d^2Z(z)}{dz^2} + \frac{1}{2}\,M\;\omega_z^2 z^2}_{3}\right) = \; E_x + E_y + E_z = E \end{aligned} \tag{9}$$

The first term is a function of x only, the second term only of y, and the third only of z. so each is a constant (call constants E_x, E_y, E_z, with E_x+E_y+E_z=E). Therefore

$$\begin{array}{lr} \left(\frac{-\hbar^2}{2M}\frac{d^2X(x)}{dx^2}+\frac{1}{2}M\,\omega_x^2x^2X(x)\right)=E_xX(x) & (a)\\ \left(\frac{-\hbar^2}{2M}\frac{d^2Y(y)}{dy^2}+\frac{1}{2}M\omega_y^2y^2Y(y)\right)=E_yY(y) & (b)\\ \left(\frac{-\hbar^2}{2M}\frac{d^2Z(z)}{dz^2}+\frac{1}{2}M\omega_z^2z^2Z(z)\right)=E_zZ(z) & (c) \end{array} \tag{10}$$

Each of these is simply the one-dimensional HO. Therefore the eigenvalues are

$$\begin{array}{lr} E_x=\left(n_x+\frac{1}{2}\right)\hbar\,\omega_x & (a)\\ E_y=\left(n_y+\frac{1}{2}\right)\hbar\,\omega_y & (b)\\ E_z=\left(n_z+\frac{1}{2}\right)\hbar\omega_z & (c) \end{array} \tag{11}$$

So, the energy levels of the three-dimensional HO are denoted by

$$E\;=\;\left(n_x+\frac{1}{2}\right)\hbar\,\omega_x\;+\;\left(n_y+\frac{1}{2}\right)\hbar\,\omega_y\;+\left(n_z+\frac{1}{2}\right)\hbar\,\omega_z \tag{12}$$

with a non-negative integer $n=\left(n_x+n_y+n_z\right)$. The corresponding eigenstates are

$$\begin{array}{lr} X(x)=\left[\sqrt{\frac{1}{2^{n_x}\,n_x!}}\left(\frac{M\omega_x}{\pi\,\hbar}\right)^{1/4}\times\exp\left[\frac{-M\omega_x\,x^2}{2\,\hbar}\right]\times H_n\left(\sqrt{\frac{M\omega_x}{\hbar}}x\right)\right] & (a)\\ Y(z)=\left[\sqrt{\frac{1}{2^{n_y}\,n_y!}}\left(\frac{M\omega_y}{\pi\,\hbar}\right)^{1/4}\times\exp\left[\frac{-M\omega_y\,y^2}{2\,\hbar}\right]\times H_n\left(\sqrt{\frac{M\omega_y}{\hbar}}y\right)\right] & (b)\\ Z(z)=\left[\sqrt{\frac{1}{2^{n_z}\,n_z!}}\left(\frac{M\omega_z}{\pi\,\hbar}\right)^{1/4}\times\exp\left[\frac{-M\omega_z\,z^2}{2\,\hbar}\right]\times H_n\left(\sqrt{\frac{M\omega_z}{\hbar}}z\right)\right] & (c) \end{array} \tag{13}$$

Hence, the general solution $\Psi(x,y,z)=\;X(x)\;Y(y)\,Z(z)$ takes the form

$$\begin{aligned} \Psi(x,y,z)=\;&\sqrt{\frac{2^{-(n_x+n_y+n_z)}}{n_x!n_y!n_z!}}\left(\frac{M^3\omega_x\,\omega_y\,\omega_z}{\pi^3\,\hbar^3}\right)^{3/4}\times\;exp\left[\frac{-M\,(\omega_x\,x^2+\omega_y y^2+\omega_z z^2)}{2\,\hbar}\right]\\ &\times H_{n_x}\left(\sqrt{\frac{M\omega_x}{\hbar}}\,x\right)\times\,H_{n_y}\left(\sqrt{\frac{M\omega_y}{\hbar}}\,y\right)\times\,H_{n_z}\left(\sqrt{\frac{M\omega_z}{\hbar}}\,z\right) \end{aligned} \tag{14}$$

The wave functions of the quantum HO contain the Gaussian form which allows them to satisfy the necessary boundary conditions at infinity. However, as in the one-dimensional case, the energy is quantized. The ground state energy is three times the one-dimensional energy, as we would expect using the analogy to three independent one-dimensional oscillators (Eisberg and Rensik, 1985). There is one further difference: in the one-dimensional case, each energy level corresponds to a unique quantum state. In three dimensions, except for the ground state, the energy level are degenerate, meaning there are several states with the same energy. For isotropic harmonic oscillator where $\omega_x=\omega_y=\omega_z=\omega$, the eigenvalues of the HO are

$$E_n\;=\;\left(n+\tfrac{3}{2}\right)\hbar\,\omega,\quad n=n_x+n_y+n_z \tag{15}$$

It can be found in different quantum mechanics books that the degeneracy D(n) of the energy level E_n is (Cohen-Tannoudji, 1977)

$$D\,(n)=\frac{(n+1)\,(n+2)}{2} \tag{16}$$

3. Separability of Schrödinger Equation in Cylindrical Coordinates

In this article, we consider a particle of mass M and charge q that moves in a magnetic field **B** and an electric field **E**, both of which are independent of time. The Lagrangian of a non-relativistic particle in an electromagnetic field is (in SI Units) is

$$L\;=\;\sum_i\tfrac{1}{2}\,M\,\dot{x}_i^2\;+\;\sum_i q\,\dot{x}_i\,A_i-q\varphi \tag{17}$$

where φ is the electric scalar potential, and the A_i are the components of the magnetic vector potential. The generalized momenta can be derived by:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = M\dot{x}_i + qA_i \tag{18}$$

