Uniform Bound on Normal Approximation of Latin Hypercube Sampling

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Abstract

Loh (Loh, W.L., 1996b) established a Berry-Esseen type bound for \( W \), the random variable based on a latin hypercube sampling, to the standard normal distribution. He used an inductive approach of Stein’s method to give the rate of convergence \( \frac{C_d}{\sqrt{n}} \) without the value of \( C_d \). In this article, we use a concentration inequality approach of Stein’s method to obtain a constant \( C_d \).

Keywords: Latin hypercube sampling, Stein’s method, Uniform bound, Berry-Esseen theorem, Concentration inequality

1. Introduction

A latin hypercube sampling (LHS) was introduced by McKay, Beckman and Conover in 1978 (McKay, M.D., 1979) as a tool to improve the efficiency of different important sampling method. After the original paper appeared, LHS has been widely used in many computer experiments. For example, it is a way to choose the points to compute the integral

\[
\mu = \int_{[0,1]^d} f(x)dx,
\]

where \( f \) is a measurable function from \([0,1]^d\) to \( \mathbb{R} \). Approximating this integral is equivalent to finding \( \mu = E(f(X)) \), where \( X \) is a random vector uniformly distributed on a unit hypercube \([0,1]^d\).

For positive integers \( n \) and \( d \), \( d \geq 2 \), a latin hypercube sample of size \( n \) (taken from the \( d \)-dimensional hypercube \([0,1]^d\)) is defined to be \( \{X(\eta_1(i), \ldots, \eta_d(i)) : 1 \leq i \leq n\} \), where

1. for all \( 1 \leq i_1, \ldots, i_d \leq n \), \( 1 \leq j \leq d \),
\[
X_j(i_1, \ldots, i_d) = (i_j - U_{i_1, \ldots, i_d,j})/n, \quad \text{and} \quad X(i_1, \ldots, i_d) = (X_1(i_1, \ldots, i_d), \ldots, X_d(i_1, \ldots, i_d));
\]
2. \( \eta_k \) is a random permutation of \([1,\ldots,n] \) each uniformly distributed over all the \( n! \) possible permutations;
3. \( U_{i_1,\ldots,i_d,j} \), \( 1 \leq i_1, \ldots, i_d \leq n \), \( 1 \leq j \leq d \), are \([0,1] \) uniform random variables;
4. the \( U_{i_1,\ldots,i_d,j} \) ’s and \( \eta_k \)’s are all stochastically independent.

Hence an unbiased estimator for \( \mu \) based on a latin hypercube sampling is

\[
\hat{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} f(X(\eta_1(k), \ldots, \eta_d(k))).
\]

McKay, Beckman and Conover (McKay, M.D., 1979) further proved that in a large number of instances, the variance of \( \hat{\mu}_n \) is substantially smaller than that the estimators based on simple random sampling. Many years later, Stein (Stein, M.L., 1987) showed that the asymptotic variance of \( \hat{\mu}_n \) is less than the asymptotic variance of an analogous estimator based on an independently and identically distributed sample. Later, Owen (Owen, A.B., 1992) gave the multivariate central limit theorem for \( \hat{\mu}_n \) when \( f \) is bounded. In addition to the LHS, there are several ways to sample \( X_i \)’s in order to estimate \( \mu \), namely, lattice sampling (Patterson, H.D., 1954), the orthogonal array sampling ((Loh, W.L., 1996a), (Neammanee, K. & Laipaporn, K., 2008), (Tang, B., 1993)), and scrambled net sampling ((Owen, A.B., 1997a), (Owen, A.B., 1997b)).

If \( \text{Var}(\hat{\mu}_n) > 0 \), we define

\[
W = \frac{\hat{\mu}_n - \mu}{\sqrt{\text{Var}(\hat{\mu}_n)}}.
\]
Then
\[ EW = 0 \text{ and } \text{Var}W = 1. \] (1)

To use the Stein’s method to approximate the distribution of \( W \) with the standard normal distribution, Loh (Loh, W.L., 1996b) wrote

\[ W = \sum_{i=1}^{n} V(\eta_1(i), \ldots, \eta_d(i)), \]

where

\[ V(i_1, \ldots, i_d) = \frac{1}{n \sqrt{\text{Var}[\eta_n]}} [f \circ X(i_1, \ldots, i_d) - \sum_{k=1}^{d} \mu_d(i_k) + (d - 1)\mu] \]
\[ \mu(i_1, \ldots, i_d) = Ef \circ X(i_1, \ldots, i_d) \text{ and } \mu_d(i_k) = \frac{1}{n^{d-1}} \sum_{j \neq k} \mu(i_1, \ldots, i_d), \]

and gave the rate of convergence \( \frac{C_d}{\sqrt{n}} \) without the value of \( C_d \) under the finiteness of the absolute third moment. Theorem 1.1 is his result.

