Strong Convergence Theorem According to Hybrid Methods for Mapping Asymptotically Quasi-Nonexpansive Types

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Abstract

The purpose of this article is to prove strong convergence theorems for mapping of asymptotically quasi-nonexpansive types in a Hilbert space according to hybrid methods. The results obtained in this paper extend and improve upon those recently announced by Qin, X., Su, Y. and Shang, M. (Qin, X. et al., 2008), and many others.

Keywords: Asymptotically quasi-nonexpansive type, Metric projection, Uniformly L-Lipschitzian

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Further, for a mapping $T : C \to C$, let $\varnothing \neq F(T)$ be the set of all fixed points of T. A mapping $T : C \to C$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ of real number with $k_n \geq 1$ and $\lim_n k_n = 1$ such that for all $x \in C,\ q \in F(T)$

$$\|T^n x - q\| \leq k_n\|x - q\|, \text{ for all } n \geq 1. \tag{1}$$

T is called asymptotically quasi-nonexpansive type (Sahu, 2003) provided T is uniformly continuous, and

$$\limsup_n\{\sup_{x\in \mathrm{C}} (\|T^n x - q\| - \|x - q\|)\} \leq 0, \text{ for all } q \in \mathrm{F(T)}. \tag{2}$$

T is called uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in \mathrm{C} \text{ and all } n \geq 1. \tag{3}$$

Fixed point iteration processes for asymptotically quasi-nonexpansive type mappings in Banach spaces have been studied extensively by many authors, (Chang, 2004; Li, 2007; Puturong, 2008 & 2009; Quan, 2006; Sahu, 2003; Saluja, 2007; Tian, 2007). The other nonlinear mappings, which are all special cases of asymptotically quasi- nonexpansive type mappings, have been also studied both in Banach spaces and Hilbert spaces. Those nonlinear mappings are nonexpansive mappings, quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive type mappings. However, In recently years, the hybrid iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors as follows:

In 2003, Nakajo, K. and Takahashi, W. introduced an iterative scheme for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily},\\ y_n &= \alpha_n x_n + (1 - \alpha_n)T x_n,\\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},\\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} &= P_{C_n\cap Q_n}(x_0), \end{aligned} \tag{4}$$

where C is a closed convex subset of H, $P_K(x_0)$ denotes the matric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (4) converges strongly to $P_{F(T)}(X_0)$. Where $F(T)$ denote the fixed points set of T.

In 2006, Kim, T. H. and Xu, H. K introduced an iterative scheme for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{5}$$

where C is bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\text{diam C})^2 \to 0 \text{ as n} \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (5) converges strongly to $P_{F(T)}(x_0)$.

They also introduced an iterative scheme for asymptotically nonexpansive semigroup $\mathfrak{I}$ in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n &= \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{6}$$

where C is bounded closed convex subset and

$$\overline{\theta_n} = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](\text{diam } C)^2 \to 0 \text{ as } n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (6) converges strongly to $P_{F(\mathfrak{I})}(x_0)$. Where $F(\mathfrak{I})$ denote the common fixed points set of $\mathfrak{I}$.

In 2006, Carlos Martinez-Yanes and Hong-Kun Xu introduced an iterative scheme for nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + (1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle)\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{7}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \to 0$, then the sequence $\{x_n\}$ generated by (7) converges strongly to $P_{F(T)}(x_0)$.

They also introduced an iterative scheme for nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{8}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\alpha_n \to 0$, then the sequence $\{x_n\}$ generated by (8) converges strongly to $P_{F(T)}(x_0)$.

In 2007, Su, Y., Wang, D. and Shang, M. introduced an iterative scheme for quasi-nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \|z - y_n\| \le \|z - x_n\|\}, \\ C_0 &= \{z \in C : \|z - y_0\| \le \|z - x_0\|\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ Q_0 &= C, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{9}$$

where C is a closed convex subset of H. They proved that if $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (9) converges strongly to $P_{F(T)}(x_0)$.

In 2008, Inchan, I. and Plubtieng, S. introduced an iterative scheme for two asymptotically nonexpansive mappings S and T in a Hilbert space H:

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n &= \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \end{aligned} \tag{10}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](\text{diam C})^2 \to 0$ as $\text{n} \to \infty$ and $0 \le \alpha_n < 1, 0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N}$. They proved that the sequence $\{x_n\}$ generated by (10) converges strongly to $P_{F(S)\cap F(T)}(x_0)$.