Rearranging, we may express the velocities in terms of the momenta, as:

$$\dot{x}_i = \frac{p_i - qA_i}{M} \tag{19}$$

If we substitute the definition of the momenta, and the definitions of the velocities in terms of the momenta, into the definition of the Hamiltonian we get (Flugge, 1971)

$$H = \left(\sum_i \dot{x}_i\, p_i\right) - L = \sum_i \frac{(p_i - qA_i)^2}{2M} + q\varphi \tag{20}$$

Rewriting Eq. (20) in terms of momentum operator gives

$$H = \frac{1}{2M}\,(-i\hbar\vec{\nabla} - q\vec{A}) \bullet (-i\hbar\vec{\nabla} - q\vec{A}) + q\varphi \tag{21}$$

The time independent Schrödinger equation for the charged particle becomes

$$\frac{1}{2M}\,(-i\hbar\vec{\nabla} - q\vec{A}) \bullet (-i\hbar\vec{\nabla} - q\vec{A})\,\Psi + q\varphi\,\Psi = E\,\Psi \tag{22}$$

or

$$-\frac{\hbar^2}{2M}\nabla^2\Psi + \frac{iq\hbar}{2M}\,(\vec{\nabla} \bullet (\vec{A}\Psi) + \vec{A} \bullet (\vec{\nabla}\Psi)\,) + \frac{q^2}{2M}A^2 + q\varphi\,\Psi = E\,\Psi \tag{23}$$

Using the identity

$$\vec{\nabla} \bullet (\vec{A}\Psi) = (\vec{\nabla} \bullet \vec{A})\Psi + \vec{A} \bullet (\vec{\nabla}\Psi) \tag{24}$$

In Eq. (23), we get

$$-\frac{\hbar^2}{2M}\nabla^2\Psi + \frac{iq\hbar}{2M}\,((\vec{\nabla} \bullet \vec{A})\Psi + 2\vec{A} \bullet (\vec{\nabla}\Psi)) + \frac{q^2}{2M}A^2 + q\varphi\,\Psi = E\,\Psi \tag{25}$$

Now, Let us consider the potentials

$$\varphi = \alpha\, z^2, \text{ and } \vec{A} = \frac{-B_o(y\,\hat{x} + x\,\hat{y})}{2} \tag{26}$$

where α is the stiffness constant, and B_o is a uniform magnetic field. Consequently, the time independent electric and magnetic fields are given respectively by

$$\vec{E} = -\vec{\nabla}\varphi = -2\,\alpha\, z\,\hat{z}, \quad \vec{B} = \vec{\nabla} \times \vec{A} = B_o\hat{z} \tag{27}$$

In addition,

$$\vec{\nabla} \bullet \vec{A} = 0, \quad \vec{A} \bullet (\vec{\nabla}\Psi) = \frac{B_o}{2}\left(x\frac{\partial\Psi}{\partial y} - y\frac{\partial\Psi}{\partial x}\right), \quad A^2 = \frac{B_o^2}{4}\left(x^2 + y^2\right) \tag{28}$$

Hence, Equation (25) becomes

$$-\frac{\hbar^2}{2M}\nabla^2\Psi + \frac{iq\hbar B_o}{2M}\left(x\frac{\partial\Psi}{\partial y} - y\frac{\partial\Psi}{\partial x}\right) + \frac{q^2B_o^2}{8M}(x^2 + y^2) + q\,\alpha\, z^2\,\Psi = E\,\Psi \tag{29}$$

But the angular momentum L_z is given by

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{30}$$

So, Eq. (29) takes the form

$$-\frac{\hbar^2}{2M}\nabla^2\Psi - \frac{qB_o}{2M}L_z\Psi + \frac{q^2B_o^2}{8M}(x^2 + y^2) + q\,\alpha\, z^2\,\Psi = E\,\Psi \tag{31}$$

Since

$$L_z\Psi = m\,\hbar\Psi \tag{32}$$

Where m is the magnetic quantum number ($m = 0, \pm 1, \pm 2, \ldots\ldots$), therfore

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + \frac{q^2B_o^2}{8M}(x^2+y^2) + q\,\alpha\,z^2\right)\Psi = \left(E + \frac{qB_o\hbar}{2M}m\right)\Psi \tag{33}$$

Now let $\omega_1 = qB_o/m$, $\omega_2 = (2\alpha q/m)^{1/2}$, and use the laplacian in cylindrical coordinates (r, φ, z):

$$\begin{gathered}-\tfrac{\hbar^2}{2M}\left[\tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial\Psi}{\partial r}\right) + \tfrac{1}{r^2}\tfrac{\partial^2\Psi}{\partial\varphi^2} + \tfrac{\partial^2\Psi}{\partial z^2}\right] + \\ \left[\tfrac{M\,\omega_1^2r^2}{8} + \tfrac{M\,\omega_2^2z^2}{2}\right]\Psi = \left(E + \tfrac{m\hbar\,\omega_1}{2}\right)\Psi\end{gathered} \tag{34}$$

