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Irreducible Unitary Representations Concerning Homogeneous Holomorphic Line Bundles Over Elliptic Orbits

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Abstract

In this paper we consider a homogeneous holomorphic line bundle over an elliptic adjoint orbit of a real semisimple Lie group, and set a continuous representation of the Lie group on a certain complex vector subspace of the complex vector space of holomorphic cross-sections of the line bundle. Then, we demonstrate that the representation is irreducible unitary.

Keywords: irreducible unitary representation, homogeneous holomorphic line bundle, elliptic adjoint orbit, complex flag manifold, real semisimple Lie group, maximal vector

1. Introduction

1.1 A Geometrical Realization of Irreducible Unitary Representations

First of all, let us recall the definition of irreducible unitary representation. Let G be a Lie group, let $\mathcal{H} = (\mathcal{H}, \langle\cdot\,,\cdot\rangle)$ be a complex Hilbert space, and let $\varrho : G \to GL(\mathcal{H})$, $g \mapsto \varrho(g)$, be a group homomorphism, where $GL(\mathcal{H})$ denotes the general linear group of $\mathcal{H}$ and it does not matter whether ϱ is continuous here. Then, the definition of irreducible unitary representation is as follows:

Definition 1.1.

(i) *ϱ is called a (continuous) representation of G on $\mathcal{H}$, if the mapping $G \times \mathcal{H} \ni (g, \phi) \mapsto \varrho(g)\phi \in \mathcal{H}$ is continuous.*

(ii) *A representation ϱ of G on $\mathcal{H}$ is said to be irreducible, if an arbitrary closed $\varrho(G)$-invariant complex vector subspace $\mathcal{H}_1 \subset \mathcal{H}$ coincides with either $\{0\}$ or $\mathcal{H}$.*

(iii) *A representation ϱ of G on $\mathcal{H}$ is said to be unitary, if $\langle\varrho(g)\phi_1, \varrho(g)\phi_2\rangle = \langle\phi_1, \phi_2\rangle$ for all $g \in G$ and $\phi_1, \phi_2 \in \mathcal{H}$.*

Here the topology for $\mathcal{H}$ is induced by the norm $\|\phi\| := \sqrt{\langle\phi, \phi\rangle}$.

It is interesting to find out irreducible unitary representations from among geometric objects for study. In this paper, we consider a homogeneous holomorphic line bundle $\iota^\sharp(G_\mathbb{C} \times_\chi \mathbb{C})$ over an elliptic (adjoint) orbit G/L of real semisimple Lie group G, and set a representation ϱ of G on a complex vector subspace $\mathcal{H}$ of the complex vector space $\mathcal{V}$ of holomorphic cross-sections of the bundle $\iota^\sharp(G_\mathbb{C} \times_\chi \mathbb{C})$. Then, we demonstrate that without any completions, $\mathcal{H}$ is a separable complex Hilbert space, and that ϱ is an irreducible unitary representation of G on $\mathcal{H}$ (see Theorem 1.2 below).

1.2 The Main Result (Theorem 1.2)

We are going to state the main result in this paper. Let $G_\mathbb{C}$ be a connected complex semisimple Lie group, let G be a connected closed subgroup of $G_\mathbb{C}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_\mathbb{C}$, and let T be a non-zero elliptic element of $\mathfrak{g}$ (see Definition 2.1 for the definition of elliptic element). Setting

$$\begin{aligned}
&L := C_G(T) = \{g \in G \mid \operatorname{Ad} g(T) = T\}, \quad L_\mathbb{C} := C_{G_\mathbb{C}}(T),\\
&\mathfrak{g}_\lambda := \{A \in \mathfrak{g}_\mathbb{C} \mid \operatorname{ad} T(A) = i\lambda A\} \text{ for } \lambda \in \mathbb{R},\\
&\mathfrak{u}_\pm := \textstyle\bigoplus_{\lambda>0} \mathfrak{g}_{\pm\lambda}, \quad U_+ := \exp \mathfrak{u}_+, \quad Q_- := \{x \in G_\mathbb{C} \mid \operatorname{Ad} x(\mathfrak{l}_\mathbb{C} \oplus \mathfrak{u}_-) \subset \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u}_-\},
\end{aligned}$$

we have an elliptic orbit G/L, a complex flag manifold[1] $G_\mathbb{C}/Q_-$ and $L = G \cap Q_-$. Moreover, we see that the mapping $\iota : G/L \to G_\mathbb{C}/Q_-$, $gL \mapsto gQ_-$, is a G-equivariant embedding whose image is a domain in $G_\mathbb{C}/Q_-$. For this reason, we

