Journal of Mathematics Research; Vol. 10, No. 4; August 2018
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Navier-Stokes Three Dimensional Equations Solutions Volume Three

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Received: May 8, 2018 Accepted: May 24, 2018 Online Published: July 25, 2018

doi:10.5539/jmr.v10n4p128 URL: https://doi.org/10.5539/jmr.v10n4p128

Abstract

The existence of smooth solution for Navier-Stokes three-dimensional equations is proved by example. The equation is solved by writing initial velocity as the sum of sine and cosine series with proper coefficients.

Keywords: Navier-Stokes, Fourier series, Vector fields, Existence, Smooth solutions, viscosity

1. Introduction

Solution of Navier-Stokes equation is found by introducing new method for solving differential equations. This new method is writing periodic scalar function in any dimensions and any dimensional vector fields as the sum of sine and cosine series with proper coefficients. The method is extension of Fourier series representation for one variable function to multi-variable functions and vector fields.

Before solving Navier-Stokes equations we introduce a new technique for writing periodic scalar functions or vector fields as the sum of cosine and sine series with proper coefficients. Fourier series representation is background for our new technique.

Periodic nature of initial velocity for Navier-Stokes problem helps us write the vector field in the form of cosine and sine series sum which simplify the problem.

1.1 Definitions

Let

- $\vec{A}$ Be period unit vectors in 3D Euclidean space.
- $\vec{R}$ Be position vector in 3D Euclidean space.
- p, U_x, U_y, U_z Smooth functions of position and time.
- a, b, c, are real number constants.
- $\vec{A} = \mathbf{i}a + \mathbf{j}b + \mathbf{k}c$
- $|\vec{A}|^2 = a^2 + b^2 + c^2 = 1$; Unit vector.
- $\vec{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$; Position vector in 3D space defined for all real number x, y, z.
- $\vec{A} \cdot \vec{R} = ax + by + cz$
- $\vec{U} = \mathbf{i}U_x + \mathbf{j}U_y + \mathbf{k}U_z$;Velocity vector field in space and time
- $\vec{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$; Operator in 3D.
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; Laplacian operator.
- $\vec{U} \cdot \vec{\nabla} = U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}$; Operator.
- $t \geq 0$;Time variable is positive real number including 0.

2. Fourier Series Sum Representation of Scalar Functions of Multi-variables and Vector Fields

Lemma

For n, m elements of positive integer.

$$\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \sin(2m\pi\vec{A}\cdot\vec{R})\ \cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx = 0 \tag{2.0}$$

For $n \neq m$ elements of positive integer.

$$\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \cos(2m\pi\vec{A}\cdot\vec{R})\ \cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx = 0 \tag{2.1}$$

$$\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \sin(2m\pi\vec{A}\cdot\vec{R})\ \sin(2n\pi\vec{A}\cdot\vec{R})\ dzdydx = 0 \tag{2.2}$$

For m element of positive integer.

$$\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \sin^2(2m\pi\vec{A}\cdot\vec{R})\quad dzdydx = 1 \tag{2.3}$$

$$\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \cos^2(2m\pi\vec{A}\cdot\vec{R})\quad dzdydx = 1 \tag{2.4}$$

The above lemma can be proved by direct integration. For easy prove refer Fourier series representation of periodic functions.

2.1 Fourier Series of Periodic Scalar Functions of Multi-Variables

Any periodic function in three dimensions can be represented as the sum of cosine and sine series with appropriate coefficients.

Let $f(x,y,z)$ is periodic scalar function of three variables x, y, z which are elements of real number, where $\vec{A}$ is period unit vector.

$$f(x,y,z) = f(x+a\,, y+b\,, z+c) \tag{2.1.0}$$

$$f(x,y,z) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\,(2n\pi\,\vec{A}\cdot\vec{R}) + b_n \sin\,(2n\pi\,\vec{A}\cdot\vec{R})\right) \tag{2.1.1}$$

By direct integration of both sides we get the following.

$$a_0 = \frac{abc}{8}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} f(x,y,z)\ \ dzdydx \tag{2.1.2}$$

Multiplying both sides of the equation, (2.1.1) with cosine function and integrating gives constant for cosine term coefficient.

$$a_n = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} f(x,y,z)\cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{2.1.3}$$

Multiplying both sides of the equation, (2.1.1) with sine function and integrating gives constant for cosine term coefficient.

$$b_n = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} f(x,y,z)\sin(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{2.1.4}$$

Detail derivation of the above result is left as an exercise for the reader because of space limitation on the modern mathematics papers. Also, since direct integration is the only tool to drive the above results.