What is notable about expression (34) is that the coordinate φ does not appear except as a variable for differentiation. When this happens in a classical Hamiltonian we say it is cyclic in the coordinate and the conjugate momentum P_φ is a constant of the motion (Goldstein, 2001). However

$$L_z = -i\,\hbar\frac{\partial}{\partial\varphi}, \quad \text{so} \quad \frac{\partial^2\Psi}{\partial\varphi^2} = -\frac{1}{\hbar^2}L_z^2\Psi = -\frac{1}{\hbar^2}m^2\hbar^2\Psi = -m^2\Psi \tag{35}$$

and using separation of variable $\Psi(r,\varphi,z) = R(r)\Phi(\varphi)Z(z)$, the expression (34) becomes

$$\begin{gathered}\tfrac{-\hbar^2}{2M}\left[\Phi Z\tfrac{1}{r}\tfrac{d}{dr}\left(r\tfrac{dR}{dr}\right) - \tfrac{m^2}{r^2}R\Phi Z + R\Phi\tfrac{d^2Z}{dz^2}\right] + \\ \left(\tfrac{1}{8}M\omega_1^2r^2 + \tfrac{1}{2}M\omega_2^2z^2\right)R\Phi Z = \left(E + \tfrac{1}{2}M\hbar\omega_1\right)R\Phi Z\end{gathered} \tag{36}$$

Divide by $R\Phi Z$ and collect the terms to get

$$\underbrace{\left\{\frac{-\hbar^2}{2M}\left[\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{m^2}{r^2}\right] + \frac{1}{8}M\omega_1^2r^2\right\}}_{1} + \underbrace{\left\{\frac{-\hbar^2}{2M}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{2}M\omega_2^2z^2\right\}}_{2} = \left(E + \frac{1}{2}M\hbar\omega_1\right) \tag{37}$$

The first term depends only on r, the second only on z, so they are both constants; call them E_r, and E_z. Therefore

$$\frac{-\hbar^2}{2M}\frac{d^2Z(z)}{dz^2} + \frac{1}{2}M\omega_2^2z^2Z(z) = E_zZ(z) \tag{38}$$

and

$$\frac{-\hbar^2}{2M}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) - \frac{m^2}{r^2}R(r)\right] + \frac{1}{8}M\omega_1^2r^2R(r) = E_rR(r) \tag{39}$$

where,$E = E_r + E_z - (m\hbar\omega_1/2)$. The z equation is a one-dimensional HO, and we can read off immediately that $E_z = (n_2 + 1/2)\hbar\omega_2$, with $n_2 = 0, 1, 2, \ldots.$, and the corresponding eigenstates are

$$Z(z) = \left[\sqrt{\frac{1}{2^{n_2}\,n_2!}}\,\left(\frac{M\omega_2}{\pi\,\hbar}\right)^{1/4} \times\ \exp\left[\frac{-M\omega_2\,z^2}{2\,\hbar}\right] \times\ H_{n_2}\left(\sqrt{\frac{M\omega_2}{\hbar}}\,z\right)\right] \tag{40}$$

On the other hand, the r equation is actually a two-dimensional HO problem and will be solved in section five using He's HPM.

4. Basic Idea of He's HPM

To illustrate the basic ideas of the new HPM, we consider the following nonlinear differential equation

$$A(u) - F(r) = 0, \quad r \in \Omega \tag{41}$$

with boundary conditions

$$B\,(u, \frac{du}{dr}) = 0, \quad r \in \Gamma \tag{42}$$

where A is a general differential operator , $F(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω . The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (40), therefore, can be written as follows (He, 2003):

$$L(u) + N(u) - F(r) = 0 \tag{43}$$

By the homotopy technique (Liao, 1997), we construct a homotopy $v(r, \delta) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$H(v, \delta) = (1 - \delta)[L(v) - L(u_o)] + \delta\,[A(v) - F(r)] = 0, \quad \delta \in [0, 1]\ ,\ r \in \Omega \tag{44}$$

$$H(v, \delta) = L(v) - L(u_o) + \delta\, L(u_o) + \delta\,[N(v) - F(r)] = 0, \quad \delta \in [0, 1]\ ,\ r \in \Omega \tag{45}$$

where $\delta \in [0, 1]$ is an embedding parameter, u_o is an initial approximation of Eq. (40), which satisfies the boundary conditions. Obviously, from Eq. (43) and Eq. (44), we have

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{46}$$

$$H(v, 1) = A(v) - F(r) = 0 \tag{47}$$

The changing process of δ from zero to unity is just that of $v(r, \delta)$ from $u_o(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_o)$, $A(v) - F(r)$ are called homotopic. We use the embedding parameter δ as a "small parameter", and assume that the solution of Eq. (44) can be written as a power series in δ:

$$v = v_0 + \delta\, v_1 + \delta^2\, v_2 + \delta^3\, v_3 + \delta^4\, v_4 + ... \tag{48}$$

The approximate solution of Eq (48), therefore, can be readily obtained

$$u = \lim_{\delta \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + ... \tag{49}$$

The series (49) is convergent for most cases. The convergence of the series (49) has been proved in [He (1999), He (2000)]. The coupling of the perturbation method is called the homotopy perturbation method, which has eliminated limitations of traditional methods. The HPM depends on the proper selection of the initial approximation v_o. However, the convergent rate depends upon the nonlinear operator $N(v)$:

a. The second derivative of $N(v)$ with respect to v must be small, because the parameter δ may be relatively large, i.e. $\delta \rightarrow 1$.

b. The norm of $L^{-1}\left(\frac{\partial N}{\partial v}\right)$ must be smaller than one, in order that the series converges.