[1] A complex flag manifold is also called a Kähler C-space or a generalized flag manifold.

can assume G/L to be a domain in $G_{\mathbb{C}}/Q_-$ and regard it as a homogeneous complex manifold of G, via ι.

$$\begin{array}{ccc} \iota^\sharp(G_{\mathbb{C}} \times_\chi \mathbb{C}) & & G_{\mathbb{C}} \times_\chi \mathbb{C} \\ \downarrow & & \downarrow \\ G/L & \xrightarrow{\iota} & G_{\mathbb{C}}/Q_- \end{array}$$

In addition, let $\chi : Q_- \to \mathbb{C}^* = GL(1, \mathbb{C})$, $q \mapsto \chi(q)$, be a holomorphic homomorphism. Denote by $G_{\mathbb{C}} \times_\chi \mathbb{C}$ the fiber bundle over $G_{\mathbb{C}}/Q_-$, with standard fiber $\mathbb{C}$ and structure group Q_-, which is associated to the principal fiber bundle $\pi_{\mathbb{C}} : G_{\mathbb{C}} \to G_{\mathbb{C}}/Q_-$, $x \mapsto xQ_-$, and denote by $\iota^\sharp(G_{\mathbb{C}} \times_\chi \mathbb{C})$ the restriction of the bundle $G_{\mathbb{C}} \times_\chi \mathbb{C}$ to the domain $G/L \subset G_{\mathbb{C}}/Q_-$. Then, one may assume that

$$\mathcal{V} := \left\{ \psi : GQ_- \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ \psi \text{ is holomorphic}, \\ (2)\ \psi(xq) = \chi(q)^{-1}\psi(x) \text{ for all } (x, q) \in GQ_- \times Q_- \end{array} \right\}$$

is the complex vector space of holomorphic cross-sections of the line bundle $\iota^\sharp(G_{\mathbb{C}} \times_\chi \mathbb{C})$. Let us set a complex vector subspace $\mathcal{H} \subset \mathcal{V}$ and a group homomorphism $\varrho : G \to GL(\mathcal{H})$, $g \mapsto \varrho(g)$, as follows:

$$\begin{aligned} &\langle \psi_1, \psi_2 \rangle := \int_G \psi_1(g)\overline{\psi_2(g)}d\mu(g), \quad \|\psi\| := \sqrt{\langle \psi, \psi \rangle} \quad \text{for } \psi_1, \psi_2, \psi \in \mathcal{V}, \\ &\mathcal{H} := \{\phi \in \mathcal{V} : \|\phi\| < \infty\}, \\ &(\varrho(g)\phi)(x) := \phi(g^{-1}x) \quad \text{for } (g, \phi) \in G \times \mathcal{H} \text{ and } x \in GQ_-, \end{aligned}$$

where μ denotes the non-zero Haar measure on G. Now, we are in a position to state

Theorem 1.2. *With the setting in Subsection* 1.2; *suppose that*

(S) $|\chi(\ell)| = 1$ *for all* $\ell \in L$ $(\subset Q_-)$.

Then, $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle)$ *is a separable complex Hilbert space and* ϱ *is an irreducible unitary representation of* G *on* $\mathcal{H}$. *Furthermore, in case of* $\mathcal{H} \neq \{0\}$, *the following two items hold*:

(I) *There exists a unique* $\varphi_{\max} \in \mathcal{H}$ *such that* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$.

(II) *There exists a non-zero* $\phi \in \mathcal{H}$ *satisfying* $\displaystyle\int_G |\langle \varrho(g)\phi, \phi \rangle|^2 d\mu(g) = \|\phi\|^6$ $(< \infty)$.

Here $(U_+ \cap GQ_-)_e$ *denotes the connected component of* $U_+ \cap GQ_-$ *containing the unit element* $e \in G_{\mathbb{C}}$.

1.3 Topics Related to the Main Result

Here are some comments on Theorem 1.2.