**Theorem 1.1** There exists a positive constant \( C_d \) which depends only on \( d \) such that for sufficiently large \( n \),

\[ \sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq C_d \beta_3, \]

where \( \Phi \) is the standard normal distribution and \( \beta_3 = \frac{1}{d} \sum_{i=1}^{n} \sum_{l=1}^{n} E[V(i_1, \ldots, i_l)]^3 \).

**Corollary 1.2** If \( E[f \circ X]^3 < \infty \), then

\[ \sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \frac{C_d}{\sqrt{n}}. \]

In 2006, Rattanawong (Rattanawong, P., 2006) showed that there exist random permutations \( \pi_1, \ldots, \pi_{d-1} \) on \( \{1, 2, \ldots, n\} \) which are uniformly distributed over all the \( n! \) possible permutations such that

\[ W = \sum_{i=1}^{n} Y(i, \pi_1(i), \ldots, \pi_{d-1}(i)) \]

and \( Y(i_1, \ldots, i_d) \)'s and \( \pi_\ell \)'s are stochastically independent. Indeed, for \( j \in \{1, \ldots, d\} \), let \( \pi_j(\omega) = \eta_{j+1}(\omega)(\eta_1(\omega)^{-1}) \) and for each \( i_1, \ldots, i_d \in \{1, \ldots, n\} \), define

\[ Y(i_1, \ldots, i_d) = \frac{1}{n \sqrt{\text{Var}[\eta_n]}} [f \circ X(i_1, \ldots, i_d) + \sum_{k=1}^{d-1} U_k(i_1, \ldots, i_d) + (-1)^d\mu], \] (2)

where

\[ \mu(i_1, \ldots, i_d) = Ef \circ X(i_1, \ldots, i_d), \]
\[ U_k(i_1, \ldots, i_d) = \frac{(-1)^k}{n^k} \sum_{1 \leq j < p < \ldots < j_k \leq d} \sum_{q_k=1}^{n} \ldots \sum_{q_2=1}^{n} \mu(i_1, \ldots, i_d), \]

and

\[ l_p = \begin{cases} q_p & \text{if } p = j_1, \ldots, j_k, \\ i_p & \text{otherwise}. \end{cases} \]

Note that the definition of \( Y(i_1, \ldots, i_d) \)'s are different from Loh (Loh, W.L., 1996b) in order that the random variable \( W \) satisfies the following property:

\[ \sum_{i_j=1}^{n} EY(i_1, \ldots, i_d) = 0 \text{ for each } j \in \{1, 2, \ldots, n\}. \] (3)
Furthermore, Neammanee and Rattanawong (Neammanee, K. & Rattanawong, P., 2008) used a concentration inequality approach of Stein’s method and assumed the finiteness of fourth moment to give a constant $C_d$. This is their result.

**Theorem 1.3** Suppose that $E(f \circ X(i_1, \ldots, i_d))^4 < \infty$, $1 \leq i_1, \ldots, i_d \leq n$. Then for $n \geq 6^d + 3$,

$$
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq (11.765 + 23.531d)\delta_4 + \frac{11.68}{n^{\frac{3}{2}}} + \frac{2.075d^{\frac{1}{2}}}{n^{\frac{3}{2}}} + \frac{(2 \sqrt{2\pi} + 10.027d^{\frac{1}{2}})}{n^{\frac{3}{2}}},
$$

where $\Phi$ is the standard normal distribution and $\delta_4 = \frac{1}{n^{d+\frac{1}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} E[Y(i_1, \ldots, i_d)]^4$.