5. Solution of Two-Dimensional HO Problem Using He's HPM

In this section, we implement the HPM, in a realistic and efficient way, to provide approximate solutions for the two dimensional HO problem which is subjected to the condition that $R(r)$ is finite at $r = 0$. For the sake of continuity, this equation is rewritten here

$$\frac{-\hbar^2}{2M}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) - \frac{m^2}{r^2}R(r)\right] + \frac{1}{8}M\omega_1^2\, r^2\, R(r) = E_r\, R(r) \tag{50}$$

that can be rewritten in the form

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{d\,R(r)}{dr} + \left(k^2 r^2 - m^2 - \beta^2 r^4\right) R(r) = 0 \tag{51}$$

where $k = (2M\, E_r/\hbar^2)^{1/2}$ and $\beta = (M\omega_1^2/4\hbar^2)^{1/2}$.

In view of Eq. (44) or (45), the homotopy for Eq. (51) can be constructed as

$$H(R, \delta) = r^2\frac{d^2R(r)}{dr^2} + r\frac{d\,R(r)}{dr} + \left(k^2 r^2 - m^2 - \delta\beta^2 r^4\right) R(r) = 0, \quad \delta \in [0, 1] \tag{52}$$

where the variation of δ from 0 to unity corresponds to the variation of $H(R, \delta)$ from $R_o(r)$.

The basic assumption of HPM is that the solution $R(r)$ can be expressed as a power of series in δ.

$$R(r) = \sum_{n=0} \delta^n R_n(r) \; = \; R_0(r) + \delta R_1(r) + \delta^2 R_2(r) + \delta^3 R_3(r).......... \tag{53}$$

The terms up to δ^3 are considered, where

$$R(r) \; \approx \; R_0(r) + \delta R_1(r) + \delta^2 R_2(r) + \delta^3 R_3(r) \tag{54}$$

$$\frac{dR(r)}{dr} \approx \frac{dR_0(r)}{dr} + \delta \frac{dR_1(r)}{dr} + \delta^2 \frac{dR_2(r)}{dr} + \delta^3 \frac{dR_3(r)}{dr}) \tag{55}$$

$$\frac{d^2R(r)}{dr^2} \approx \frac{d^2R_0(r)}{dr^2} + \delta \frac{d^2R_1(r)}{dr^2} + \delta^2 \frac{d^2R_2(r)}{dr^2} + \delta^3 \frac{d^2R_3(r)}{dr^2} \tag{56}$$

Substitution of Eqs. (54-56) into Eq. (52) yields

$$\begin{aligned} & r^2 \left(\tfrac{d^2R_0(r)}{dr^2} + \delta \tfrac{d^2R_1(r)}{dr^2} + \delta^2 \tfrac{d^2R_2(r)}{dr^2} + \delta^3 \tfrac{d^2R_3(r)}{dr^2} \right) + \\ & r \left(\tfrac{dR_0(r)}{dr} + \delta \tfrac{dR_1(r)}{dr} + \delta^2 \tfrac{dR_2(r)}{dr} + \delta^3 \tfrac{dR_3(r)}{dr} \right) + \\ & \left[k^2 r^2 - m^2 - \delta \beta^2 r^4 \right] \left(R_0(r) + \delta R_1(r) + \delta^2 R_2(r) + \delta^3 R_3(r) \right) = \; 0 \end{aligned} \tag{57}$$

Summing up the coefficient of like power of δ gives

$$\begin{aligned} & \delta^0 \left(r^2 \tfrac{d^2R_0(r)}{dr^2} + r \tfrac{dR_0(r)}{dr} + (k^2r^2 - m^2) R_0(r) \right) + \\ & \delta^1 \left(r^2 \tfrac{d^2R_1(r)}{dr^2} + r \tfrac{dR_1(r)}{dr} + (k^2r^2 - m^2) R_1(r) - \beta^2 r^4 R_0(r) \right) + \\ & \delta^2 \left(r^2 \tfrac{d^2R_2(r)}{dr^2} + r \tfrac{dR_2(r)}{dr} + (k^2r^2 - m^2) R_2(r) - \beta^2 r^4 R_1(r) \right) + \\ & \delta^3 \left(r^2 \tfrac{d^2R_3(r)}{dr^2} + r \tfrac{dR_3(r)}{dr} + (k^2r^2 - m^2) R_3(r) - \beta^2 r^4 R_2(r) \right) = 0 \end{aligned} \tag{58}$$

where

$$\begin{aligned} & \delta^0 : r^2 \tfrac{d^2R_0(r)}{dr^2} + r \tfrac{dR_0(r)}{dr} + (k^2r^2 - m^2) R_0(r) = 0 && (a_0) \\ & \delta^1 : r^2 \tfrac{d^2R_1(r)}{dr^2} + r \tfrac{dR_1(r)}{dr} + (k^2r^2 - m^2) R_1(r) = \beta^2 r^4 R_0(r) && (a_1) \\ & \delta^2 : r^2 \tfrac{d^2R_2(r)}{dr^2} + r \tfrac{dR_2(r)}{dr} + (k^2r^2 - m^2) R_2(r) = \beta^2 r^4 R_1(r) && (a_2) \\ & \delta^3 : r^2 \tfrac{d^2R_3(r)}{dr^2} + r \tfrac{dR_3(r)}{dr} + (k^2r^2 - m^2) R_3(r) = \beta^2 r^4 R_2(r) && (a_3) \\ & \vdots \\ & \delta^p : r^2 \tfrac{d^2R_p(r)}{dr^2} + r \tfrac{dR_p(r)}{dr} + (k^2r^2 - m^2) R_p(r) = \beta^2 r^4 R_{(p-1)}(r) && (a_p) \end{aligned} \tag{59}$$