2.2 Fourier Series of Periodic Vector Fields

Let $\vec{\mathrm{U}}^0(x,y,z)$ be vector field defined in 3 dimensional Euclidean space variables of x, y, z elements of real numbers. The field is periodic with $\vec{A}$ then,

$$\vec{\mathrm{U}}^0(x,y,z) = \vec{\mathrm{U}}^0(x+a\,,y+b\,,z+c) \tag{2.2.0}$$

$$\vec{\mathrm{U}}^0(x,y,z) = \vec{\mathrm{a}}_0 + \sum_{n=1}^{\infty}\left(\vec{\mathrm{a}}_\mathrm{n}\cos\left(2n\pi\,\vec{A}\cdot\vec{R}\right) + \vec{\mathrm{b}}_\mathrm{n}\sin\left(2n\pi\,\vec{A}\cdot\vec{R}\right)\right) \tag{2.2.1}$$

Direct integration of both sides of equation (2.2.1) gives us the following.

$$\vec{\mathrm{a}}_0 = \frac{abc}{8} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)dzdydx \tag{2.2.2}$$

Multiplying both sides of the equation (2.2.1) by cosine function and integrating give coefficient vector for cosine function term in the sum.

$$\vec{\mathrm{a}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)\cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{2.2.3}$$

Multiplying both sides of the equation (2.2.1) by sine function and integrating give coefficient vector for sine term in the sum.

$$\vec{\mathrm{b}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)\sin(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{2.2.4}$$

The above results can be extended to periodic tensors in general. It is huge project by itself to extend these results to periodic tensor fields. Without doubt the results have power to revolutionize the mathematics and off course physics. We will leave this for other papers and focus on our goal, which is proving the existence of solution for Navier-Stokes equations. The next topic will wind up our journey to prove of the existence of smooth solutions for Navier-Stokes equations.

3. Solutions for Navier-Stokes 3D Equations

Apply all results derived above to the equation and find simple solution of Navier-Stokes equations so that the existence of smooth solution is proved by example.

3.1 Statement of the Problem

Navier-Stokes equation and conditions for physically reasonable solutions as posted by clay mathematics institute are listed below.

Navier-Stokes equations are shown below.

$$\frac{\partial \vec{U}}{\partial t}+\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}=\nu\nabla^2\vec{U}-\vec{\nabla}\mathrm{p}+\vec{\mathrm{F}} \tag{3.1.0}$$

$$\frac{\partial \vec{U}}{\partial t}+\left(u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}+u_z\frac{\partial}{\partial z}\right)\vec{U}=\nu\nabla^2\vec{U}-\vec{\nabla}\mathrm{p}+\vec{\mathrm{F}} \tag{3.1.1}$$

Divergence free velocity vector field.

$$\vec{\nabla}\cdot\vec{\mathrm{U}}=0 \tag{3.1.2}$$

Initial condition, velocity vector field at time zero is given.

$$\vec{\mathrm{U}}|_{\mathrm{t=0}}=\vec{\mathrm{U}}^0 \tag{3.1.3}$$

Periodic Nature of initial velocity vector field is showed.

$$\vec{\mathrm{U}}^0(\mathrm{x,y,z})=\vec{\mathrm{U}}^0(\mathrm{x+a\,,y+b\,,z+c}) \tag{3.1.4}$$

Periodic nature of Velocity vector field at all times.

