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Results on Existence and Uniqueness of Solution Of Impulsive Neutral Integro-Differential System

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Abstract

In this work, the solution of the impulsive neutral integro-differential system is analysed. The Du Bois–Reymond's assumptions on solution variation of piecewise smooth functions are used to establish the existence of only the impulsive term as a solution of the system at the points of discontinuity. The theories of infinitesimal generator of a strongly continuous compact semigroup is used to formulate theorems on existence and uniqueness of system solution, and proves are provided using an approximate piecewise continuous, compact operator and a continuous positive non decreasing function $\psi:[0,\infty)\to(0,\infty)$. Results obtained are improvement on the qualitative analysis of impulsive neutral integro-differential system

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1. Introduction

An impulsive neutral integro-differential equation is an equation which involves both the integral and derivatives of the unknown function, with time lag incorporated in both the state and derivative of the system (which describe the historical value of the rate of change and the state) and a coupled impulsive term showing the abrupt changes of the state at certain moments of time between intervals of continuous evolution. This equation has wide application in various evolutionary processes including population dynamics, aeronautics, economics and engineering. There have been increasing interests in the analysis of the qualitative properties of the impulsive neutral integro-differential equation, mostly on the theories of existence and uniqueness of the system solution (see; Bainov, Myshkis & Zahariev, 1987; Bainov & Simeonov, 1985; Benchohra & Ntouyas, 2006; Haddad, Chellaboina & Nersesor, 2008; Hale & Kato, 1987; Igobi, Eni, Eteng & Atsu, 2011; Isaac & Lipscey, 2010; Jiang & Shen, 2011).

Some researchers have employed the approximation of the impulsive neutral integro-differential equation as an integro-differential equation coupled with a difference equation to be satisfied at certain fixed or variable impulse times. The resulting solutions are thereby piecewise continuous (with discontinuity at the impulse times). This approach ensures that the well-established results for integro-differential equations are utilised to developed theories on existence and uniqueness of the system solution (Agawal & Saker, 2001; Ballinger, 1999; Benchohra & Ntouyas, 2006; Bao & Hou, 2010; Diop, Ezzinbe & Zene, 2015; Diop, Ezzinbe & Lo, 2012; Kavilla, Arjunan & Ravichandran, 2014; Li & N'Guerekata, 2010).

Alternative approach used by Halanay and Wexler (1968) and Pandit and Deo (1982) involves defining a measure differential equation (incorporating Dirac delta functions) where the derivative involved is a distributional derivative. The points at which impulses occur are fixed, generalized functions are considered and the resulting solutions are of bounded variation. The disadvantage of this approached is that most classical theory cannot be applied to these types of systems.

In this research, the first approach is employed to analyse the solution of the impulsive neutral integro-differential system in the Banach space (X). The Du Bois–Reymond's assumptions on solution variation of piecewise smooth functions are used to establish the impulsive term as the only solution term of the system at the points of discontinuity. The theories of infinitesimal generator of a strongly continuous compact semigroup is used to formulate theorems on existence and uniqueness of system solution, and proves are provided using an approximate piecewise continuous, compact operator and a continuous positive non decreasing function $\psi:[0,\infty)\to(0,\infty)$.

2. Preliminary Results

Consider a piecewise continuous linear space $PC(J,X)$, for $J(t_0,T)\subset R_+$ *and* X a Banach space. Let $x_t = x(t-r)\in PC([J,X)$ defined a delay function, with a delay constant $r>0$, and $f: J\times X\to R$ be a piecewise continuous, compact operator in X such that for $x(t)\in PC(J,X)$, the impulsive neutral integro-differential equation is of the form

$$\begin{aligned} \dot{x}(t)+Ax(t) &= Dx_t+\int_{t_0-r}^{t} f(s,\dot{x}(s))ds \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)) \\ x(t_0) &= \phi_0, \end{aligned} \tag{2.1}$$

where $\phi_0\in PC(J,R^n)$, is the initial data at any time t_0, and for $I: J\times X\to R^n$, $\Delta x|_{t=t_k} = I_k(x(t_k^-)) = x(t_k^+)-x(t_k^-): k=1,\ldots,m$, defined the impulsive term experienced at points t_k, $k=1,2,\ldots,m$, such that $x(t_k^+)$, $x(t_k^-)$ represent the right and the left limit of $x(t)$ at $t=t_k$ respectively and D is a differential operator of the delay function x_t. A is an infinitesimal generator of a strongly continuous compact semigroup $R(.)$ in X.

