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Existence and Uniqueness Fourth-order Differential Equations in Banach Space Having Fourier Type

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Abstract

The aim of this work is to study the existence of a periodic solutions of differential equations $\frac{d^4}{dt^4}x(t) = Ax(t) + f(t)$. Our approach is based on the M-boundedness of linear operators, Fourier type, $B^s_{p,q}$-multipliers and Besov spaces.

Mathematics Subject Classification: 45N05, 45D05, 43A15, 47D99

Keywords: integro-differential equations, Fourier type, $B^s_{p,q}$-multipliers

1. Introduction

The aim of this paper is to study the existence and uniqueness of solutions for some differential equations by using methods of maximal regularity in spaces of Besov space. Motivated by the fact that functional differential equations arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers (Arendt, W. & Bu, S., 2004; Bahloul, R., & Ezzinbi, K., 2016; Hernan, R. H., 2012; Keyantuo, V. & et al., 2009) and the references listed therein for information on this subject. One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the subjet, we refer to Poblete (2009), Lizama (2006), Hernan (2012), Ezzinbi et al (2016) and Arendt-Bu (2004). We characterize the existence of periodic solutions for the following integro-differential equations in vector-valued spaces and Besov. Our results involve only M-boundedness of the resolvent.

In this work, we study the existence of periodic solutions for the following differential equations

$$\frac{d^4}{dt^4}x(t) = Ax(t) + f(t), \tag{0.1}$$

where $A : D(A) \subseteq X \to X$ is a linear closed operator on Banach space $(X, \|.\|)$, and $f \in L^p(\mathbb{T}, X)$ for all $p \geq 1$.

For example:

$$\frac{\partial^4}{\partial t^4}w(t,x) = \frac{\partial^2}{\partial t^2}w(t,x) + g(t,x)$$

Put $y(t)(x) = w(t,x)$, $f(t)(x) = g(t,x)$ and $A\varphi = \varphi''$

Then we have

$$\frac{d^4}{dt^4}y(t) = Ay(t) + f(t)$$

In (2016), Bahloul et al established the existence of a periodic solution for the following partial functional differential equation.

$$\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + f(t)$$

where $A : D(A) \subseteq X \to X$ is a linear closed operator on Banach space $(X, \|.\|)$ and L and G are in $B(L^p([-r_{2\pi}, 0],\ X);\ X)$.

In (2004), Arendt gave necessary and sufficient conditions for the existence of periodic solutions of the following evolution equation.

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \text{ for } t \in \mathbb{R},$$

where A is a closed linear operator on an UMD-space Y.

In (2016), Sylvain Koumla, Khalil Ezzinbi, Rachid Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay in Frchet spaces

$$\frac{d}{dt}x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) + h(t, x_t)$$

This work is organized as follows : After preliminaries in the second section, we give a main result and the conclusion.

2. Vector-valued Space and Preliminaries

Let X be a Banach Space. Firstly, we denote By $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.

Given $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of 2π-periodic locally p-integrable functions from $\mathbb{R}$ into X, with the norm:

$$\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p \, dt \right)^{1/p}$$

For $f \in L^p(\mathbb{T}; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the k-th Fourier coefficient of f that is defined by:

$$\mathcal{F}(f)(k) = \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt \text{ for } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For $1 \leq p < \infty$, the periodic vector-valued space is defined by.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let D($\mathbb{T}$) be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_n = \sup_{x \in \mathbb{T}} |f^{(n)}(x)|$ for $n \in \mathbb{N}$. Let $D'(\mathbb{T}; X) = \mathcal{L}(D(\mathbb{T}), X)$. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \leq 2\}, I_k = \{t \in \mathbb{R}, 2^{k-1} < |t| \leq 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I}_k$, for each $k \in \mathbb{N}$, $\sum_{k \in \mathbb{N}} \phi_k(x) = 1$.