**Corollary 1.4** Suppose that $E(f \circ X(i_1, \ldots, i_d))^4 < \infty$, $1 \leq i_1, \ldots, i_d \leq n$. If $\delta_4 \sim \frac{1}{\sqrt{n}}$, then

$$
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \frac{28.729 + 35.633d}{\sqrt{n}}.
$$

In this article, we use a concentration inequality approach of Stein’s method with ideas of Neammanee and Rattanawong ((Neammanee, K. & Rattanawong, P., 2008), (Neammanee, K. & Rattanawong, P., 2009b)) and Neammanee and Rerkth羌irat (Neammanee, K. & Rerkth羌irat, N.) to obtain a constant $C_d$ by assuming the finiteness of the absolute third moment. Theorem 1.5 is our main result.

**Theorem 1.5** Suppose that $E(f \circ X(i_1, \ldots, i_d))^3 < \infty$, $1 \leq i_1, \ldots, i_d \leq n$. For $n \geq 6^d$,

$$
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq (22.88 + 28.99d)\delta_3 + \frac{3.88 + 2.09d}{\sqrt{n}} + 1.03\delta_3 + \left(\frac{C_d}{n^{\frac{3}{2}}} + \frac{Cn^2}{n^{\frac{3}{2}}} + \frac{Cn^2\delta_3^3}{n^{\frac{3}{2}}}\right) + O\left(\frac{1}{n}\right),
$$

where

$$
\delta_3 = \frac{1}{n^{d+\frac{1}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} E[Y(i_1, \ldots, i_d)]^3
$$

and the definition of $Y(i_1, \ldots, i_d)$ is given by (2).

**Corollary 1.6** Suppose that $E(f \circ X(i_1, \ldots, i_d))^3 < \infty$, $1 \leq i_1, \ldots, i_d \leq n$. If $n \geq 6^d$ and $\delta_3 \sim \frac{1}{\sqrt{n}}$, then

$$
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \frac{26.76 + 31.08d}{\sqrt{n}} + \frac{2.92d}{n} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).
$$

**Example.** In the case of $d = 2$, we observe that this is a special case of the combinatorial central limit theorem (For more detail see Von Bahr (Von Bahr, B., 1976), Ho and Chen (Ho, S.T., 1978)). Under the finiteness of absolute third moment, Neammanee and Suntornchost (Neammanee, K. & Suntornchost, J., 2005) gave the uniform rate of convergence and obtained the rate $\frac{198}{\sqrt{n}}$. Recently, Neammanee and Rerkth羌irat (Neammanee, K. & Rerkth羌irat, N.) improve the constant to be 78.36. For this work, Corollary 1.6 yields the constant 93.17. Although this constant is not shaper than the previous result, we establish a uniform bound on a generalization of a combinatorial central limit theorem by assuming the finiteness of absolute third moment.

2. Auxiliary Results

In this section, we will give some lemmas which are used in the next section. Almost of them, we generalize the results of Neammanee and Rerkth羌irat (Neammanee, K. & Rerkth羌irat, N.) and improve the results of Neammanee and Rattanawong (Neammanee, K. & Rattanawong, P., 2009b) under the finiteness of absolute third moment. Throughout this work, we let

$$
\delta_2 = \frac{1}{n^{d+\frac{1}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} E[Y(i_1, \ldots, i_d)]^2 \quad \text{and} \quad \delta_3 = \frac{1}{n^{d+\frac{1}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} E[Y(i_1, \ldots, i_d)]^3.
$$

**Lemma 2.1** Suppose that $E[f \circ X(i_1, \ldots, i_d)]^3 < \infty$, $1 \leq i_1, \ldots, i_d \leq n$. For $n \geq 36$,

$$
\delta_2 \leq \frac{1.02943}{\sqrt{n}}.
$$
Proof. By (1), we have

\[ 1 = EW^2 \]
\[ = \sum_{i=1}^{n} EY^2(i, \pi_1(i), \ldots, \pi_{d-1}(i)) + \sum_{i=1}^{n} \sum_{j=1 \atop j \neq i}^{n} EY(i, \pi_1(i), \ldots, \pi_{d-1}(i))Y(j, \pi_1(j), \ldots, \pi_{d-1}(j)) \]
\[ = \sqrt{n} \delta_2 + \frac{1}{n^{d-1}(n-1)^{d-2}} \sum_{i=1}^{n} \sum_{j=1 \atop j \neq i}^{n} \sum_{k=1 \atop k \neq j \neq i}^{n} EY(i_1, \ldots, i_d)EY(j_1, \ldots, j_d) \]
\[ = \sqrt{n} \delta_2 + \frac{(-1)^d}{n^{d-1}(n-1)^{d-2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} [EY(i_1, \ldots, i_d)]^2. \]