They also introduced an iterative scheme for two asymptotically nonexpansive semigroups $\mathscr{S} = \{S(t) : 0 \le t \le \infty\}$ and $\mathscr{T} = \{T(t) : 0 \le t \le \infty\}$ in a Hilbert space H:

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ z_n &= \beta_n x_n + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \widetilde{\theta_n}\} \\ x_{n+1} &= P_{C_{n+1}}(x_0), \quad n \in \mathbb{N}, \end{aligned} \tag{11}$$

where $\widetilde{\theta_n} = (1-\alpha_n)\left[(\widetilde{t_n}^2 - 1) + (1-\beta_n)\widetilde{t_n}^2(\widetilde{s_n}^2 - 1)\right](\text{diam } C)^2 \to 0$ as $n \to \infty$ ($\widetilde{t_n} = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$ and $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_t^S dt$), $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t_n \to \infty, s_n \to \infty$.

They proved that the sequence $\{x_n\}$ generated by (11) converges strongly to $P_{F(\mathscr{S})\cap F(\mathscr{T})}(x_0)$.

In 2008, Qin, X., Su, Y. and Shang, M. introduced an iterative scheme for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ z_n &= \beta_n x_n + (1-\beta_n)T^n x_n, \\ y_n &= \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ &\qquad + (1-\alpha_n)[k_n^2\|z_n\|^2 - \|x_n\|^2 + (k_n^2 - 1)M + 2\langle x_n - k_n^2 z_n, v\rangle]\}, \\ Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{12}$$

where M is a appropriate constant such that $M > \|v\|^2$ for each $v \in C_n$. They proved that if $\{k_n\}$ is a sequence such that $k_n \to 1$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \le 1 - \delta$ for all n and for some $\delta \in (0,1]$ and $\beta_n \to 1$, then the sequence $\{x_n\}$ generated by (12) converges strongly to $P_{F(T)}(x_0)$.

They also introduced an iterative scheme for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)T^n x_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + (k_n^2 - 1 - \alpha_n k_n^2)\|x_n\|^2 \\ &\quad + \alpha_n\|x_0\|^2 - 2\langle \alpha_n x_0 + (k_n^2 - 1 - \alpha_n k_n^2)x_n, v\rangle + (1-\alpha_n)(k_n^2-1)M\}, \\ Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{13}$$

where M is a appropriate constant such that $M > \|v\|^2$ for each $v \in C_n$. They proved that if $\{k_n\}$ is a sequence such that $k_n \to 1$ as $n \to \infty$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ generated by (13) converges strongly to $P_{F(T)}(x_0)$.

The purpose of this paper is to prove strong convergence theorems for mapping of asymptotically quasi-nonexpansive types in Hilbert space. The results obtained in this paper extend and improve upon those recently announced by X. Qin, Y. Su, and M.Shang and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$; let C be a nonempty closed convex subset of H. In a real Hilbert space H, we have

$$\begin{aligned} \langle x+y, z\rangle &= \langle x, z\rangle + \langle y, z\rangle, \\ \langle \alpha x, y\rangle &= \alpha\langle x, y\rangle, \\ \langle x, y\rangle &= \overline{\langle y, x\rangle}, \\ \|x+y\|^2 &= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2, \\ \|x-y\|^2 &= \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2, \end{aligned}$$

for all $x, y, z \in H$.

Let $\{x_n\}$ be a sequence of H and let $x \in H$. Then, $\{x_n\}$ is said to converge weakly to x, denoted by $x_n \rightharpoonup x$, if for any $y \in H$, $\langle x_n, y\rangle \to \langle x, y\rangle$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for any $y \in C$. Such a P_C is called the metric projection of H onto C.

Recalling a well-known concept, and the following essential lemma, in order to prove our main results:

Lemma 2.1 *There holds the identity in a Hilbert space H:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.2 [Qin, 2008] *Let C be a closed convex subset of a real Hilbert space H and let P_c be the metric projection from H onto C. Given $x \in H$ and $z \in C$. Then $z = P_C(x)$ if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \le 0 \quad \text{for all } y \in C.$$

Lemma 2.3 [Takahashi, 2009] *Let H be a real Hilbert space and let $\{x_n\}$ be a bounded sequence of H such that $x_n \to x$. Then the following inequality hold:*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Lemma 2.4 [Takahashi, 2009] *Let H be a real Hilbert space and let $\{x_n\}$ be a sequence of H. If $x_n \to x$*, then $x_n \rightharpoonup x$.