$$\vec{\mathrm{U}}\,(\mathrm{x,y,z,t})=\vec{\mathrm{U}}\,(\mathrm{x+a\,,y+b\,,z+c,t}) \tag{3.1.5}$$

External force is set to zero for simplicity.

$$\vec{\mathrm{F}}=0 \tag{3.1.6}$$

Velocity vector field and scalar pressure are defined for all real positions and positive time.

$$\vec{\mathrm{U}}\,,\mathrm{p}\ \in \mathrm{C}^\infty\left(\mathrm{R}^\mathrm{n}\times[0,\infty)\right) \tag{3.1.7}$$

3.2 Initial Condition

Because initial velocity is periodic it can be written as infinite sum of sine and cosine series.

$$\vec{\mathrm{U}}^0(x,y,z)=\vec{\mathrm{a}}_0+\sum_{n=1}^{\infty}\left(\vec{\mathrm{a}}_\mathrm{n}\cos\,(2n\pi\,\vec{A}\cdot\vec{R})+\vec{\mathrm{b}}_\mathrm{n}\sin\,(2n\pi\,\vec{A}\cdot\vec{R})\right) \tag{3.2.0}$$

Coefficients of the sine and cosine terms are shown below.

$$\vec{\mathrm{a}}_0=\frac{abc}{8}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}}\vec{\mathrm{U}}^0(x,y,z)dzdydx \tag{3.2.1}$$

$$\vec{\mathrm{a}}_\mathrm{n}=\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}}\vec{\mathrm{U}}^0(x,y,z)\cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{3.2.2}$$

$$\vec{\mathrm{b}}_\mathrm{n}=\frac{abc}{4}\iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}}\vec{\mathrm{U}}^0(x,y,z)\sin(2n\pi\vec{A}\cdot\vec{R})\ dzdydx \tag{3.2.3}$$

Initial velocity vector field is divergent free.

$$\vec{\nabla} \cdot \vec{\mathrm{U}}^0(\mathrm{x,y,z}) = 0 \tag{3.2.4}$$

$$\vec{\nabla} \cdot \vec{\mathrm{U}}^0(x,y,z) = \sum_{n=1}^{\infty}\left(-2n\pi\, \vec{\mathrm{a}}_\mathrm{n} \cdot \vec{A} \sin\,(2n\pi\, \vec{A} \cdot \vec{R}) + 2n\pi\, \vec{\mathrm{b}}_\mathrm{n} \cdot \vec{A} \cos\,(2n\pi\, \vec{A} \cdot \vec{R})\right) \tag{3.2.5}$$

Multiply both sides of the equation (3.2.5) by sine function and integrate.

$$-2n\pi\vec{A} \cdot\ \vec{\mathrm{a}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\nabla} \cdot \vec{\mathrm{U}}^0(x,y,z)\sin(2n\pi\vec{A} \cdot \vec{R})\ dzdydx \tag{3.2.6}$$

Since

$$\vec{\nabla} \cdot \vec{\mathrm{U}}^0(x,y,z) = 0 \tag{3.2.7}$$

Therefore

$$-2\mathrm{n}\pi\vec{\mathrm{A}} \cdot\ \vec{\mathrm{a}}_\mathrm{n} = 0 \tag{3.2.8}$$

Multiply both sides of the equation (3.2.5) by cosine function and integrate.

$$2n\pi\vec{A} \cdot\ \vec{\mathrm{b}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\nabla} \cdot \vec{\mathrm{U}}^0(x,y,z)\cos(2n\pi\vec{A} \cdot \vec{R})\ dzdydx \tag{3.2.9}$$

Since

$$\vec{\nabla} \cdot \vec{\mathrm{U}}^0(x,y,z) = 0 \tag{3.2.10}$$

Therefore

$$2n\pi\vec{A} \cdot\ \vec{\mathrm{b}}_\mathrm{n} = 0 \tag{3.2.11}$$

Dot product of initial velocity with its period vector $\vec{A}$ is showed.