We will use the symbol $R_{p,m}(r)$ to denote the solution of the differential equation number p for the discussed case m. The equations group (59) will be solved for the case $m = 0$. Hence, $R_0(r) = R_{0,0}(r)$, $R_1(r) = R_{1,0}(r)$, $R_2(r) = R_{2,0}(r)$, and $R_p(r) = R_{p,m}(r)$. In this case, Eq.(59-a_0) can be rewritten as:

$$r^2 \frac{d^2R_{0,0}(r)}{dr^2} + r \frac{dR_{0,0}(r)}{dr} + k^2 r^2 R_{0,0}(r) = 0 \tag{60}$$

The general solution of Eq. (60) using MATLAB is given by

$$R_{0,0}(r) = \; c_1 \, J_0(\sqrt{k}\, r) \; + \; c_2 \, Y_0(\sqrt{k}\, r) \tag{61}$$

where J_0 is the first kind Bessel function of order 0, which is nonsingular at the origin, and it is sometimes called cylinder function or cylindrical harmonic. Y_0 is the second kind Bessel function of order 0, which is singular at the origin. c_1 and c_2 are arbitrary constants. In order to get a bounded solution on an interval $[0, a]$, we may take the conditions:

1. $R(r)$ is finite (or bounded) as $r \to 0$: i.e, $\lim_{r \to 0} R(r) \; < \infty$, thus $c_2 = 0$.

2. $R(0) = 1$ in order to have $c_1 = 1$, since $J_0(0) = 1$. Thus,

$$R_{0,0}(r) = \; J_0(\sqrt{k}\, r) \tag{62}$$

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The series form of $J_m(\sqrt{k}\,r)$ is given by

$$J_m(\sqrt{k}\,r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{\sqrt{k}\,r}{2}\right)^{2n+m} \tag{63}$$

where the factorials can be generalized to gamma functions for non-integral m, therefore Eq. (63) takes the form

$$J_m(\sqrt{k}\,r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+m+1)} \left(\frac{\sqrt{k}\,r}{2}\right)^{2n+m} \tag{64}$$

If $J_0(r)$ is approximated by:

$$J_0(r) \approx 1 - \left(\frac{r}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{r}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{r}{2}\right)^6 + \frac{1}{(4!)^2}\left(\frac{r}{2}\right)^8 \tag{65}$$

In the interval $[0, a]$, the absolute value of the maximum error will not exceed $\frac{a^{10}}{2^{10}(5!)^2}$. If we take the interval to be $[0, 4]$, the maximum error will not exceed 0.071111.

Hence, $R_{0,0}(\sqrt{k}\,r)$ take the form

$$\underset{Approx}{R_{0,0}(\sqrt{k}\,r)} \approx 1 - \left(\frac{\sqrt{k}\,r}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\sqrt{k}\,r}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\sqrt{k}\,r}{2}\right)^6 + \frac{1}{(4!)^2}\left(\frac{\sqrt{k}\,r}{2}\right)^8 \tag{66}$$

For $k = 1$ and $m = 0$, the solution $R_{0,0}(r)$ becomes

$$\underset{Approx}{R_{0,0}(r)} \approx 1 - \frac{1}{4}r^2 + \frac{1}{64}r^4 - \frac{1}{2304}r^6 + \frac{1}{147\,456}r^8 \tag{67}$$

Some selected value of the approximated solution of $R_{0,0}(r)$, for $k = 1$ and $m = 0$, are listed in Table.1, and the solution $R_{0,0}(r)$ is also illustrated in Fig. (2). Similarly, Eq. (59-a_1) can be written as:

$$r^2\frac{d^2R_{1,0}(r)}{dr^2} + r\frac{dR_{1,0}(r)}{dr} + (k^2r^2 - m^2)R_{1,0}(r) = \beta^2 r^4 R_{0,0}(r) \tag{68}$$

Substituting Eq. (66) in Eq. (68) leads to

$$\begin{aligned} &r^2\tfrac{d^2R_{1,0}(r)}{dr^2} + r\tfrac{dR_{1,0}(r)}{dr} + (k^2r^2 - m^2)R_{1,0}(r) = \\ &\beta^2 r^4\left(1 - k\left(\tfrac{r}{2}\right)^2 + \tfrac{1}{(2!)^2}k^2\left(\tfrac{r}{2}\right)^4 - \tfrac{1}{(3!)^2}k^3\left(\tfrac{r}{2}\right)^6 + \tfrac{1}{(4!)^2}k^4\left(\tfrac{r}{2}\right)^8\right) \end{aligned} \tag{69}$$

Using MATLAB, the general solution of Eq. (69) is

$$\begin{aligned} &R_{1,0}(r) = c_3\,J_0(\sqrt{k}\,r) + c_4\,Y_0(\sqrt{k}\,r) + \\ &\tfrac{\beta^2}{k^2}\left(-220 + 55\,k\,r^2 - \tfrac{27}{8}\,k^2\,r^4 + \tfrac{25}{288}\,k^3\,r^6 - \tfrac{41}{36864}\,k^4\,r^8 + \tfrac{1}{147456}k^5 r^{10}\right) \end{aligned} \tag{70}$$