Definition 2.0

A matrix function $t\to\phi(t)\in PC(J,R^n)$ is called a matrix solution of the homogeneous linear system (2.1) if each of its columns is a vector solution. A matrix solution ϕ is called the fundamental matrix solution of (2.1) if its columns form a fundamental set of solutions whose columns are linearly independent, and $\phi(t_0)$ is invertible.

Definition 2.1

Let A be an infinitesimal generator of a strongly continuous compact semigroup $R(.)$ satisfying the Chapman-Kolmogorov identities

i. $R(t,t)=I$

ii. $R(t,s)R(s,u)=R(t,u)$

iii. $R(t,s)^{-1}=R(s,t)$.

Then, there exists a strongly continuous exponentially bounded family $R(t,t_0)=\phi(t)\phi(t_0)^{-1}=e^{-A(t-t_0)}$ known as the resolvent matrix of equation (2.1)

Definition 2.2

The function $x(t)\in PC(J,X)$, is said to be a mild solution of (2.1) if

i. $x(t)$ is continuous at each $t\in J(t_0,T)\subset R_+$

ii. the derivative of $x(t)$ exists and satisfies equation (2.1) for $t\in[t_{0,}t_k]$, and

$\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k=1,\ldots,m$ and

iii. there exists a continuous function $\eta: t\to x_t$ and a continuous positive non decreasing function

$\psi:[0,\infty)\to(0,\infty)$, such that for any $p\in L^1(J,R_+)$, $\|\eta(t,x_t)\|\le p(t)\psi(\|x_t\|_X)$,

so that

$$x(t)=R(t,t_0)\phi_0+\int_{t_0}^{t}R(t,s)\big(Dx_s+\eta(s,x_s)\big)ds\quad+\sum_{k=1}^{n}I(t_k x(t_k^-)), \tag{2.2}$$

where $\eta(t,x_t)=\int_{t_0-r}^{t}f(s,\dot{x}(s))ds$.

Hypothesis 2.1

Assume the following Du Bois–Reymond's assumptions on solution variation of piecewise smooth function hold

$$\left.\begin{aligned} c_k &= \frac{1}{t_k-t_{k-1}}\int_{t_{k-1}}^{t_k}R(t,s)ds,\\ x_{t_k} &= \int_{t_{k-1}}^{t_k}(R(t,s)-c_k)ds\quad for\ t\in(t_0,t_k),\ t\neq t_k,\ k=1,2,\ldots \end{aligned}\right\} \tag{2.3}$$

for $t\in(t_0,t_k),\ k=1,2,3,\ldots$, and $R(t,s)$ the resolvent matrix.

Lemma 2.1

Let $x(t)\in PC(J,X)$ be a mild solution of system (2.1), then for all $x_t\in v_0(t,t_k)\subseteq J(t_0-r,t)$, $R(t,t_0)$ is constant at finite number of points $t=t_k:k=1,2,\ldots$, and $\int_{t_0}^{t_k}R(t,s)Dx_s ds=0$ Moreover, there exists a constant c_k satisfying hypothesis (2.1) such that for $t\in[t_0,t_k)$, $R(t,t_0)=c$ at $t=t_k:k=1,2,\ldots$, and $\int_{t_0}^{t}R(t,s)Dx_s ds+\int_{t_0}^{t}R(t,s)\eta(s,x_s)ds=0$.