Lemma 2.5 [Takahashi, 2009] *Let H be a real Hilbert space and suppose $x_n \rightharpoonup x$.* Then $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ for all $y \in H$ with $x \neq y$.

3. Main Results

In this section, we provide proof strong convergence theorems for a mapping of asymptotically quasi-nonexpansive type in a Hilbert space by hybrid methods.

Lemma 3.1 *Let C be a nonempty closed convex subset of Hilbert space H, and $T : C \to C$ be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set $F(T)$ and T is uniformly L-Lipschitzian. Put*

$$G_n = \max\{0, \sup_{x\in C}[\|T^n x - p\| - \|x - p\|]\}$$

for all $n \geq 1$, for all $p \in F(T)$ so that $\sum_{n=1}^{\infty} G_n < \infty$. Then $F(T)$ is a closed and convex.

Proof We first show that $F(T)$ is closed. To see this, let $\{x_n\}$ be a sequence in $F(T)$ with $x_n \to x$, we shall prove that $x \in F(T)$. Then we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx$$

and hence $x \in F(T)$. This implies that $F(T)$ is closed. Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\lambda \in (0, 1)$. We show $z = \lambda x + (1-\lambda)y \in F(T)$. From Lemma 2.1, we have

$$\begin{aligned}
\|z - T^n z\|^2 &= \|\lambda x + (1-\lambda)y - \lambda T^n z - (1-\lambda)T^n z\|^2 \\
&= \|\lambda(x - T^n z) + (1-\lambda)(y - T^n z)\|^2 \\
&= \lambda\|x - T^n z\|^2 + (1-\lambda)\|y - T^n z\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\
&= \lambda\left[(\|T^n z - x\| - \|z - x\|) + \|z - x\|\right]^2 \\
&\quad + (1-\lambda)\left[(\|T^n z - y\| - \|z - y\|) + \|z - y\|\right]^2 - \lambda(1-\lambda)\|x-y\|^2 \\
&= \lambda\Big[(\|T^n z - x\| - \|z - x\|)^2 + 2\left(\|T^n z - x\| - \|z - x\|\right) \\
&\quad (\|z - x\|) + \|z - x\|^2\Big] + (1-\lambda)\Big[(\|T^n z - y\| - \|z - y\|)^2 \\
&\quad + 2\left(\|T^n z - y\| - \|z - y\|\right)(\|z - y\|) + \|z - y\|^2\Big] - \lambda(1-\lambda)\|x-y\|^2 \\
&\leq \lambda\left[\sup_{q\in C}(\|T^n q - x\| - \|q - x\|)\right]^2 + 2\lambda\left[\sup_{q\in C}(\|T^n q - x\| - \|q - x\|)\right](\|z - x\|) \\
&\quad + \lambda\|z - x\|^2 + (1-\lambda)\left[\sup_{q\in C}(\|T^n q - y\| - \|q - y\|)\right]^2 \\
&\quad + 2(1-\lambda)\left[\sup_{q\in C}(\|T^n q - y\| - \|q - y\|)\right](\|z - y\|) + (1-\lambda)\|z - y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\
&\leq \lambda G_n^2 + 2\lambda G_n(\|z - x\|) + (1-\lambda)G_n^2 + 2(1-\lambda)G_n(\|z - y\|) + \|\lambda(z - x) + (1-\lambda)(z - y)\|^2 \\
&= G_n^2 + 2\lambda G_n(\|z - x\|) + 2(1-\lambda)G_n(\|z - y\|).
\end{aligned}$$

Which implies that $\|z - T^n z\| \to 0$. We obtain,

$$\|z - Tz\| \leq \|z - T^{n+1}z\| + \|T^{n+1}z - Tz\| \leq \|z - T^{n+1}z\| + L\|T^n z - z\| \to 0,$$

which implies that $Tz = z$. So, we get $z \in F(T)$. This implies that $F(T)$ is convex.