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $(\phi_j)_{j \geq 0} \in \phi(\mathbb{R})$ the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}; X) = \{f \in D'(\mathbb{T}; X) : \|f\|_{B^s_{p,q}} := (\sum_{j \geq 0} 2^{sjq} \| \sum_{k \in \mathbb{Z}} e_k \phi_j(k) \hat{f}(k) \|^q_p)^{1/q} < \infty\}.$$

Proposition 0.1. *(Keyantuo, V. & et al., 2009)*
1) $B^s_{p,q}((0, 2\pi); X)$ is a Banach space;

2) Let $s > 0$. Then $f \in B^{s+1}_{p,q}((0, 2\pi); X)$ in and only if f is differentiale and $f' \in B^s_{p,q}((0, 2\pi); X)$

3) Let $s > 0$. Then $f \in B^{s+4}_{p,q}((0, 2\pi); X)$ in and only if f is differentiale four times and $f^{(4)} \in B^s_{p,q}((0, 2\pi); X)$

Definition 0.1. (Keyantuo, V. & et al., 2009)
For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in Z} \subset B(X, Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists $u \in B^s_{p,q}(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Definition 0.2. (Arendt, W. & Bu, S. 2004)
The Banach space X has Fourier type $r \in]1, 2]$ if there exists $C_r > 0$ such that

$$\|\mathcal{F}(f)\|_{r'} \leq C_r \|f\|_r, \; f \in L^r(\mathbb{R}, X)$$

where $\frac{1}{r'} + \frac{1}{r} = 1$.

Definition 0.3. (Keyantuo, V. & et al., 2009)
Let $\{M_k\}_{k \in Z} \subseteq B(X, Y)$ be a sequence of operators. $\{M_k\}_{k \in Z}$ is M-bounded of order 1(or M-bounded) if

$$\sup_k \|M_k\| < \infty \text{ and } \sup_k \|k(M_{k+1} - M_k)\| < \infty \tag{0.2}$$

Theorem 0.1. *(Arendt, W. & Bu, S. 2004)*
Let X and Y be Banach spaces having Fourier type $r \in]1, 2]$ and let $\{M_k\}_{k\in Z} \subseteq B(X, Y)$ be a sequence satisfying (0.2). Then for $1 \le p, q < \infty, s \in \mathbb{R}, \{M_k\}_{k\in Z}$ is an $B^s_{p,q}$-multiplier.

3. Main Result

For convenience, we introduce the following notations:

$$a_k = \frac{k}{(k+1)^4},\quad b_k = 4 + \frac{6}{k} + \frac{4}{k^2} + \frac{1}{k^4} \text{ and } c_k = \frac{k^4}{(k+1)^4}$$

Definition 0.4. : Let $1 \le p, q < \infty$ and $s > 0$. We say that a function $x \in B^s_{p,q}(\mathbb{T}; X)$ is a strong $B^s_{p,q}$-solution of (0.1) if $x(t) \in D(A), x(t) \in B^{s+4}_{p,q}(\mathbb{T}; X)$ and equation (0.1) holds for a.e $t \in \mathbb{T}$.

We prove the following result.

Lemma 0.2. *: Let X be a Banach space and A be a linear closed and bounded operator. Suppose that $(k^4 I - A)$ is bounded invertible and $k^4(k^4 I - A)^{-1}$ is bounded. Then $\left\{(k^4 I - A)^{-1}\right\}_{k\in Z}$ and $\left\{k^4(k^4 I - A)^{-1}\right\}_{k\in Z}$ are M-bounded.*

Proof. Let $S_k = k^4 N_k$ and $N_k = (k^4 I - A)^{-1}$.

Now, we are going to show that

$$\begin{cases} \sup_k \|k(N_{k+1} - N_k)\| < \infty, \\ \\ \sup_k \|k(S_{k+1} - S_k)\| < \infty \end{cases} \tag{0.3}$$

By hypothesis we have, $\{N_k\}_{k\in\mathbb{Z}}$ and $\{S_k\}_{k\in\mathbb{Z}}$ are bounded. Then We have

$$\begin{aligned} \sup_{k\in\mathbb{Z}} \|k(N_{k+1} - N_k)\| &= \sup_{k\in\mathbb{Z}} \left\|kN_{k+1}[(k^4 I - A) - ((k+1)^4 I - A)]N_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-kN_{k+1}[4k^4 + 6k^3 + 4k^2 + k]N_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-kN_{k+1}[4 + \frac{6}{k} + \frac{4}{k^2} + \frac{1}{k^4}]k^4 N_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-\frac{k}{(k+1)^4} S_{k+1}[4 + \frac{6}{k} + \frac{4}{k^2} + \frac{1}{k^4}]S_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \|-a_k S_{k+1} b_k S_k\| \end{aligned}$$