Thus

\[ \sqrt{n} \delta_2 \leq 1 + \frac{1}{n^{d-1}(n-1)^{d-2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} [EY(i_1, \ldots, i_d)]^2 \leq 1 + \frac{\sqrt{n} \delta_2}{(n-1)^{d-1}}. \]

This implies that for \( n \geq 36 \) and \( d \geq 2, \)

\[ \sqrt{n} \delta_2 \leq 1 + \frac{\sqrt{n} \delta_2}{35^2} = 1 + 0.02858 \sqrt{n} \delta_2 \]

and hence

\[ \delta_2 \leq \frac{1}{(1 - 0.02858) \sqrt{n}} \leq \frac{1.02943}{\sqrt{n}}. \]

For each \( i_1, \ldots, i_d \in \{1, 2, \ldots, n\}, \) we define

\[ Y_0(i_1, \ldots, i_d) = Y(i_1, \ldots, i_d)1(\|Y(i_1, \ldots, i_d)\| > 1), \quad \tilde{Y}_0(i_1, \ldots, i_d) = Y(i_1, \ldots, i_d)1(\|Y(i_1, \ldots, i_d)\| \leq 1), \quad \text{and} \]

\[ \tilde{Y}(\pi) = \sum_{i=1}^{n} \tilde{Y}_0(i_1, \pi_1(i), \ldots, \pi_{d-1}(i)), \]

where \( 1(\cdot) \) is the indicator function, i.e., for a nonempty set \( A, \) the indicator function of \( A \) is defined by

\[ 1(A)(\omega) = \begin{cases} 
1 & \text{if } \omega \in A, \\
0 & \text{if } \omega \notin A.
\end{cases} \]

Next, we note that for any integer \( m, n \) and \( r \) which \( m \geq 0, \) and \( n, r > 0, \)

\[ E[Y^m(i_1, \ldots, i_d)Y_0^n(i_1, \ldots, i_d)] \leq E[Y^m(i_1, \ldots, i_d)Y_0^n(i_1, \ldots, i_d)]^r \leq E[Y(i_1, \ldots, i_d)]^{m+n+r}. \]

\[ (4) \]

Lemma 2.2 Suppose that \( E[f \circ X(i_1, \ldots, i_d)] < \infty, \) \( 1 \leq i_1, \ldots, i_d \leq n. \) If \( n \geq 36, \) then

\[ E[\sum_{i=1}^{n} \sum_{k=1}^{n} Y(i, \pi_1(k), \ldots, \pi_{d-1}(k))]^2 \leq 1.02943n. \]

Proof. Observe that

\[ E[\sum_{i=1}^{n} \sum_{k=1}^{n} Y(i, \pi_1(k), \ldots, \pi_{d-1}(k))]^2 \]
\[ = \sum_{i=1}^{n} \sum_{k=1}^{n} EY^2(i, \pi_1(k), \ldots, \pi_{d-1}(k)) + \sum_{i=1}^{n} \sum_{j=0}^{m} \sum_{m=0}^{n} \sum_{i,j \neq (i,m)}^{n} EY(i, \pi_1(k), \ldots, \pi_{d-1}(k))Y(l, \pi_1(m), \ldots, \pi_{d-1}(m)) \]
\[
\begin{align*}
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{n} \sum_{l=1}^{n} \sum_{m=k}^{n} EY^2(i, \pi_1(k), \ldots, \pi_{d-1}(k)) + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{m \neq k}^{n} EY(i, \pi_1(k), \ldots, \pi_{d-1}(k))Y(l, \pi_1(m), \ldots, \pi_{d-1}(m)) \\
&\quad + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{m \neq k}^{n} EY(i, \pi_1(k), \ldots, \pi_{d-1}(k))Y(l, \pi_1(k), \ldots, \pi_{d-1}(k)) \\
&= \frac{1}{n^{d-2}} \sum_{i=1}^{n} \sum_{j=1}^{n} EY^2(i_1, i_2) + \frac{1}{n(n-1)^{d-2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{k \neq i}^{n} EY(i_1, i_2) EY(i_3, i_4) \\
&\quad + \frac{1}{n^{d-2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{k \neq i}^{n} EY(i_1, i_2, i_3) EY(i_4). \\
\end{align*}
\]