Theorem 3.1 *Let C be a nonempty bounded closed convex subset of Hilbert space H and let $T : C \to C$ be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set $F(T)$ and T is uniformly L-Lipschitzian. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that $\alpha_n \leq 1 - \delta$ for all n and for some $\delta \in (0, 1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
x_0 &\in C \text{ chosen arbitrarily}, \\
z_n &= \beta_n x_n + (1-\beta_n)T^n x_n, \\
y_n &= \alpha_n x_n + (1-\alpha_n)T^n z_n, \\
C_n &= \{v \in C : \|y_n - v\|^2 \leq \alpha_n\|x_n\|^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)\|z_n\|^2 \\
&\qquad - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, v\rangle + \|v\|^2\}, \\
Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\
x_{n+1} &= P_{C_n\cap Q_n}(x_0),
\end{aligned}$$

where $M = 2(1-\alpha_n)(diam\ C)$ and $G_n = \max\{0, \sup_{x\in C}[\|T^n x - p\| - \|x - p\|]\}$ for all $n \geq 1$, for all $p \in F(T)$ so that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\{x_n\}$ converges to $P_{F(T)}x_0$.

Proof It is obvious that for $n \in \mathbb{N} \cup \{0\}$, $\mathbb{Q}_n$ is closed and convex. We show that C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. Let $\{v_m\}_{m=1}^{\infty} \subseteq C_n \subset C$ with $v_m \to v$ as $m \to \infty$. Since C is closed and $v_m \in C_n$, we have $v \in C$ and

$$\|y_m - v_m\|^2 \leq \alpha_n\|x_n\|^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)\|z_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, v_m\rangle + \|v_m\|^2.$$

Then

$$\begin{aligned}||y_n - v||^2 &= ||y_n - v_m + v_m - v||^2\\ &= ||y_n - v_m||^2 + ||v_m - v||^2 + 2\langle y_n - v_m, v_m - v\rangle\\ &\le ||y_n - v_m||^2 + ||v_m - v||^2 + 2||y_n - v_m||||v_m - v||\\ &\le \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, v_m\rangle\\ &\quad + ||v_m||^2 + ||v_m - v||^2 + 2||y_n - v_m||||v_m - v||.\end{aligned}$$

Taking $m \to \infty$

$$||y_n - v||^2 \le \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, v\rangle + ||v||^2.$$

Then $v \in C_n$ and hence C_n is closed.

Let $x, y \in C_n \subset C$ with $z = \lambda x + (1-\lambda)y$ where $\lambda \in (0,1)$. Since C is convex, $z \in C$. Thus, we have

$$||y_n - x||^2 \le \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, x\rangle + ||x||^2.$$

and

$$||y_n - y||^2 \le \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, y\rangle + ||y||^2.$$

Hence

$$\begin{aligned}||y_n - z||^2 &= ||y_n - (\lambda x + (1-\lambda)y)||^2\\ &= ||\lambda(y_n - x) + (1-\lambda)(y_n - y)||^2\\ &= \lambda||y_n - x||^2 + (1-\lambda)||y_n - y||^2 - \lambda(1-\lambda)||y - x||^2\\ &\le \lambda\alpha_n||x_n||^2 + \lambda(1-\alpha_n)G_n^2 + \lambda MG_n + \lambda(1-\alpha_n)||z_n||^2 - 2\lambda\langle \alpha_n x_n + (1-\alpha_n)z_n, x\rangle\\ &\quad + \lambda||x||^2 + (1-\lambda)\alpha_n||x_n||^2 + (1-\lambda)(1-\alpha_n)G_n^2 + (1-\lambda)MG_n + (1-\lambda)(1-\alpha_n)||z_n||^2\\ &\quad - 2(1-\lambda)\langle \alpha_n x_n + (1-\alpha_n)z_n, y\rangle + (1-\lambda)||y||^2 - \lambda(1-\lambda)||y - x||^2\\ &= \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, \lambda x + (1-\lambda)y\rangle\\ &\quad + ||\lambda x + (1-\lambda)y||^2\\ &= \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, z\rangle + ||z||^2.\end{aligned}$$

It follows that $z \in C_n$ and hence C_n is closed and convex. Then $C_n \cap Q_n$ is closed and convex.