Then the mild solution

$$x(t)=\sum_{k=1}^{n}I(t_k x(t_k^-))\ \ on\ t_k;\ k=1,2,\ldots$$

Proof

Assume hypothesis (2.1) hold so that,

$$\begin{aligned} x_{t_1} &= \int_{t_0}^{t_1}(R(t,s)-c_1)ds=\int_{t_0}^{t_1}(R(t,s)ds-c_1(t_1-t_0)\\ &= \int_{t_0}^{t_1}(R(t,s)ds-\int_{t_0}^{t_1}(R(t,s)ds=0\\ x_{t_2} &= \int_{t_1}^{t_2}(R(t,s)-c_2)ds=\int_{t_1}^{t_2}(R(t,s)ds-c_2(t_2-t_1)\\ &= \int_{t_1}^{t_2}(R(t,s)ds-\int_{t_1}^{t_2}(R(t,s)ds=0\\ x_{t_k} &= \int_{t_{k-1}}^{t_k}(R(t,s)-c_k)ds=\int_{t_{k-1}}^{t_k}(R(t,s)ds-c_k(t_k-t_{k-1})\\ &= \int_{t_{k-1}}^{t_k}(R(t,s)ds-\int_{t_{k-1}}^{t_k}(R(t,s)ds=0. \end{aligned}$$

Hence

$$\int_{t_0}^{t_k} R(t,s)Dx_s ds = 0 \text{ , for } \quad x_{t_k} = 0, \quad k = 1,\ 2,\ 3,\ ....$$

and so for

$$\int_{t_0}^{t} R(t,s)Dx_s ds = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} R(t,s)Dx_s ds = 0, \quad for\ t = t_k, k = 1, 2, ...$$

Again, assume there exists a monotone function $g(t) \le c_k$ such that

$$(t_k - t_{k-1})\,\dot{g}(t) \le R(t,t_0),$$

and

$$g(t) \le \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} R(t,s)ds = \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} cd \quad = c\,.$$

Then by equation (2.2)

$$\begin{aligned} x(t) &= R(t,t_0)\phi_0 + (t_k - t_{k-1})\left(\int_{t_0}^{t_k} \dot{g}(s)Dx_s ds \ + \ \int_{t_0}^{t_k} \dot{g}(s)\eta(s,x_s ds \right) + \sum_{k=1}^{n} I(t_k, x(t_k^-)). \\ &= R(t,t_0)\phi_0 + (t_k - t_{k-1})\left(\int_{t_0}^{t_k} \dot{g}(s)Dx_s ds \ + \ \left[g(s)\eta(s,x_s)\right]_t^{t_k} - \int_{t_0}^{t_k} g(s)\dot{\eta}(s,x_s ds \right) + \sum_{k=1}^{n} I(t_k, x(t_k^-)). \\ &= R(t,t_0)\phi_0 + \frac{(t_k - t_{k-1})}{(t_k - t_{k-1})}\left(\int_{t_0}^{t_k} R(t,s)Dx_s ds \right) + (t_k - t_{k-1})\left(\ \left[c\eta(s,x_s)\right]_t^{t_k} - \int_{t_0}^{t_k} c\,\dot{\eta}(s,x_s ds \right) + \sum_{k=1}^{n} I(t_k, x(t_k^-)). \\ &= R(t,t_0)\phi_0 \ + \sum_{k=1}^{n} I(t_k, x(t_k^-)). \end{aligned}$$

For $R(t,t_0) = c = 0, at\ t = t_k,\ k = 1,\ 2,\ 3....,$

$$x(t) = \sum_{k=1}^{n} I(t_k, x(t_k^-))$$

Corollary 2.0

Given an initial data $\phi_0 \in PC([J,X),$ and $R(t,t_0)$ satisfying definition (2.1), then the mild solution $x(t) \in PC(J,X)$ is,

i. $$x(t) = R(t,t_0)\phi_0 + \int_{t_0}^{t} R(t,s)\left(Dx_s + \eta(s,x_s)\right)ds \quad + \sum_{k=1}^{n} I(t_k x(t_k^-)), \quad t = [t_0, T)$$