We obtain:

$$\sup_{k\in\mathbb{Z}} \|k(N_{k+1} - N_k)\| < \infty \tag{0.4}$$

On the other hand, we have

$$\begin{aligned} \sup_{k\in\mathbb{Z}} \|k(S_{k+1} - S_k)\| &= \sup_{k\in\mathbb{Z}} \left\|k[(k+1)^4 N_{k+1} - k^4 N_k]\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|kN_{k+1}[(k+1)^4(k^4 I - A) - k^4((k+1)^4 I - A)]N_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-kN_{k+1}[4k^4 + 6k^3 + 4k^2 + k]AN_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-\frac{k^5}{(k+1)^4} S_{k+1}[4 + \frac{6}{k} + \frac{4}{k^2} + \frac{1}{k^4}]AN_k\right\| \\ &= \sup_{k\in\mathbb{Z}} \|-c_k S_{k+1} b_k A k N_k\| \\ &= \sup_{k\in\mathbb{Z}} \left\|-c_k S_{k+1} b_k \frac{1}{k^3} S_k\right\| \end{aligned}$$

Then

$$\sup_{k\in\mathbb{Z}} \|k(S_{k+1} - S_k)\| < \infty$$

so, $(N_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are M-bounded. □

Theorem 0.3. *Let $1 \leq p, q < \infty$ and $s > 0$. Let X be a Banach space having Fourier type $r \in]1, 2]$ and A be a linear closed and bounded operator. If $(k^4 I - A)$ is bounded invertible and $k^4(k^4 I - A)^{-1}$ is bounded. Then for every $f \in B^s_{p,q}(\mathbb{T}, X)$ there exist a unique strong $B^s_{p,q}$-solution of (**0.1**).*

Proof. Define $S_k = k^4 N_k, N_k = (k^4 I - A)^{-1}$ for $k \in \mathbb{Z}$. Since by Lemma(0.2), $(S_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are M-bounded, we have by Theorem 0.1 that $(S_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are an $B^s_{p,q}$-multipliers. Since $S_k - AN_k = I$ (because $((k^4 I - A)N_k = I)$, we deduce AN_k is also an $B^s_{p,q}$-multiplicateur.

Now let $f \in B^s_{p,q}(\mathbb{T}, X)$. Then there exist $u, v, w \in B^s_{p,q}(\mathbb{T}, X)$, such that

$\hat{u}(k) = N_k \hat{f}(k), \hat{v}(k) = S_k \hat{f}(k)$ and $\hat{w}(k) = AN_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. So, We have $\hat{u}(k) \in D(A)$ and $A\hat{u}(k) = \hat{w}(k)$ for all $k \in \mathbb{Z}$, we deduce that $u(t) \in D(A)$. On the other hand $\exists v \in B^s_{p,q}(\mathbb{T}, X)$ such that $\hat{v}(k) = S_k \hat{f}(k) = k^4 N_k \hat{f}(k) = k^4 \hat{u}(k)$. Then we obtain $\frac{d^4}{dt^4} u(t) = v(t)$ a.e. Since $u(t) \in B^{s+4}_{p,q}(\mathbb{T}, X)$.

We have $\widehat{\frac{d^4}{dt^4} u}(k) = k^4 \hat{u}(k)$ for all $k \in \mathbb{Z}$, It follows from the identity

$$k^4 N_k - AN_k = I$$

that

$$\frac{d^4}{dt^4} u(t) = Au(t) + f(t)$$

For the uniqueness we suppose two solutions u_1 and u_2, then $u = u_1 - u_2$ is strong L^p-solution of equation (0.1) corresponding to the function $f = 0$, taking Fourier transform, we get $(k^4 I - A)\hat{u}(k) = 0$, which implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and $u(t) = 0$. Then $u_1 = u_2$. The proof is completed. □

4. Conclusion

We are obtained necessary and sufficient conditions to guarantee existence and uniqueness of periodic solutions to the equation $\frac{d^4}{dt^4} u(t) = Au(t) + f(t)$ in terms of either the M-boundedness of the modified resolvent operator determined by the equation. Our results are obtained in the Besov space.

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