By Lemma 2.1, (3) and (4), we have

\[
E[\sum_{i=1}^{n} \sum_{k=1}^{n} Y(i, \pi_1(k), \ldots, \pi_{d-1}(k))]^2 \leq 1.02943n.
\]

\[\square\]

From now, we use the following system giving by Ho and Chen (Ho, S.T., 1978) and Neammanee and Rattanawong (Neammanee, K. & Rattanawong, P., 2008). Let \(I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}\) be uniformly distributed random variables on \([1, 2, \ldots, n]\) and \(\rho_1, \ldots, \rho_{d-1}\) and \(\tau_1, \ldots, \tau_{d-1}\) are random permutations of \([1, 2, \ldots, n]\). Assume that

\[
\begin{align*}
&\{I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}, \rho_1, \ldots, \rho_{d-1}, \tau_1, \ldots, \tau_{d-1}\} \text{ is independent of } Y(i_1, i_2) \text{‘s,} \\
&(I, K), (L_1, M_1), \ldots, (L_{d-1}, M_{d-1}), \text{ are uniformly distributed on } \{(i, k)|i, k = 1, 2, \ldots, n \text{ and } i \neq k\}, \\
&(I, K), (L_1, M_1), \ldots, (L_{d-1}, M_{d-1}) \text{ and } \tau_1, \ldots, \tau_{d-1} \text{ are mutually independent,} \\
&\text{ and } \rho_1, \ldots, \rho_{d-1} \text{ are mutually independent, and} \\
&\begin{align*}
\tau_i(\alpha) &= \begin{cases} 
\tau_i & \text{if } \alpha = I, \\
\tau_i^{-1}(L) & \text{if } \alpha = K, \\
\tau_i^{-1}(M) & \text{if } \alpha = \tau_i^{-1}(L), \\
\tau_i^{-1}(M) & \text{if } \alpha = \tau_i^{-1}(M), 
\end{cases} \\
\rho_i(\alpha) &= \begin{cases} 
M_i & \text{if } \alpha = I, \\
I & \text{if } \alpha = K, \\
\tau_i^{-1}(L) & \text{if } \alpha = \tau_i^{-1}(L), \\
\tau_i^{-1}(M) & \text{if } \alpha = \tau_i^{-1}(M), 
\end{cases}
\end{align*}
\]

where \(\rho_i(\tau_i^{-1}(\alpha)) = \rho_i(\tau_i^{-1}(\alpha)) = \alpha\) for \(i = 1, 2, \ldots, d - 1\). Now, we define some notations;

\[
\begin{align*}
\tilde{Y}(\rho) &= \tilde{Y}(\rho) - \tilde{S}_{1,0} - \tilde{S}_{2,0} + \tilde{S}_{3,0} + \tilde{S}_{4,0}, \\
\tilde{Y}(\rho) &= \sum_{i=1}^{n} \tilde{Y}_0(i, \rho_1(i), \ldots, \rho_{d-1}(i)).
\end{align*}
\]

where

\[
\begin{align*}
\tilde{S}_{1,0} &= \tilde{Y}_0(I, \rho_1(I), \ldots, \rho_{d-1}(I)), \\
\tilde{S}_{2,0} &= \tilde{Y}_0(K, \rho_1(K), \ldots, \rho_{d-1}(K)), \quad \tilde{S}_{3,0} = \tilde{Y}_0(I, \rho_1(K), \ldots, \rho_{d-1}(K)), \\
\tilde{S}_{4,0} &= \tilde{Y}_0(K, \rho_1(I), \ldots, \rho_{d-1}(I)).
\end{align*}
\]

It is easy to see that \(\tilde{S}_{1,0}, \tilde{S}_{2,0}, \tilde{S}_{3,0}, \tilde{S}_{4,0}\) have the same distribution.