otherwise,

ii. $x(t) = \sum_{k=1}^{n} I(t_k, x(t_k^-))$, for $t = t_k,\ k = 1, 2, ...$

3. Main Result

Hypothesis 3.0

H_1. Let $\eta : J \times X \to R$ be a piecewise continuous, compact operator in X, such that for any continuous function $\mu \in X$, $\|x(t)\| \le \|\mu(t)\|,\ t \in [t_0, T]$ and $\|Dx_t + \eta(t, x_t)\| \le \|\eta(t,\mu)\|$.

H₂. For $I : J \times X \to R^n$, such that $\Delta x|_{t=t_k} = I_k(x(t_k^-));\ k = 1, \ldots, m$, there exists a constant $B_k > 0$ such that $\sup_{t\in J}\|I(x(t_k^-))\| \le B_k$ and $\|I(x(t_k^-)) - I(x^*(t_k^-))\| \le B_k |x - x^*|$.

H₃. Given the coefficient matrix A, and for $t, t_0 \in J$, then $\|R(t,t_0)\| = e^{\|A(t-t_0)\|} \le L$ for $L \ge 0$

Theorem 3.0

Suppose assumptions H_1, H_2, H_3 of (3.0) hold, such that for any continuous positive non decreasing function $\psi : [0,\infty) \to (0,\infty)$, and $p \in L^1(J, R_+)$, $\|\eta(t,\mu)\| \le p(t)\psi(\|\mu\|_X)$ satisfies

$$\int_{N_{k-1}+B_k}^{\mu_k(t)} \frac{d\mu}{\psi(\|\mu\|)} \le \int_{t_{k-1}}^{t} R(t,s)p(s)ds, \quad k = 1,\ldots,m+1,$$

where $N_0 = L\|\phi_0\|$, and for $k = 2,\ldots,m+1,\quad N_{k-1} = M_{k-1} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-1}}^{t} R(t,s)p(s)ds\right)$,

$$\Gamma_k(z) = \int_{N_k+B_{k+1}}^{z} \frac{d\mu}{\psi(\|\mu\|)}, \quad z \ge N_{k-1},\ k = 1,\ldots,m+1.$$

Then, there exists a mild solution $x(t)$ of (2.3), such that

$$\sup\{\|x(t)\| : t \in [t_{k-1}, t_k]\} \le M_{k-1},\ k = 1,\ldots,m+1.$$

Consequently, for each possible mild solution $x(t)$ of (2.2)

$$\|x(t)\| \le \max\{L\|\phi_0\|,\ N_{k-1}, k = 2,\ldots,m+1\} := b^*$$

Proof

By equation (2.2),

$$x(t) = R(t,t_0)\phi_0 + \int_{t_0}^{t} R(t,s)\left(Dx_s + \eta(s,x_s)\right)ds + \sum_{k=1}^{n} I(t_k x(t_k^-)), \quad t \in [t_0, t_k] ,$$

so that

$$\begin{aligned}
\|x(t)\| &= \|R(t,t_0)\varphi_0\| + \int_{t_0}^{t} \|R(t,s)\left(Dx_s + \eta(s,x_s)\right)\|ds + \|I(t_k, x(t_k^-))\|, \\
&\le \|R(t,t_0)\|\|\varphi_0\| + \int_{t_0}^{t} \|R(t,s)\eta(t,u)\|ds + \|I(t_k, x(t_k^-))\| \\
&\le \|R(t,t_0)\|\|\varphi_0\| + \int_{t_0}^{t} \|R(t,s)\|\|\eta(t,u)\|ds + \|I(t_k, x(t_k^-))\| \\
&\le L\|\varphi_0\| + L\int_{t_0}^{t} p(s)\psi(\|\mu\|)ds + \|I(t_k, x(t_k^-))\|
\end{aligned} \tag{3.1}$$