In order to get a bounded solution as $r \to 0$, we choose $c_4 = 0$, and with the condition $R_{1,0}(0) = 1$ we get ($c_3 = 1 + \frac{220\beta^2}{k^2}$). Thus,

$$\begin{aligned} &R_{1,0}(r) \approx \left(1 + \tfrac{220\beta^2}{k^2}\right) J_0(\sqrt{k}\,r) + \\ &\tfrac{\beta^2}{k^2}\left(-220 + 55\,k\,r^2 - \tfrac{27}{8}\,k^2\,r^4 + \tfrac{25}{288}\,k^3\,r^6 - \tfrac{41}{36864}\,k^4\,r^8 + \tfrac{1}{147456}k^5 r^{10}\right) \end{aligned} \tag{71}$$

Using the approximation given by Eq. (66), we get

$$\begin{aligned} &\underset{Approx}{R_{1,0}(r)} \approx \left(1 + \tfrac{220\beta^2}{k^2}\right)\left(1 - k\left(\tfrac{r}{2}\right)^2 + \tfrac{k^2}{(2!)^2}\left(\tfrac{r}{2}\right)^4 - \tfrac{k^3}{(3!)^2}\left(\tfrac{r}{2}\right)^6 + \tfrac{k^4}{(4!)^2}\left(\tfrac{r}{2}\right)^8\right) + \\ &\tfrac{\beta^2}{k^2}\left(-220 + 55\,k\,r^2 - \tfrac{27}{8}\,k^2\,r^4 + \tfrac{25}{288}\,k^3\,r^6 - \tfrac{41}{36864}\,k^4\,r^8 + \tfrac{1}{147456}k^5 r^{10}\right) \end{aligned} \tag{72}$$

After collecting the similar terms, we get the solution of Eq. (59-a_1)

$$\begin{aligned} R_{1,0}(r) \; &\approx 1-\left(\tfrac{k}{4}\right)r^2+\left(\tfrac{k^2}{64}+\tfrac{\beta^2}{16}\right)r^4 \\ &-\left(\tfrac{k^3}{2304}+\tfrac{5\,k\beta^2}{576}\,r^6\right)+\left(\tfrac{k^4}{147456}+\tfrac{7\,k^2\beta^2}{18432}\right)r^8+\tfrac{\beta^2\,k^3}{147456}\,r^{10} \end{aligned} \tag{73}$$

For $k = 1$ and $m = 0$, the solution $R_{1,0}(r)$ becomes :

$$\underset{Approx}{R_{1,0}(r)} \approx 1-\frac{1}{4}r^2+\frac{5}{64}r^4-\frac{7}{768}\,r^6+\frac{19}{49152}r^8+\frac{1}{147456}\,r^{10} \tag{74}$$

Some selected value of the approximated solution of $R_{1,0}(r)$, for $k = 1, \beta = 1$, and $m = 0$, are listed in Table.2, and the solution $R_{1,0}(r)$ is also illustrated in Fig. (3). Similarly, Eq. (59-a_2) can be written as:

$$r^2\frac{d^2R_{2,0}(r)}{dr^2}+r\,\frac{dR_{2,0}(r)}{dr}+k^2r^2R_{2,0}(r)=\beta^2r^4R_{1,0}(r) \tag{75}$$

Substituting Eq. (72) in Eq. (75) leads to

$$\begin{aligned} &r^2\tfrac{d^2R_{2,0}(r)}{dr^2}+r\,\tfrac{dR_{2,0}(r)}{dr}+k^2r^2R_{2,0}(r)= \\ &\beta^2r^4\left[\begin{array}{c} 1-\left(\tfrac{k}{4}\right)\,r^2+\left(\tfrac{k^2}{64}+\tfrac{\beta^2}{16}\right)\,r^4-\left(\tfrac{k^3}{2304}+\tfrac{5\,k\beta^2}{576}\right)r^6+ \\ \left(\tfrac{k^4}{147456}+\tfrac{7\,k^2\beta^2}{18432}\right)\,r^8+\tfrac{\beta^2\,k^3}{147456}\,r^{10} \end{array}\right] \end{aligned} \tag{76}$$

that can be simplified to

$$\begin{aligned} &r^2\tfrac{d^2R_{2,0}(r)}{dr^2}+r\,\tfrac{dR_{2,0}(r)}{dr}+k^2r^2R_{2,0}(r)= \\ &\tfrac{\beta^4\,k^3}{147456}\,r^{14}+\left(\tfrac{k^4\beta^2}{147456}+\tfrac{7\,k^2\beta^4}{18432}\right)\,r^{12}-\left(\tfrac{k^3\beta^2}{2304}+\tfrac{5\,k\beta^4}{576}\right)r^{10}+\left(\tfrac{k^2\beta^2}{64}+\tfrac{\beta^4}{16}\right)r^8-\left(\tfrac{k\beta^2}{4}\right)\,r^6+\beta^2r^4 \end{aligned} \tag{77}$$

Using MATLAB, the general solution of Eq. (77) is

$$\begin{aligned} R_{2,0}(r) = \; &c_5\,J_0(\sqrt{k}\,r)+c_6\,Y_0(\sqrt{k}\,r)+\tfrac{\beta^4k^2}{147456}\,r^{12}+\tfrac{(\beta^2k^3-88\beta^4k)}{147456}\,r^{10}+ \\ &\tfrac{(7520\beta^4-164\beta^2k^2)}{147456}\,r^8+\tfrac{(-472064\beta^4+12800\beta^2k^2)}{147456\,k}\,r^6+\tfrac{(16994304\beta^4-497664\beta^2k^2)}{147456\,k^2}\,r^4+ \\ &\tfrac{(8110080\beta^2k^2-271908864\beta^4)}{147456\,k^3}r^2+\tfrac{(1087635456\beta^4-32440320\beta^2k^2)}{147456\,k^4} \end{aligned} \tag{78}$$