$$\vec{A} \cdot\ \vec{\mathrm{U}}^0(x,y,z) = \vec{A} \cdot \vec{\mathrm{a}}_0 + \sum_{n=1}^{\infty}\left(\vec{A} \cdot \vec{\mathrm{a}}_\mathrm{n} \cos\,(2n\pi\, \vec{A} \cdot \vec{R}) + \vec{A} \cdot \vec{\mathrm{b}}_\mathrm{n} \sin\,(2n\pi\, \vec{A} \cdot \vec{R})\right) \tag{3.2.12}$$

Since

$$-2n\pi\vec{A} \cdot\ \vec{\mathrm{a}}_\mathrm{n} = 0 \tag{3.2.13}$$

And

$$2n\pi\vec{A} \cdot\ \vec{\mathrm{b}}_\mathrm{n} = 0 \tag{3.2.14}$$

Then

$$\vec{\mathrm{A}} \cdot\ \vec{\mathrm{U}}^0(\mathrm{x,y,z}) = \vec{\mathrm{A}} \cdot \vec{\mathrm{a}}_0 \tag{3.2.15}$$

3.3 Velocity Vector Field and Scalar Pressure Solutions

Writing periodic initial velocity help find velocity vector easily as a function of space variables and time. After writing initial velocity as the sum of sine and cosine series, plug time component as product of coefficients of constant, sine and cosine terms. Finally substitute this velocity vector field to Navier-Stokes equations so that time component and pressure can be easily deciphered.

Predicted velocity vector field as solution is shown below.

$$\vec{\mathrm{U}}\,(x,y,z,t) = \vec{\mathrm{a}}_0 h(t) + \sum_{n=1}^{\infty} \mathrm{H(n,t)}\left(\vec{\mathrm{a}}_\mathrm{n} \cos(2n\pi\, \vec{A} \cdot \vec{R}) + \vec{\mathrm{b}}_\mathrm{n} \sin(2n\pi\, \vec{A} \cdot \vec{R})\right) \tag{3.3.0}$$

Split Navier-Stokes equations in to different parts and substitute the above solution (3.3.0) to each.

First time derivative of the velocity vector ($\frac{\partial \vec{U}}{\partial t}$), is simplified to the following expression.

$$\frac{\partial \vec{U}}{\partial t} = \vec{a}_0 \frac{\partial}{\partial t} h(t) + \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \mathrm{H(n,t)} \left(\vec{a}_n \cos(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n \sin(2n\pi \vec{A} \cdot \vec{R}) \right) \tag{3.3.1}$$

The next term$(\vec{U} \cdot \vec{\nabla})\vec{U}$, is simplified to the following equation.

$$\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) \vec{U}$$

$$= \vec{U} \cdot \vec{A} \sum_{n=1}^{\infty} \mathrm{H(n,t)} \left(-\vec{a}_n 2n\pi \sin(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n 2n\pi \cos(2n\pi \vec{A} \cdot \vec{R}) \right) \tag{3.3.2}$$

The final term $\nabla^2 \vec{U}$, is simplified to the following expression.

$$\nu \nabla^2 \vec{U} = \nu \sum_{n=1}^{\infty} -(2n\pi)^2 |\vec{A}|^2 \mathrm{H(n,t)} \left(\vec{a}_n \cos\,(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n \sin\,(2n\pi \vec{A} \cdot \vec{R}) \right) \tag{3.3.3}$$

Collecting all results of simplification we get the following equation.

$$\vec{a}_0 \frac{\partial}{\partial t} h(t) + \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \mathrm{H(n,t)} \left(\vec{a}_n \cos(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n \sin(2n\pi \vec{A} \cdot \vec{R}) \right)$$

$$+ \vec{A} \cdot \vec{U} \sum_{n=1}^{\infty} \mathrm{H(n,t)} \left(-\vec{a}_n\, 2n\pi \sin\,(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n\, 2n\pi \cos\,(2n\pi \vec{A} \cdot \vec{R}) \right)$$

$$= \nu \sum_{n=1}^{\infty} -(2n\pi)^2 |\vec{A}|^2 \mathrm{H(n,t)} \left(\vec{a}_n \cos\,(2n\pi \vec{A} \cdot \vec{R}) + \vec{b}_n \sin\,(2n\pi \vec{A} \cdot \vec{R}) \right) - \vec{\nabla} \mathrm{p} \tag{3.3.4}$$