Next, we show that $F(T) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Indeed, let $p \in F(T)$, we have

$$\begin{aligned}||y_n - p||^2 &= ||\alpha_n(x_n - p) + (1-\alpha_n)(T^n z_n - p)||^2\\ &= \alpha_n||x_n - p||^2 + (1-\alpha_n)||T^n z_n - p||^2 - \alpha_n(1-\alpha_n)||x_n - T^n z_n||^2\\ &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)||T^n z_n - p||^2\\ &= \alpha_n||x_n - p||^2 + (1-\alpha_n)[(||T^n z_n - p|| - ||z_n - p||) + ||z_n - p||]^2\\ &= \alpha_n||x_n - p||^2 + (1-\alpha_n)(||T^n z_n - p|| - ||z_n - p||)^2\\ &\quad + 2(1-\alpha_n)(||T^n z_n - p|| - ||z_n - p||)(||z_n - p||) + (1-\alpha_n)||z_n - p||^2\\ &\le \alpha_n||x_n - p||^2 + (1-\alpha_n)[\sup_{x\in C}(||T^n x - p|| - ||x - p||)]^2\\ &\quad + 2(1-\alpha_n)[\sup_{x\in C}(||T^n x - p|| - ||x - p||)](||z_n - p||) + (1-\alpha_n)||z_n - p||^2\\ &\le \alpha_n||x_n||^2 - 2\langle \alpha_n x_n, p\rangle + \alpha_n||p||^2 + (1-\alpha_n)G_n^2\\ &\quad + 2(1-\alpha_n)G_n(diam\ C) + (1-\alpha_n)||z_n - p||^2.\\ &= \alpha_n||x_n||^2 - 2\langle \alpha_n x_n, p\rangle + \alpha_n||p||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle (1-\alpha_n)z_n, p\rangle\\ &\quad + ||p||^2 - \alpha_n||p||^2\\ &= \alpha_n||x_n||^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)||z_n||^2 - 2\langle \alpha_n x_n + (1-\alpha_n)z_n, p\rangle + ||p||^2. \qquad (14)\end{aligned}$$

It follows from (14) that $p \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, we get $F(T) \subset C_n$. Let us show by induction that $F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. In the case of $n = 0$, we have $F(T) \subset C_0$ and $Q_0 = C$. So, we get $F(T) \subset C_0 \cap Q_0$. Suppose that $F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. Since $C_k \cap Q_k$ is closed and convex, we can define $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. From $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, by Lemma 2.2 we have

$$\langle x_0 - x_{k+1}, x_{k+1} - z\rangle \geq 0 \quad \text{for all } z \in C_k \cap Q_k.$$

Since $F(T) \subset C_k \cap Q_k$, we also have

$$\langle x_0 - x_{k+1}, x_{k+1} - u\rangle \geq 0 \quad \text{for all } u \in F(T).$$

So, we get $F(T) \subset Q_{k+1}$. Then we obtain $F(T) \subset C_{k+1} \cap Q_{k+1}$. Next, let us show that $\{x_n\}$ is bounded. Since $F(T)$ is a closed and convex. Put $z_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, we get

$$\|x_{n+1} - x_0\| \leq \|z - x_0\| \quad \text{for all } z \in C_n \cap Q_n.$$

From $z_0 \in F(T) \subset C_n \cap Q_n$, we also have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

and hence $\{x_n\}$ is bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and from the definition of Q_n we have $x_n = P_{Q_n}(x_0)$, we get $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. From boundedness of $\{x_n\}$, we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. So, we obtain $(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2) \to 0$. On the other hand, from $x_{n+1} \in Q_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0.$$

So, for all $n \in \mathbb{N} \cup \{0\}$ we get

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) - (x_{n+1} - x_0)\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+1} - x_0\rangle + \|x_{n+1} - x_0\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_{n+1}, x_0 - x_n\rangle \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

This implies

$$\|x_{n+1} - x_n\| \to 0. \tag{15}$$

However, since $\lim_{n\to\infty} \beta_n = 1$ and $\{x_n\}$ is bounded, we obtain

$$\|z_n - x_n\| = \|\beta_n x_n - x_n\| + (1 - \beta_n)\|T^n x_n - x_n\| \to 0. \tag{16}$$