**Lemma 2.3** Suppose that \(E[f \circ X(i_1, \ldots, i_d)]^3 < \infty\, , 1 \leq i_1, \ldots, i_d \leq n\). Then

\[
\begin{align*}
(1) \quad &E\tilde{Y}^2(\rho) \leq \frac{1}{n} + \frac{2\sqrt{\delta_2}}{(n - 1)^{d - 1}} + \frac{n^{d - 1}\delta_1^3}{(n - 1)^{d - 1}}. \\
(2) \quad &\text{For } n \geq 6d \text{ and } \delta_1 \leq \frac{1}{30}, \text{ we have } E\tilde{Y}^2(\pi) \leq C(n + n^2\delta_2). \\
(3) \quad &E[\tilde{Y}(\rho) - \tilde{Y}(\rho)]^2 = \frac{4}{n} + R, \text{ where} \\
&\frac{4}{n} \leq 36\delta_2 \sqrt{n(n - 1)} + \frac{8\delta_3}{n - 1} + \frac{4n^{d - 1}\delta_2^3}{(n - 1)^{d - 1}} + \frac{4n^{d - 2}\delta_2^3}{(n - 1)^{d - 1}}.
\end{align*}
\]
Proof. By using the same argument of Neammanee and Rerkruthairat (Neammanee, K. & Rerkruthairat, N.), we have this lemma. □

Corollary 2.4 Suppose that \(E|f \circ X(i_1, ..., i_d)|^3 < \infty\), \(1 \leq i_1, ..., i_d \leq n\). For \(n \geq 6^d\) and \(\delta_3 \leq \frac{1}{30}\), we get that

\[
\begin{align*}
(1) \quad & E\hat{Y}^3(\pi) \leq 1.05998. \\
(2) \quad & E|\hat{Y} - \hat{Y}(\pi)|^2 \leq \frac{4.27125}{n-1} + O\left(\frac{1}{n^2}\right).
\end{align*}
\]

Proof. We apply the idea of Neammanee and Rattanawong (Neammanee, K. & Rattanawong, P., 2008) and the fact that \(\frac{d-1}{n-1} \leq \frac{1}{35}\) for all \(n \geq 6^d\), we have

\[
\left(\frac{n}{n-1}\right)^{d-1} \leq 1 + \sum_{r=1}^{\infty} \left(\frac{d-1}{n-1}\right)^r \leq 1.03. \tag{5}
\]

By Lemma 2.1 and (5), we obtain this corollary. □

3. Proof of Theorem 1.5

We will prove this theorem by using ideas in two papers of Neammanee and Rertanawong ((Neammanee, K. & Rattanawong, P., 2008), (Neammanee, K. & Rattanawong, P., 2009b)) and a paper of Neammanee and Rerkruthairat (Neammanee, K. & Rerkruthairat, N.). Since \(|P(W \leq z) - \Phi(z)| \leq 0.55\) (Chen, L.H.Y., 2001, p. 246), we can assume \(\delta_3 \leq \frac{1}{30}\). Assume that \(z > 0\). In case of \(z < 0\), we use the fact that \(\Phi(z) = 1 - \Phi(-z)\) and then apply the result to \(-W\). Using the same argument of Neammanee and Rertanawong (Neammanee, K. & Rattanawong, P., 2009b), we have

\[
|P(W \leq z) - \Phi(z)| \leq P(W \neq \hat{Y}(\pi)) + |P(\hat{Y}(\pi) \leq z) - \Phi(z)| \leq \delta_3 + |T_1| + |T_2| + |T_3| + |T_4|
\]

where

\[
\begin{align*}
T_1 &= Eg_z(\hat{Y}(\tau)) \int_{-\infty}^{\infty} K(t)dt - E \int_{-\infty}^{\infty} g_z'(\hat{Y}(\rho) + t)K(t)dt, \\
T_2 &= Eg_z(\hat{Y}(\tau))E \int_{-\infty}^{\infty} K(t)dt - Eg_z(\hat{Y}(\tau))E \int_{-\infty}^{\infty} K(t)dt, \\
T_3 &= Eg_z(\hat{Y}(\tau)) - Eg_z(\hat{Y}(\tau))E \int_{-\infty}^{\infty} K(t)dt, \\
T_4 &= \frac{1}{n}Eg_z(\hat{Y}(\rho)) \sum_{i=1}^{n} \sum_{k=1}^{n} \hat{Y}_0(i, \rho_1(k), \ldots, \rho_{d-1}(k)), \\
& \quad \cdot \sum_{k=1}^{n} K(t) = \frac{n}{4} (\hat{Y}(\rho) - \hat{Y}(\rho))(\|0 \leq t \leq \hat{Y}(\rho) - \hat{Y}(\rho)\) - \|\hat{Y}(\rho) - \hat{Y}(\rho) < 0))
\end{align*}
\]

and \(g_z\) is the solution of the Stein’s equation for normal distribution function

\[
g'(w) - wg(w) = \mathbb{I}(w \leq z) - \Phi(z), \quad \text{for all } w \in \mathbb{R}.
\]