(c.1) Theorem 1.2 and Godement's result (Godement, 1947) ensure that the representation ϱ is square integrable or discrete series in case of $\mathcal{H} \neq \{0\}$ (ref. Shucker, 1983, also).

(c.2) The vector $\varphi_{\max} \in \mathcal{H}$ is a maximal vector of weight χ whenever G is compact and L is a maximal torus in G.

(c.3) If G has a compact Cartan subgroup, then the supposition (S) always holds (see Remark 2.15). In particular, it does hold in the case where L is compact.

(c.4) If M is a homogeneous pseudo-Kähler manifold of G and G acts on M almost effectively, then M becomes an elliptic orbit of G. Conversely, any elliptic orbit of G is a homogeneous pseudo-Kähler manifold of G. cf. Dorfmeister-Guan, 1991.

(c.5) We assert that $\varrho : G \to GL(\mathcal{H})$ is irreducible unitary, on weak assumptions for G/L and $\chi : Q_- \to \mathbb{C}^*$. Nobody has completely proved that on the same assumptions as ours, as far as the authors know. Needless to say, one has already shown that on strong assumptions. For example, Borel-Weil (cf. Serre, 1995) deals with the case where G is compact, Harish-Chandra (Harish-Chandra, 1956) does the case where L is a compact Cartan subgroup of G, etc.

(c.6) It is known in some cases that there are sufficient conditions for $\mathcal{H}$ to be not equal to $\{0\}$. e.g. Harish-Chandra, 1956; Serre, 1995.

Let us give two examples.

Example 1.3. Let $G_{\mathbb{C}} = SL(2,\mathbb{C})$,

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \,\middle|\, \alpha,\beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}, \quad T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

This T is a non-zero elliptic element of $\mathfrak{g} = \mathfrak{su}(1,1)$ because the linear transformation $\operatorname{ad} T : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is represented by

$$\operatorname{ad} T = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

relative to a complex basis $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ of $\mathfrak{g}_{\mathbb{C}}$. Moreover, a direct computation yields

$$L = S(U(1)\times U(1)), \quad U_+ = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C} \right\}, \quad Q_- = \left\{ \begin{pmatrix} \beta & 0 \\ \gamma & 1/\beta \end{pmatrix} \,\middle|\, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C} \right\}.$$

Define a holomorphic homomorphism $\chi : Q_- \to \mathbb{C}^*$ by

$$\chi \begin{pmatrix} \beta & 0 \\ \gamma & 1/\beta \end{pmatrix} := 1/\beta^2 \quad \text{for} \begin{pmatrix} \beta & 0 \\ \gamma & 1/\beta \end{pmatrix} \in Q_-.$$

Then, Theorem 1.2 implies that ϱ is an irreducible unitary representation of G on $\mathcal{H}$ because the supposition (S) holds. In this case $\mathcal{H} \neq \{0\}$. Indeed; let us consider a holomorphic function $\varphi_{\max} : GQ_- \to \mathbb{C}$ defined by

$$\varphi_{\max} \begin{pmatrix} u & v \\ z & w \end{pmatrix} := 1/w^2 \quad \text{for} \begin{pmatrix} u & v \\ z & w \end{pmatrix} \in GQ_-,$$

where we remark that $w \neq 0$ for all $\begin{pmatrix} u & v \\ z & w \end{pmatrix} \in GQ_-$. For this $\varphi_{\max}$ one can confirm that $\varphi_{\max}(xq) = \chi(q)^{-1}\varphi_{\max}(x)$ for all $(x,q) \in GQ_- \times Q_-$. So, $\varphi_{\max} \in \mathcal{V}$. Taking an Iwasawa decomposition of $G = SU(1,1)$ into account, we deduce

$$\begin{aligned} \|\varphi_{\max}\|^2 &= \int_G |\varphi_{\max}(g)|^2 d\mu(g) \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dn \int_0^{4\pi} dk \left| \varphi_{\max}\left(\begin{pmatrix} \cosh\left(\frac{a}{2}\right) & \sinh\left(\frac{a}{2}\right) \\ \sinh\left(\frac{a}{2}\right) & \cosh\left(\frac{a}{2}\right) \end{pmatrix} \begin{pmatrix} 1+\frac{in}{2} & -\frac{in}{2} \\ \frac{in}{2} & 1-\frac{in}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{ik}{2}} & 0 \\ 0 & e^{-\frac{ik}{2}} \end{pmatrix} \right) \right|^2 = 4\pi < \infty. \end{aligned}$$