Rearrange the above equation (3.3.4) give the following equation.

$$-\vec{\nabla} \mathrm{p} = \vec{a}_0 \frac{\partial}{\partial t} h(t)$$

$$+ \sum_{n=1}^{\infty} \Big(((\frac{\partial}{\partial t} \mathrm{H(n,t)} + \nu (2n\pi)^2\, \mathrm{H}) \vec{a}_n + 2n\pi\, (\vec{A} \cdot \vec{U}) \mathrm{H} \vec{b}_n) \cos(2n\pi \vec{A} \cdot \vec{R}) + ((\frac{\partial}{\partial t} \mathrm{H(n,t)}$$

$$+ \nu (2n\pi)^2\, \mathrm{H}) \vec{b}_n - 2n\pi\, (\vec{A} \cdot \vec{U}) \mathrm{H} \vec{a}_n) \sin(2n\pi \vec{A} \cdot \vec{R}) \Big) \tag{3.3.5}$$

Assume p has both periodic and non periodic components.

Let periodic component of p be$f$, then Fourier series representation is shown below.

$$f(x,y,z) = s_0 + \sum_{n=1}^{\infty} \left(s_n \cos\,(2n\pi \vec{A} \cdot \vec{R}) + c_n \sin\,(2n\pi \vec{A} \cdot \vec{R}) \right) \tag{3.3.6}$$

The gradient of the periodic component of the pressure solution is shown below.

$$\vec{\nabla}\mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z}) = 0 + \vec{\mathrm{A}}\sum_{\mathrm{n}=1}^{\infty} 2\mathrm{n}\pi\big(\mathrm{c_n}\cos\left(2\mathrm{n}\pi\vec{\mathrm{A}}\cdot\vec{\mathrm{R}}\right) - \mathrm{s_n}\sin\left(2\mathrm{n}\pi\vec{\mathrm{A}}\cdot\vec{\mathrm{R}}\right)\big) \tag{3.3.7}$$

Equating $\vec{\nabla} f(x,y,z)$ and periodic part of $-\vec{\nabla}\mathrm{p}$ we arrive to the following two simultaneous equations.

$$(\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t}))\vec{\mathrm{a}}_\mathrm{n} + 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t})\vec{\mathrm{b}}_\mathrm{n} = \vec{\mathrm{A}}\,2n\pi\;\; c_n \tag{3.3.8}$$

$$(\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t}))\vec{\mathrm{b}}_\mathrm{n} - 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t})\vec{\mathrm{a}}_\mathrm{n} = \vec{\mathrm{A}}\,2n\pi\;\; s_n \tag{3.3.9}$$

Dot product of both sides of the equation (3.3.8) with $\vec{A}$ results the following.

$$2\mathrm{n}\pi\;\;\mathrm{c_n} = 0 \tag{3.3.10}$$

Dot product of both sides of the equations (3.3.9) with $\vec{A}$ results the following.

$$2n\pi\;\; s_n = 0 \tag{3.3.11}$$

Then the two equations, (3.3.8) and (3.3.9) are reduced to the following equations.

$$(\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t}))\vec{\mathrm{a}}_\mathrm{n} + 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t})\vec{\mathrm{b}}_\mathrm{n} = 0 \tag{3.3.12}$$

$$(\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t}))\vec{\mathrm{b}}_\mathrm{n} - 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t})\vec{\mathrm{a}}_\mathrm{n} = 0 \tag{3.3.13}$$

Solve the above simultaneous equations (3.3.12) and (3.3.13).

$$\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t})) = \mp\sqrt{-1}\;\; 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t}) \tag{3.3.14}$$

Our solution has to be only real as mentioned in the equation (3.1.7).