By assumption (3.0) of H_1, there exists a function $\mu_1 \in X$ such that $\|x(t)\| \le \|\mu(t)\|,\ t \in [t_0, t_1]$, and equation (3.2) implies

$$\|\mu(t)\| \le N_0 + B_1 + L\int_{t_0}^{t} p(s)\psi(\|\mu_1\|)ds. \tag{3.2}$$

By Gronwall's inequality,

$$\int_{N_0+B_1}^{\mu_1(t)} \frac{d\mu}{\psi(\|\mu\|)} \le L\int_{t_0}^{t} p(s)ds, \tag{3.3}$$

and

$$\|\mu_1(t^*)\| \le \Gamma_1^{-1}\left(L\int_{t_0}^{t} p(s)ds\right) := M_1,\ t^* \in [t_0, t_1], \tag{3.4}$$

then for every $t \ne t_1,\ t \in [t_0, t_1]$, we have $\sup_{t\in[t_0,t_1]} \|x(t)\| = M_1$.

Also, for $t \in [t_1, t_2]$,

$$\|\mu(t)\| \le N_1 + B_2 + L\int_{t_1}^{t} p(s)\psi(\|\mu\|_X)ds, \quad t \in [t_1, t_2]. \tag{3.5}$$

and

$$\int_{N_1+B_2}^{\mu_2(t)} \frac{d\mu}{\psi(\|\mu\|)} \le L\int_{t_1}^{t} p(s)ds. \tag{3.6}$$

so that

$$\|\mu_2(t^*)\| \le \Gamma_2^{-1}\ L\left(\int_{t_0}^{t} p(s)ds\right) := M_2,\ t^* \in [t_1, t_2]. \tag{3.7}$$

Then, for every $t \in [t_1, t_2]$, $\sup_{t\in[t_1,t_2]} |x(t)| = M_2$.

Continuing this process for $t \ne t_k,\ t \in [t_k, T]$,

$$\int_{N_{k-1}+B_k}^{\mu_k(t)} \frac{d\mu}{\psi(\mu)} \le L\int_{t_k}^{T} p(s)ds. \tag{3.8}$$

Then,

$$\|\mu_k(t^*)\| \le \Gamma_{k+1}^{-1}\left(L\int_{t_k}^{T} p(s)ds\right) := M_k. \tag{3.9}$$

Since for every $t \in [t_k, T]$, then

$$\sup_{t\in[t_k,T]} \|x(t)\| := M_k \tag{3.10}$$

$$\|x(t^*)\| \le \sup_{t\in[t_k,t^*_k]} |x(t)| = M_k := N_k.$$

Therefore, for each possible mild solution $x(t)$ of equation (2.2),

$$\|x\| \le \max\{L\|\phi_0\|,\ N_{k-1} : k = 1,\dots,m+1\} := b^*, \tag{3.11}$$

Hence, $x(t)$ is bounded.

Hypothesis 3.1

H_4. Let $\eta : X \to R$ satisfies H_1 such that $\sup_{t\in J}\|\eta(t,\ 0)\| = \alpha$ and $\|\eta(t,u) - \eta(t,0)\| \le K$ for $K > 0$, then there exists

$$\varepsilon \geq L\left(\|\phi_0\| + a[K+\alpha] + \frac{B}{L}\right). \tag{3.12}$$

H_5. Let there exists

$$\lambda \geq L\int_{t_0}^{t} l(s)ds + B_k$$

for $0 \leq \lambda < 1$ such that $\partial_H \|(\beta x_1)(t) - (\beta x_2)(t)\| \leq \lambda \|x_1 - x_2\|$, for each $x_1,\ x_2 \in \Gamma_\in$.

Theorem 3.1

Let hypothesis (3.0) and (3.1) hold such that for each bounded $\Gamma_\varepsilon \subseteq X$, for $t \in J$, the set

$$\left\{R(t,t_0)\phi_0 + \int_{t_0}^{t} R(t,s)\big(Dx_s + \eta(s,x_s)\big)ds \quad + \sum_{k=1}^{n} I(t_k x(t_k^-)),\ x \in \Omega\right\}$$

is relatively compact in X. Then, system (2.1) has at least one solution.