Therefore $\varphi_{\max} \in \mathcal{H}$, and $\mathcal{H} \neq \{0\}$. Incidentally, $\varphi_{\max}(u) = 1$ for all $u \in U_+ \cap GQ_-$,

$$\int_G |\langle \varrho(g)\varphi_{\max}, \varphi_{\max}\rangle|^2 d\mu(g) = 64\pi^3 < \infty,$$

and the representation space $\mathcal{H}$ of ϱ corresponds to the complex Hilbert space of square integrable holomorphic 1-forms ω on the open unit disk in $\mathbb{C}$.

Example 1.4. Let $G_{\mathbb{C}} = SL(2,\mathbb{C})$, $G = SU(2)$ and

$$T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

This T is a non-zero elliptic element of $\mathfrak{g} = \mathfrak{su}(2)$ and

$$L = S(U(1)\times U(1)), \quad U_+ = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C} \right\}, \quad Q_- = \left\{ \begin{pmatrix} \beta & 0 \\ \gamma & 1/\beta \end{pmatrix} \,\middle|\, \beta \in \mathbb{C}^*, \gamma \in \mathbb{C} \right\}.$$

Define holomorphic homomorphisms $\chi_+ : Q_- \to \mathbb{C}^*$ and $\chi_- : Q_- \to \mathbb{C}^*$ by

$$\chi_+\begin{pmatrix}\beta & 0\\ \gamma & 1/\beta\end{pmatrix} := \beta^2, \quad \chi_-\begin{pmatrix}\beta & 0\\ \gamma & 1/\beta\end{pmatrix} := 1/\beta^2 \quad \text{for} \begin{pmatrix}\beta & 0\\ \gamma & 1/\beta\end{pmatrix} \in Q_-,$$

respectively. In each case Theorem 1.2 assures that ϱ is an irreducible unitary representation of G on $\mathcal{H}$, since the supposition (S) holds. Note that $GQ_- = G_{\mathbb{C}}$ and $\mathcal{V} = \mathcal{H}$.

• In the case of χ_+, one has $\mathcal{H} \neq \{0\}$. Indeed; if we define a holomorphic function $\varphi_{\max} : G_{\mathbb{C}} \to \mathbb{C}$ by

$$\varphi_{\max}\begin{pmatrix}u & v\\ z & w\end{pmatrix} := w^2 \quad \text{for} \begin{pmatrix}u & v\\ z & w\end{pmatrix} \in G_{\mathbb{C}},$$

then it turns out that $\varphi_{\max}(xq) = \chi_+(q)^{-1}\varphi_{\max}(x)$ for all $(x, q) \in G_{\mathbb{C}} \times Q_-$ and $\varphi_{\max}(u) = 1$ for all $u \in U_+$, so that $0 \neq \varphi_{\max} \in \mathcal{V} = \mathcal{H}$.

• In the case of χ_-, one has $\mathcal{H} = \{0\}$. Let us explain the reason why. For a given holomorphic function $\psi : G_{\mathbb{C}} \to \mathbb{C}$, suppose it to satisfy $\psi(xq) = \chi_-(q)^{-1}\psi(x)$ for all $(x, q) \in G_{\mathbb{C}} \times Q_-$. Then for $M := \sup\{|\psi(g)| : g \in G\}$, we assert that $M < \infty$ because $G = SU(2)$ is compact. Moreover, by the supposition we conclude