Again, in order to get a bounded solution as $r \to 0$, we choose $c_6 = 0$, and with the condition $R_{2,0}(0) = 1$ we get $c_5 = 1 - \frac{1087635456\beta^4-32440320\beta^2k^2}{147456k^4}$. Thus,

$$\begin{aligned} R_{2,0}(r) = &\left(1-\tfrac{1087635456\beta^4-32440320\beta^2k^2}{147456k^4}\right)\,J_0(\sqrt{k}\,r)+\tfrac{\beta^4k^2}{147456}\,r^{12}+\tfrac{(\beta^2k^3-88\beta^4k)}{147456}\,r^{10}+ \\ &\tfrac{(7520\beta^4-164\beta^2k^2)}{147456}\,r^8+\tfrac{(-472064\beta^4+12800\beta^2k^2)}{147456\,k}\,r^6+\tfrac{(16994304\beta^4-497664\beta^2k^2)}{147456\,k^2}\,r^4+ \\ &\tfrac{(8110080\beta^2k^2-271908864\beta^4)}{147456\,k^3}r^2+\tfrac{(1087635456\beta^4-32440320\beta^2k^2)}{147456\,k^4} \end{aligned} \tag{79}$$

Using the approximation given by Eq. (66), we get

$$\begin{aligned} \underset{Approx}{R_{2,0}(r)} \approx &\left(1-\tfrac{1087635456\beta^4-32440320\beta^2k^2}{147456k^4}\right)\left(1-k\left(\tfrac{r}{2}\right)^2+\tfrac{k^2}{(2!)^2}\left(\tfrac{r}{2}\right)^4-\tfrac{k^3}{(3!)^2}\left(\tfrac{r}{2}\right)^6+\tfrac{k^4}{(4!)^2}\left(\tfrac{r}{2}\right)^8\right) \\ &+\tfrac{\beta^4k^2}{147456}\,r^{12}+\tfrac{(\beta^2k^3-88\beta^4k)}{147456}\,r^{10}+\tfrac{(7520\beta^4-164\beta^2k^2)}{147456}\,r^8+\tfrac{(-472064\beta^4+12800\beta^2k^2)}{147456\,k}r^6+ \\ &\tfrac{(16994304\beta^4-497664\beta^2k^2)}{147456\,k^2}\,r^4+\tfrac{(8110080\beta^2k^2-271908864\beta^4)}{147456\,k^3}r^2+\tfrac{(1087635456\beta^4-32440320\beta^2k^2)}{147456\,k^4} \end{aligned} \tag{80}$$

For $k = 1$ and $m = 0$, the solution $R_{2,0}(r)$ becomes :

$$\underset{Approx}{R_{2,0}(r)} \approx 1-\frac{1}{4}r^2+\frac{5}{64}r^4-\frac{7}{768}r^6+\frac{67}{49152}r^8-\frac{29}{49152}r^{10}+\frac{1}{147456}\,r^{12} \tag{81}$$

Some selected value of the approximated solution of $R_{2,0}(r)$, for $k = 1$, $\beta = 1$ and $m = 0$, are listed in Table.3, and the solution $R_{2,0}(r)$ is also illustrated in Fig. (3).

As a result, according to Eq. (49), the general solution $R(r)$ can be approximated by

$$\begin{aligned} R(r) &\approx R_{0,0} + R_{1,0} + R_{2,0} + ... \\ &\approx 3 - \tfrac{3}{4}r^2 + \tfrac{11}{64}r^4 - \tfrac{43}{2304}r^6 + \tfrac{259}{147456}r^8 - \tfrac{43}{73728}r^{10} + \tfrac{1}{147456}\ r^{12} + ... \end{aligned} \tag{82}$$

However, this work will be extended to compare this approximate solution with some analytical solutions. In addition solution of Eq. (52) for the case $m = 1$ will be also investigated.

6. Conclusion

An approximate solution of the time-independent Schrödinger equation for a two-dimensional harmonic oscillator has been obtained using an effective and realistic method, which is He's HPM. HPM works successfully in handling the differential equation directly and produces the solutions in terms of convergent series with easily computable components. HPM requires less computational work when compared with other methods. Consequently, this research has thrown up many questions in need of further investigation, and will serve as a base for future studies.

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Table 1. Some selected values of the approximated of $R_{0,0}(r)$, $k = 1$ and $m = 0$.

r	$R_{0,0}(r)$ Approx	r	$R_{0,0}(r)$ Approx
0.2	0.9900249722	2.2	0.1105364805
0.4	0.9603982266	2.4	0.0029209600
0.6	0.912004863	2.6	−0.0958910360
0.8	0.8462873600	2.8	−0.1831324400
1.0	0.765197753	3.0	−0.2562866210
1.2	0.6711331600	3.2	−0.3130683733
1.4	0.5668570555	3.4	−0.3513561277
1.6	0.4554094933	3.6	−0.3690652400
1.8	0.340010088	3.8	−0.3639515027
2.0	0.2239583333	4.0	−0.3333333333

Table 2. Some selected values of the approximated of $R_{1,0}(r)$, $k = 1, \beta = 1$ and $m = 0$.