Proof

Consider the operator $\beta : \Omega \to \Omega$ defined by

$$\beta(x)(t) = R(t,t_0)\phi_0 + \int_{t_0}^{t} R(t,s)\big(Dx_s + \eta(s,x_s)\big)ds \ + I(t_k x(t_k^-)), \quad t \in [t_0, T], \tag{3.13}$$

and a close nonempty and convex set $\Gamma_\varepsilon = \left\{x(t) \in C(J) : |t - t_0| \leq \alpha,\ \|x\|_\infty \leq \varepsilon\right\}$.

Then, for any $x \in \Gamma_\varepsilon$,

$$\begin{aligned}
\|\beta(x(t))\| &= \|R(t,t_0)\phi_0\| + \int_{t_0}^{t} \|R(t,s)(Dx_s + \eta(s,x_s))\| ds + \left\|I(t_k, x(t_k^-))\right\| \\
&\leq \|R(t,t_0)\phi_0\| + \int_{t_0}^{t} \|R(t,s)\eta(s,u)\| ds + \left\|I(t_k, x(t_k^-))\right\| \\
&\leq L\|\phi_0\| + L\int_{t_0}^{t} \|\eta(s,u) - \eta(s,0)\| + \|\eta(s,0)\| ds + \left\|I(t_k, x(t_k^-))\right\| \\
&\leq L\left(\|\phi_0\| + a[K+\alpha] + \frac{B}{L}\right) \\
&\leq \varepsilon
\end{aligned}$$

Showing that β is equicontinuous in X:

Let $t_1, t_2 \in J,\ t_1 < t_2$, and for $x \in X$, then,

$$(\beta x)(t_2)-\beta(x)(t_1)=\left(R(t_2,t_0)\phi_0+\int_{t_0}^{t_2}R(t_2,s)\eta(s,u)ds+I_2(t_2,x(t_2^-))\right)$$
$$-\left(R(t_1,t_0)\phi_0+\int_{t_0}^{t_1}R(t_1,s)\eta(s,u)ds+I_1(t_1,x(t_1^-))\right)$$

$$\|(\beta x)(t_2)-\beta(x)(t_1)\|\le\|R(t_2,t_0)-R(t_1,t_0)\|\|\phi_0\|+\left\|\int_{t_0}^{t_2}R(t_2,s)\eta(s,u)ds-\int_{t_0}^{t_1}R(t_1,s)\eta(s,u)ds\right\|$$
$$+\left|I_2(t_2,x(t_2^-))-I_1(t_1,x(t_1^-))\right|$$
$$\le\|R(t_2,t_0)-R(t_1,t_0)\|\|\phi_0\|+\int_{t_1}^{t_2}\|R(t_2,s)\|\|\eta(s,u)\|ds+\int_{t_0}^{t_1-\lambda}\|R(t_2,s)-R(t_1,s)\|\|\eta(s,u)\|ds$$
$$+\int_{t_1}^{t_1-\lambda}\|R(t_2,s)-R(t_1,s)\|\|\eta(s,u)\|ds+\left|I_2(t_2,x(t_2^-))-I_1(t_1,x(t_1^-))\right|.$$

Therefore $\|(\beta x)(t_2)-(\beta x)(t_1)\|\to 0$ as $t_2\to t_1$ independent of x, for a sufficiently small λ, since $\eta(t,u)$ is compact and continuous in the uniform operator topology for $t>0$.