$$\begin{aligned}\left|\psi\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\right| &= \left|\psi\left(\begin{pmatrix}1/\sqrt{1+|\alpha|^2} & \alpha/\sqrt{1+|\alpha|^2}\\ -\overline{\alpha}/\sqrt{1+|\alpha|^2} & 1/\sqrt{1+|\alpha|^2}\end{pmatrix}\begin{pmatrix}1/\sqrt{1+|\alpha|^2} & 0\\ \overline{\alpha}/\sqrt{1+|\alpha|^2} & \sqrt{1+|\alpha|^2}\end{pmatrix}\right)\right|\\ &= \left|\chi_-\begin{pmatrix}1/\sqrt{1+|\alpha|^2} & 0\\ \overline{\alpha}/\sqrt{1+|\alpha|^2} & \sqrt{1+|\alpha|^2}\end{pmatrix}^{-1}\psi\begin{pmatrix}1/\sqrt{1+|\alpha|^2} & \alpha/\sqrt{1+|\alpha|^2}\\ -\overline{\alpha}/\sqrt{1+|\alpha|^2} & 1/\sqrt{1+|\alpha|^2}\end{pmatrix}\right| \leq \frac{M}{1+|\alpha|^2} \leq M\end{aligned}$$

for all $\alpha \in \mathbb{C}$. Since $\mathbb{C}$ and U_+ are biholomorphic, Liouville's theorem on entire functions implies that ψ is the constant function with the value $\psi(e)$ on U_+. This constant must be zero because $|\psi(e)| \leq M/(1 + |\alpha|^2)$ for all $\alpha \in \mathbb{C}$. Consequently it follows from the supposition that $\psi(a) = 0$ for all $a \in U_+Q_-$. Hence, the theorem of identity tells us that $\psi = 0$ on the whole $G_{\mathbb{C}}$, because U_+Q_- is an open subset in $G_{\mathbb{C}}$. For this reason one has $\{0\} = \mathcal{V} = \mathcal{H}$. Incidentally, in the case of χ_+ (resp. χ_-) the representation space $\mathcal{H}$ of ϱ corresponds to the complex vector space of holomorphic vector fields X (resp. holomorphic 1-forms ω) on the complex projective space CP^1.

1.4 Outline of this Paper

This paper consists of three sections.

§2 Preliminaries

We recall the definition of elliptic element and establish Theorem 2.3 which will play a role in the next section. Besides, we review elementary facts about Haar measures, complex flag manifolds, homogeneous holomorphic line bundles and so on.

§3 Proof of the main result

We construct an element $\hat{\psi}_\lambda \in \mathcal{V}$ from $\psi \in \mathcal{V}$ and (3.3), and then clarify some properties of $\hat{\psi}_\lambda$, especially its integral $\int_0^{2\pi} \hat{\psi}_\lambda d\lambda$. Making use of the properties we conclude almost all of the propositions and lemmas in this section, and finally complete the proof of Theorem 1.2.

2. Preliminaries

2.1 Notation

Throughout this paper, for a Lie group G, we denote its Lie algebra by the corresponding Fraktur small letter $\mathfrak{g}$, and utilize the following notation:

(n1) $i := \sqrt{-1}$,

(n2) Ad, ad : the adjoint representation of G, $\mathfrak{g}$,

(n3) $C_G(T) := \{g \in G \mid \operatorname{Ad} g(T) = T\}$ for an element $T \in \mathfrak{g}$,

(n4) $N_G(\mathfrak{m}) := \{g \in G \mid \operatorname{Ad} g(\mathfrak{m}) \subset \mathfrak{m}\}$ for a vector subspace $\mathfrak{m} \subset \mathfrak{g}$,

(n5) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,

(n6) $f|_S$: the restriction of a mapping f to a set S.

2.2 The Definition of Elliptic Element and Theorem 2.3

Throughout Subsection 2.2, we denote by $\mathfrak{l}_{\mathbb{C}}$ the complexification of a real Lie algebra $\mathfrak{l}$. Let $\mathfrak{g}$ be a real semisimple Lie algebra. We recall the definition of elliptic element.

Definition 2.1 (cf. Kobayashi, 1998, p.5)**.** *An element $T \in \mathfrak{g}$ is said to be elliptic, if $\operatorname{ad} T$ is a semisimple linear transformation of $\mathfrak{g}$ and all the eigenvalues of $\operatorname{ad} T$ in $\mathfrak{g}_{\mathbb{C}}$ are purely imaginary.*

For an elliptic element $T \in \mathfrak{g}$, we set

$$\mathfrak{g}_\lambda(T) := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\} \quad \text{for } \lambda \in \mathbb{R}; \quad \mathfrak{u}_\pm(T) := \bigoplus_{\lambda>0} \mathfrak{g}_{\pm\lambda}(T). \tag{2.2}$$