First, we bound \(T_4\) by using Lemma 2.2 and the fact that \(0 \leq g_z(w) \leq \min\left(\frac{\sqrt{2\pi}}{4}, \frac{1}{|w|}\right)\) (Chen, L.H.Y., 2001, p. 246). Indeed,

\[
\begin{align*}
|T_4| &\leq \frac{1}{n}E|g_z(\hat{Y}(\rho))| \sum_{i=1}^{n} \sum_{k=1}^{n} Y(i, \rho_1(k), \ldots, \rho_{d-1}(k)) + \frac{1}{n}E|g_z(\hat{Y}(\rho))| \sum_{i=1}^{n} \sum_{k=1}^{n} Y_0(i, \rho_1(k), \ldots, \rho_{d-1}(k)) \\
&\leq \frac{1}{n}E|g_z^2(\hat{Y}(\rho))| \left[ E \left( \sum_{i=1}^{n} \sum_{k=1}^{n} Y(i, \rho_1(k), \ldots, \rho_{d-1}(k)) \right) \right]^2 + \frac{\sqrt{2\pi}}{4n} \sum_{i=1}^{n} \sum_{k=1}^{n} E|Y_0(i, \rho_1(k), \ldots, \rho_{d-1}(k))| \\
&\leq \frac{\sqrt{2\pi}}{4n} \left[ \sqrt{1.02943n} + \frac{1}{4n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} E|Y(i, \ldots, i_d)|^2 \right] \\
&\leq 0.63582 \sqrt{n} + 0.62666\delta_3.
\end{align*}
\]
We apply Lemma 2.3(3) and the fact that \( |g'_c(w)| \leq 1 \) (Stein, C.M., 1986, p. 23) to obtain

\[
|T_3| \leq E[|g'_c(\hat{Y}(\tau))||1-E] \int_{-\infty}^{\infty} K(t)dt| \leq |1 - \frac{(n-1)E(\hat{Y}(\rho)-Y(\rho))^2}{4}| \leq 2\delta_3 + 1.03\delta_3^2 + O(\frac{1}{n}).
\]

Next, we will bound \( T_2 \). Let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by

\[
\{I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}, Y(i_1, \ldots, i_d) : 1 \leq i_1, i_2, \ldots, i_d \leq n\}
\]

\[ A = \{\tau(I) \neq I, \tau(K) \neq K, \tau(I) \neq M_i, \tau(K) \neq L_i, i = 1, \ldots, d-1\}, \text{ and} \]

\[
G = \hat{Y}_0(I, M_1, \ldots, M_{d-1}) + \hat{Y}_0(K, L_1, \ldots, L_{d-1}) - \hat{Y}_0(I, L_1, \ldots, L_{d-1}) - \hat{Y}_0(K, M_1, \ldots, M_{d-1}).
\]

Observe that this is a generalization of the definition of Neammanee and Rerkruthairat (Neammanee, K. & Rerkruthairat, N.). By the same argument of Neammanee and Rattanawong (Neammanee, K. & Rattanawong, P., 2008, p. 24-25), we have

\[
E^R(A') \leq \frac{4}{n} \sum_{r=1}^{n} \left(d - 1\right) \leq \frac{4}{n} (2d-1 - 1) \text{ and } |T_2| \leq \frac{n-1}{2} E[G^R(A')].
\]

By Lemma 2.3(3), we get that

\[
|T_2| \leq \frac{2(n-1)}{n} \sum_{r=1}^{n} \left(d - 1\right) E[|Y(\rho) - \hat{Y}(\rho)|^2] = O(\frac{1}{n}).
\]