Therefore we have to get rid of complex component.

$$\mp\sqrt{-1}\;\; 2\mathrm{n}\pi\,(\vec{\mathrm{A}}\cdot\vec{\mathrm{U}})\mathrm{H}(\mathrm{n},\mathrm{t}) = 0 \tag{3.3.15}$$

$$\frac{\partial}{\partial \mathrm{t}}\mathrm{H}(\mathrm{n},\mathrm{t}) + \nu(2\mathrm{n}\pi)^2\,\mathrm{H}(\mathrm{n},\mathrm{t})) = 0 \tag{3.3.16}$$

Solve equation 3.3.15 we get the following results.

$$\vec{\mathrm{A}}\cdot\vec{\mathrm{U}} = 0 \tag{3.3.17}$$

$$\vec{\mathrm{A}}\cdot\vec{\mathrm{a}}_0 = 0 \tag{3.3.18}$$

Solve equation 3.3.16 we get the following result.

$$\mathrm{H}(\mathrm{n},\mathrm{t}) = \mathrm{e}^{-\upsilon(2\mathrm{n}\pi)^2\mathrm{t}} \tag{3.3.19}$$

Since periodic part of the pressure is zero our gradient solution is the following.

$$-\vec{\nabla}\mathrm{p} = \vec{\mathrm{a}}_0\frac{\partial}{\partial t}h(t) \tag{3.3.20}$$

4. Result

Final solutions of Navier-Stokes equation for a given initial velocity $\vec{\mathrm{U}}^0(x,y,z)$ and viscosity υ are shown below.

$$\vec{\mathrm{U}}\,(x,y,z,t) = \vec{\mathrm{a}}_0 h(t) + \sum_{n=1}^{\infty}\mathrm{e}^{-\upsilon(2n\pi)^2\mathrm{t}}\big(\vec{\mathrm{a}}_\mathrm{n}\cos\left(2n\pi\vec{A}\cdot\vec{R}\right) + \vec{\mathrm{b}}_\mathrm{n}\sin\left(2n\pi\vec{A}\cdot\vec{R}\right)\big) \tag{3.3.21}$$

$$-\mathrm{p} = \vec{\mathrm{a}}_0\cdot\vec{\mathrm{R}}\frac{\partial}{\partial t}h(t) \tag{3.3.22}$$

Where $\vec{\mathrm{A}}\cdot\vec{\mathrm{R}} = \mathrm{ax} + \mathrm{by} + \mathrm{cz}$

$$\lim_{t\to\infty} h(t) < \infty \ and \ h(0) = 1$$

$$\vec{\mathrm{a}}_0 = \frac{abc}{8} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)dzdydx$$

$$\vec{\mathrm{a}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)\cos(2n\pi\vec{A}\cdot\vec{R})\ dzdydx$$

$$\vec{\mathrm{b}}_\mathrm{n} = \frac{abc}{4} \iiint_{-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}}^{\frac{1}{a},\frac{1}{b},\frac{1}{c}} \vec{\mathrm{U}}^0(x,y,z)\sin(2n\pi\vec{A}\cdot\vec{R})\ dzdydx$$

It is straight forward to show the solution fulfill all the conditions listed section 3.1.

1. Velocity vector field and scalar pressure are solutions of Navier-Stokes equations.
2. Velocity Vector is divergence free.
3. Velocity vector field at time zero is equal to initial velocity vector field given.
4. Velocity vector field is periodic.
5. Velocity vector field and scalar pressure are defined in all space coordinates and positive time.

Therefore, the existence of smooth solution for Navier-Stokes equations is proved by example.

5. Conclusion

- Navier-Stokes equations have smooth solutions.
- Non linear part of Navier-Stokes equation is zero for periodic velocity solution.
- Periodic vector fields can be represented as the infinite sum of sine and cosine series.
- For periodic divergent free vector field the coefficients of sine and cosine terms are orthogonal to period vector.
- Initial velocity given has to be orthogonal to period vector.
- Velocity vector field solution is orthogonal to period vector.

6. Remark

- Fourier series representation can be generalized to periodic tensors of any rank.
- More general solution can be derived by considering Fourier transform of vector fields in multi dimensional space.

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