Since $\|z_n - x_{n+1}\| \leq \|z_n - x_n\| + \|x_n - x_{n+1}\|$. It follows from (15) and (16) that

$$\|z_n - x_{n+1}\| \to 0. \tag{17}$$

On the other hand, it follows from $x_{n+1} \in C_n$ that

$$\begin{aligned}
\|y_n - x_{n+1}\|^2 &\leq \alpha_n\|x_n\|^2 + (1 - \alpha_n)G_n^2 + MG_n + (1 - \alpha_n)\|z_n\|^2 - 2\langle \alpha_n x_n + (1 - \alpha_n)z_n, x_{n+1}\rangle + \|x_{n+1}\|^2 \\
&= \alpha_n\|x_n\|^2 + (1 - \alpha_n)\|z_n\|^2 + \|x_{n+1}\|^2 - 2\alpha_n\langle x_n, x_{n+1}\rangle - 2\langle z_n, x_{n+1}\rangle \\
&\quad + 2\alpha_n\langle z_n, x_{n+1}\rangle + (1 - \alpha_n)G_n^2 + MG_n \\
&= \alpha_n\|x_n\|^2 + \|z_n\|^2 - \alpha_n\|z_n\|^2 + \|x_{n+1}\|^2 - 2\alpha_n\langle x_n, x_{n+1}\rangle - 2\langle z_n, x_{n+1}\rangle \\
&\quad + 2\alpha_n\langle z_n, x_{n+1}\rangle + (1 - \alpha_n)G_n^2 + MG_n \\
&= \|z_n\|^2 - 2\langle z_n, x_{n+1}\rangle + \|x_{n+1}\|^2 + \alpha_n\|x_n\|^2 - 2\alpha_n\langle x_n, x_{n+1}\rangle + \alpha_n\|x_{n+1}\|^2 \\
&\quad - \alpha_n\|z_n\|^2 + 2\alpha_n\langle z_n, x_{n+1}\rangle - \alpha_n\|x_{n+1}\|^2 + (1 - \alpha_n)G_n^2 + MG_n \\
&= \|z_n - x_{n+1}\|^2 + \alpha_n\|x_n - x_{n+1}\|^2 - \alpha_n\|z_n - x_{n+1}\|^2 + (1 - \alpha_n)G_n^2 + MG_n.
\end{aligned}$$

It follows from (15), (17) and $\sum_{n=1}^{\infty} G_n < \infty$ that

$$\|y_n - x_{n+1}\| \to 0 \tag{18}$$

It follows from (15) and (18) that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{19}$$

Again, noticing that $T^n z_n = y_n + \alpha_n \|T^n z_n - x_n\|$, we have

$$\|T^n z_n - x_n\| = \|y_n - x_n\| + \alpha_n \|T^n z_n - x_n\|.$$

It follows that $\|T^n z_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\|$. Since $\alpha_n \le 1-\delta$ and T is uniformly L-Lipschitzian, we have

$$\begin{aligned}\|T^n x_n - x_n\| &\le \|T^n x_n - T^n z_n\| + \|T^n z_n - x_n\| \\ &\le L\|x_n - z_n\| + \frac{1}{\delta}\|y_n - x_n\|.\end{aligned}$$

Therefore, it follows from (16) and (19) that

$$\|T^n x_n - x_n\| \to 0.$$

We obtain

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\le L\|x_n - T^n x_n\| + (L+1)\|x_n - x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\|,\end{aligned}$$

which implies that

$$\|Tx_n - x_n\| \to 0.$$

Since $\{x_n\}$ is bounded, there exists a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$. Assume $q \ne Tq$. From Opial's theorem (Lemma 2.5), we have

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - q\| &< \liminf_{i\to\infty} \|x_{n_i} - Tq\| \\ &\le \liminf_{i\to\infty} (\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - Tq\|) \\ &\le \liminf_{i\to\infty} (L\|x_{n_i} - q\|) \\ &= (L)\liminf_{i\to\infty} \|x_{n_i} - q\|.\end{aligned}$$

This is a contradiction. So, we have $q = Tq$, and hence $q \in F(T)$. From $z_0 = P_{F(T)}x_0$, Lemma 2.3, 2.4 and $\|x_{n+1} - x_0\| \le \|z_0 - x_0\|$, we have

$$\|x_0 - z_0\| \le \|x_0 - q\| \le \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \le \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \le \|x_0 - z_0\|.$$

So, we get $\lim_{i\to\infty}\|x_0 - x_{n_i}\| = \|x_0 - q\| = \|x_0 - z_0\|$.

It follows that $x_0 - x_{n_i} \to x_0 - z_0$; hence, $x_{n_i} \to z_0$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to z_0$. This complete the proof.