Finally, we will bound \( T_1 \). Denote \( \Delta Y = \hat{Y}(\rho) - \bar{Y}(\tau) \). Again, by using the same argument of Neammanee and Rerkruthairat (Neammanee, K. & Rerkruthairat, N.), we have

\[
T_1 \leq B_1 + B_2 + B_3 + B_4 + B_5,
\]

where

\[
B_1 = E \int_{Y(\tau) \neq \bar{Y}(\tau)} K(t)dt, \quad B_2 = E \int_K |\bar{Y}(\tau)||\Delta Y|K(t)dt, \quad B_3 = E \int_K |\bar{Y}(\tau)||\hat{Y}(\tau)|dt,
\]

\[
B_4 = \frac{\sqrt{2\pi}}{4} E \int_K |\Delta Y|K(t)dt, \quad B_5 = \frac{\sqrt{2\pi}}{4} E \int_K |\hat{Y}(\tau)|dt, \quad \text{and}
\]

\[
|B_2 + B_3 + B_4 + B_5| \leq 26.9862d\delta_3 + 13.2496d\delta_3 + \left(\frac{C_d}{n}\right) \left[\frac{C_d^3}{n^3} + Cn^2\delta_3^2\right] + O\left(\frac{1}{n}\right).
\]

Hence, it suffices to bound \( B_1 \). For each \( \delta \geq 0 \) and \( a, b \in \mathbb{R} \), where \( a < b \), we define the function \( f_\delta \) by

\[
f_\delta(t) = \begin{cases} \frac{1}{2}(b-a) - \delta & \text{if } t < a - \delta, \\ \frac{1}{2}(b+a) + t & \text{if } a - \delta \leq t \leq b + \delta, \\ \frac{1}{2}(b-a) + \delta & \text{if } b + \delta < t. \end{cases}
\]

Then

\[
|f_\delta(t)| \leq \frac{1}{2}(b-a) + \delta \text{ for every } t \in \mathbb{R}.
\]

Note that \( E[|\Delta Y|^k] \) is bounded by a sum of \( (4d)^k \) terms each of the form \( E[|\bar{Y}_0(I, \rho(I), \ldots, \rho_{d-1}(I))|^k] \). This implies

\[
E[|\Delta Y|^k] \leq 4d \quad \text{and} \quad E[|\Delta Y|^k] \leq \frac{(4d)^k\delta_3}{n^k}
\]

for \( k \in [2, 3] \). Similar to \( T_4 \), we obtain

\[
|\Delta f_\delta(\bar{Y}(\rho))| \leq \frac{1.01461}{\sqrt{n}} (E[f_\delta^2(\bar{Y}(\rho))] + (2d + 6)\delta_3).
\]

By the same argument of Neammanee and Rerkruthairat (Neammanee, K. & Rerkruthairat, N.) Lemma 2.1, 2.3(1, 3) and (8) to (10), we have

\[
B_1 \leq \frac{3.12695}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + (2d + 6)\delta_3 + O\left(\frac{1}{n}\right).
\]
We use (6), (7) and (11) to conclude that

\[
T_1 \leq \frac{3.23695}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + 28.9862d\delta_3 + 19.24969\delta_3 + \left(\frac{C_d}{n^{\pi\delta_3^2}}\right) + \frac{Cn^{\frac{\delta_3^2}{n^2}}}{n^2} + O\left(\frac{1}{n}\right). \tag{12}
\]

By the same argument of (12), we have

\[
T_1 \geq -\frac{3.23695}{\sqrt{n}} - \frac{2.08919d}{\sqrt{n}} - 28.9862d\delta_3 - 19.24969\delta_3 - \left(\frac{C_d}{n^{\pi\delta_3^2}}\right) - \frac{Cn^{\frac{\delta_3^2}{n^2}}}{n^2} - O\left(\frac{1}{n}\right)
\]

(see Neammanee, K. & Rattanawong, P., 2008, p. 13) for more detail). Hence

\[
|T_1| \leq \frac{3.23695}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + 28.9862d\delta_3 + 19.24969\delta_3 + \left(\frac{C_d}{n^{\pi\delta_3^2}}\right) + \frac{Cn^{\frac{\delta_3^2}{n^2}}}{n^2} + O\left(\frac{1}{n}\right).
\]

Therefore,

\[
|P(W \leq z) - \Phi(z)| \leq 22.87635\delta_3 + \frac{3.87227}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + 28.9862d\delta_3 + 1.036\delta_3 + \left(\frac{C_d}{n^{\pi\delta_3^2}}\right) + \frac{Cn^{\frac{\delta_3^2}{n^2}}}{n^2} + O\left(\frac{1}{n}\right).
\]

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References