r	$R_{0,0}(r)$ Approx	r	$R_{0,0}(r)$ Approx
0.2	0.9901244176	2.2	0.8168524930
0.4	0.9619629207	2.4	0.8786768896
0.6	0.9197062836	2.6	0.9674550064
0.8	0.8696762482	2.8	1.1110582081
1.0	0.8194037543	3.0	1.3702392578
1.2	0.7764881104	3.2	1.8590684615
1.4	0.7473973310	3.4	2.7729654818
1.6	0.7364307740	3.6	4.4261964496
1.8	0.7451374457	3.8	7.3010302439
2.0	0.7725694444	4.0	12.111111111

Table 3. Some selected values of the approximated of $R_{2,0}(r)$, $k = 1, \beta = 1$ and $m = 0$.

r	$R_{0,0}(r)$ Approx	r	$R_{0,0}(r)$ Approx
0.2	0.9901244201	2.2	−0.145136575
0.4	0.9619634982	2.4	−1.582538338
0.6	0.9197190923	2.6	−4.770734773
0.8	0.8697764745	2.8	−11.30131161
1.0	0.8197903103	3.0	−23.85821533
1.2	0.7770524613	3.2	−46.77714119
1.4	0.7449314374	3.4	−86.80331713
1.6	0.7146651325	3.6	−154.0849165
1.8	0.6475182442	3.8	−263.4353635
2.0	0.4392361111	4.0	−435.8888889

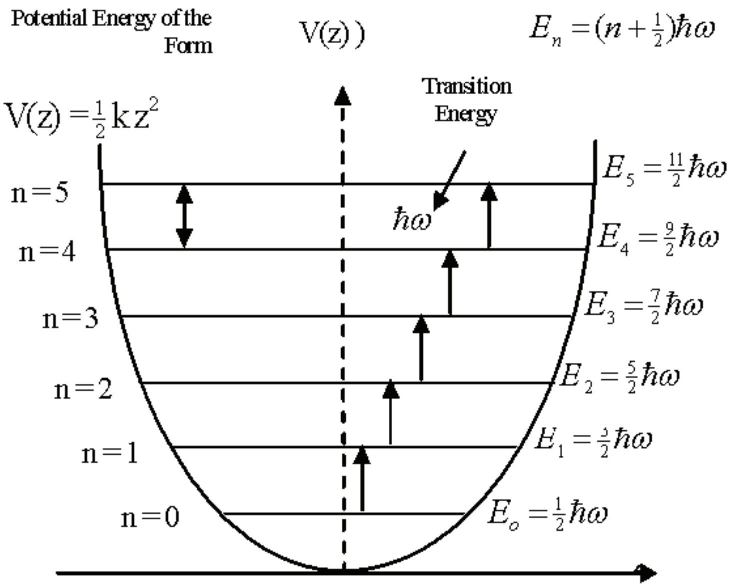

Figure 1. Energy Levels of the one-dimensional harmonic oscillator

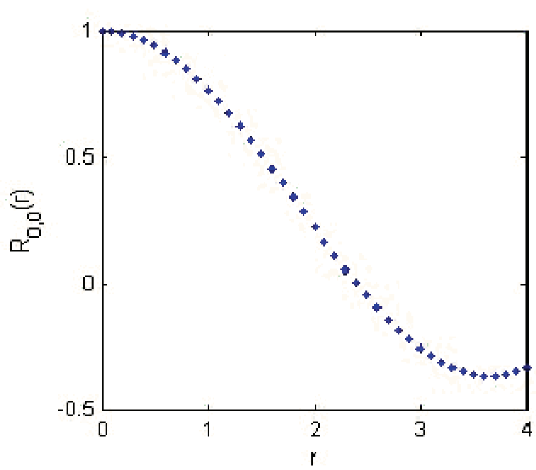

Figure 2. Approximated solutions $R_{0,0}(r)$ of Eq.(59-a0): $k = 1, \beta = 1$, and $m = 0$

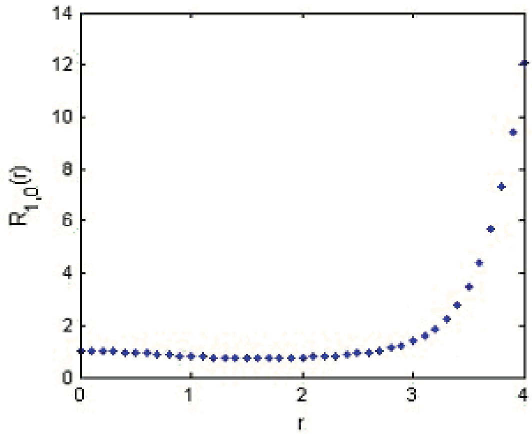

Figure 3. Approximated solutions $R_{1,0}(r)$ of Eq.(59-a1): $k = 1, \beta = 1$, and $m = 0$

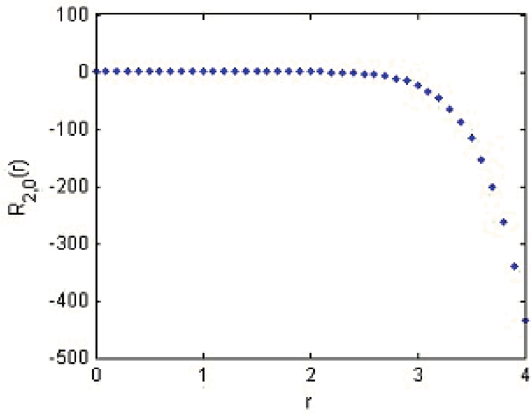

Figure 4. Approximated solutions $R_{2,0}(r)$ of Eq.(59-a2): $k = 1, \beta = 1$, and $m = 0$.