Theorem 3.2 *Let C be a nonempty bounded closed convex subset of Hilbert space H and let $T : C \to C$ be an asymptotically quasi-nonexpansive type mapping with nonempty fixed point set $F(T)$ and T is uniformly L-Lipschitzian. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \to 0$ as $n \to \infty$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)T^n x_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \le \alpha_n\|x_0\|^2 + (1-\alpha_n)G_n^2 + MG_n + (1-\alpha_n)\|x_n\|^2 \\ &\qquad - 2\langle \alpha_n x_0 + (1-\alpha_n)x_n, v\rangle + \|v\|^2\}, \\ Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned}$$

where $M = 2(1-\alpha_n)(diam\ C)$ and $G_n = \max\{0, \sup_{x\in C}[\|T^n x - p\| - \|x - p\|]\}$ for all $n \ge 1$, for all $p \in F(T)$ so that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\{x_n\}$ converges to $P_{F(T)}x_0$.

Proof Similarly as in the proof of Theorem 3.1, we can get the proof is completed.

References

Chang, S. S., Kim, K. & Kang, S. M. (2004). Approximating fixed points of asymptotically quasi-nonexpansive type mappings by the Ishikawa iterative sequences with mixed errors.*Dynamic Systems and Applications*, 13, 179-186.

Inchan, I. & Plubtieng, S. (2008). Strong convergence theorems of hybrid methods for two asymptotically nonexpansive mappings in Hilbert spaces. *Nonlinear Analysis*, 2, 1125-1135.

Kim, T. H. & Xu, H. K. (2006). Strong convergence of modified Mann iterations for asymptotically mappings and semigroups. *Nonlinear Analysis*, 64, 1140-1152.

Li, J., Huang, N. J. & Cho, Y. J. (2007). Stability of iterative sequences approximating common fixed point for a system of asymptotically quasi-nonexpansive type mappings. *Kyungpook Mathematical Journal*, 47, 81-89.

Martinez-Yanesa, C. & Xu, H. K. (2006). Strong convergence of the CQ method for fixed point Iteration processes. *Nonlinear Analysis*, 64, 2400-2411.

Nakajo, K. & Takahashi, W. (2003). Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *Journal of Mathematical Analysis and Application*, 279, 372-379.

Puturong, N. (2008). Convergence of three step mean value iteratives scheme of theorems for a mapping of asymptotically quasi-nonexpansive type. *Silpakon University Science an Technology Journal*, 2, 29-36.

Puturong, N. (2009). Approximating common fixed points of one-step iterative scheme with error for asymptotically quasi-nonexpansive type nonself-mappings. *Kyungpook Mathematical Journal*, 49, 667-674.

Qin, X., Su, Y. & Shang, M. (2008). Strong convergence theorems for asymptotically nonexpansive mappings by hybrid methods. *Kyungpook Mathematical Journal*, 48, 133-142.

Quan, J., Chang, S. S. & Long, X. J. (2006). Approximation common fixed points of asymptotically quasi-nonexpansive type mappings by the finite steps iteratives sequences. *Fixed Point Theory and Applications*, 2006, 1-8.

Sahu, D. R. & Jung, J. S. (2003). Fixed point iteration processes for non-Lipschitzian mappings of asymptotically quasi-nonexpansive type. *International Journal of Mathematics and Mathematical Sciences*, 2003, 2075-2081.

Saluja, G. S. (2007). Strong convergence theorems of three step iterative sequences with errors for asymptotically quasi-nonexpansive type nonself mappings. *Advances in Applied Mathematical Analysis*, 2, 57-69.

Schu, J. (1991). Iterative construction of fixed point of asymptotically nonexpansive mappings. *Journal of Mathematical Analysis and Applications*, 158, 407-413.

Su, Y., Wang, D. & Shang, M. (2007). Strong convergence of monotone hybrid algorithm for quasi-nonexpansive mappings. *International Journal of Mathematics Analysis*, 25, 1235-1241.

Takahashi, W. (2009) *Introduction to Nonlinear and Convex Analysis*. Yokohama: Publishers Japan.

Tian, Y. X., Chang, S. S. & Huang, J. L. (2007). On the approximation problem of common fixed points for a finite family of non-self asymptotically quasi-nonexpansive type mappings in Banach spaces. *Computers and Mathematices with Application*, 53, 1847-1853.