Showing that β is continuous:

Consider the sequence $x_n\in\Gamma_\varepsilon$, such that $x_n\to x$ for each $x\in X$. Let there exists a $v_n\in\eta(t,\mu(t))$ measurable in Γ_ε such that $\|v_n(t)-v(t)\|\le l(t)|x_n-x|$, and for any $H_n(t)\in(\beta x_n)(t)$ defined as

$$H_n(t)=R(t,t_0)\varphi_0+\int_{t_0}^{t}R(t,s)v_n(s)ds+I(t_k,x_n(t_k^-)),\quad t\in[t_0,T]. \tag{3.14}$$

Then,

$$\|H_n(t)-H(t)\|\le\left\|\int_{t_0}^{t}R(t,s)v_n-vds\right\|+\left\|I_k(t_k,x_n(t_k^-))-I_k(t_k,x(t_k^-))\right\|$$
$$\le\int_{t_0}^{t}\|R(t,s)\|\|v_n-v\|ds+\left\|I_k(t_k,x_n(t_k^-))-I_k(t_k,x(t_k^-))\right\|$$
$$\le L\int_{t_0}^{t}l(s)\|x_n-x\|ds+B_k\|x_n-x\|$$
$$\le\left(L\int_{t_0}^{t}l(s)ds+B_k\right)\|x_n-x\|$$

By the continuity of x_n in $J\times X$, $\lim_{n\to\infty}\|H_n(t)-H(t)\|=0$, which implies that $(\beta x_n)(t)\to(\beta x)(t)$ and $x_n\to x$, Hence $x\in\Gamma_\varepsilon$ is the fixed solution of $(\beta x)(t)$

Proving Uniqueness of $(\beta x)(t)$

Let $x_1(t),\, x_2(t)\in\Gamma_\varepsilon$ be two solutions of system (2.1) satisfying hypothesis H_5. Then, for any $H_1(t),\, H_2(t)\in(\beta x)(t)$, there exists a $v_1(t),\, v_1(t)\in\eta(t,u(t))$ measureable in Γ_ε such that $\|v_1(t)-v_2(t)\|\le l(t)\|x_1(t)+x_2(t)\|$.

Defining $H_1(t),\, H_2(t)\in(\beta x)(t)$ as

$$H_1(t)=R(t,t_0)\phi_0+\int_{t_0}^{t}R(t,s)v_1(s)ds+I(x_1(t_k^-)),\ t[t_0,T],$$

$$H_2(t) = R(t,t_0)\phi_0 + \int_{t_0}^{t} R(t,s)v_2(s)ds + I(x_2(t_k^-)),\ t[t_0,T],$$

respectively, then

$$\begin{aligned}\|H_1(t) - H_2(t)\| &\le \int_{t_0}^{t} \|R(t,s)(v_1(s) - v_2(s))\|ds + \|I(t,x_1(t_k^-)) - I(t,x_2(t_k^-))\| \\ &\le \int_{t_0}^{t} \|R(t,s)\|\|v_1(s) - v_2(s)\|ds + \|I(t,x_1(t_k^-)) - I(t,x_2(t_k^-))\| \\ &\le L\int_{t_0}^{t} l(s)\|x_1 - x_2\|ds + B_k\|x_1 - x_2\| \\ &\le \left(L\int_{t_0}^{t} l(s)ds + B_k\right)\|x_1 - x_2\| \\ &\le \lambda\|x_1 - x_2\|.\end{aligned}$$

Then $H(t) \in (\beta x)(t)$ satisfies the contraction mapping principle on (J, X), and therefore $x(t) \in (J, X)$ is a unique fixed point which is the unique solution of equation (2.1).

4. Conclusion

The analysis of the solution of the impulsive neutral integro-differential system in the Banach space X was considered. Theorem and prove on the existence of only the impulsive term as a solution of the system equations at the points of discontinuity was presented using the Du Bois–Reymond's assumptions on solution variation of piecewise smooth functions. Theorems on existence and uniqueness of the system solution were formulated using the rich theories of an infinitesimal generator A of a strongly continuous compact semigroup $R(.)$, and proves were provided using an approximate piecewise continuous, compact operator and a continuous positive non decreasing function $\psi:[0,\infty)\to(0,\infty)$. Results obtained are improvement on the qualitative analysis of impulsive neutral integro-differential system

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