Our goal in this subsection is to demonstrate

Theorem 2.3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then for any elliptic element $T \in \mathfrak{g}$, there exists an elliptic element $T' \in \mathfrak{g}$ such that*

(i) *all the eigenvalues of $\operatorname{ad} iT'$ in $\mathfrak{g}_{\mathbb{C}}$ are integer,*

(ii) *$C_G(T) = C_G(T')$,*

(iii) *$\mathfrak{u}_+(T) = \mathfrak{u}_+(T')$, $\mathfrak{u}_-(T) = \mathfrak{u}_-(T')$.*

Remark 2.4. Theorem 2.3 allows us to suppose that all the eigenvalues of $\operatorname{ad} iT$ in $\mathfrak{g}_{\mathbb{C}}$ are integer for the elliptic element T concerning Theorem 1.2, because there are no changes in the topological group structures on L, U_+ and Q_- even if we substitute T' for T. Remark here that the $\mathfrak{g}_\lambda(T)$ and $\mathfrak{u}_\pm(T)$ in (2.2) correspond to the $\mathfrak{g}_\lambda$ and $\mathfrak{u}_\pm$ in Subsection 1.2, respectively.

We want to first prove Lemma 2.5, next deduce Proposition 2.6 from the lemma, and finally obtain the goal from the proposition.

Lemma 2.5. *Let $\mathfrak{g}$ be a real simple Lie algebra with $2 \leq \dim_{\mathbb{R}} \mathfrak{g}$. Then for any elliptic element $T \in \mathfrak{g}$, there exists an elliptic element $T' \in \mathfrak{g}$ such that*

(i) *all the eigenvalues of $\operatorname{ad} iT'$ in $\mathfrak{g}_{\mathbb{C}}$ are integer,*

(ii′) *$\mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T')$,*

(iii) *$\mathfrak{u}_+(T) = \mathfrak{u}_+(T')$, $\mathfrak{u}_-(T) = \mathfrak{u}_-(T')$.*

Proof. It is obvious in case of $T = 0$. Thus, we suppose $T \neq 0$ hereafter. Since $T \in \mathfrak{g}$ is elliptic, there exists a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}$ containing T. Furthermore, there exists a maximal torus $\mathfrak{t} \subset \mathfrak{k}$ such that $T \in \mathfrak{t}$. Let us investigate the following cases (a) and (b) individually:

(a) $\mathfrak{k}$ is not semisimple, (b) $\mathfrak{k}$ is semisimple.

Henceforth $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ denote the complex subalgebras of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{t}$ and $\mathfrak{k}$, respectively. Note that $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ in case (a) because $\mathfrak{g}$ is simple.

Case (a): Denote by $\triangle$ the set of non-zero roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$, and put $\mathfrak{t}_{\mathbb{R}} := \{H \in \mathfrak{t}_{\mathbb{C}} \mid \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \triangle\}$. In this case we obtain

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\triangle} \mathfrak{g}_\alpha, \quad T \in \mathfrak{t} = i\mathfrak{t}_{\mathbb{R}},$$

where $\mathfrak{g}_\alpha := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} H(A) = \alpha(H)A \text{ for all } H \in \mathfrak{t}_{\mathbb{C}}\}$ for $\alpha \in \triangle$. Let $\triangle^+$ denote the set of positive roots in $\triangle$ for the lexicographic linear ordering with respect to a real basis

$$-iT =: H_1, H_2, \ldots, H_l$$

of $\mathfrak{t}_{\mathbb{R}}$. In view of this ordering we conclude that

$$\beta(H_1) \geq 0 \quad \text{for all } \beta \in \triangle^+. \tag{a.1}$$

Let $\{\alpha_a\}_{a=1}^l$ ($\subset \triangle^+$) be a fundamental root system of $\triangle$, and $\{Z_a\}_{a=1}^l$ its dual basis. Then $\mathfrak{t}_\mathbb{R} = \mathrm{span}_\mathbb{R}\{Z_a\}_{a=1}^l$, so that (a.1) allows us to express the element $H_1 \in \mathfrak{t}_\mathbb{R}$ as follows: $H_1 = \sum_{a=1}^l h_a Z_a$, $h_a \geq 0$. Changing the indexes of $\alpha_1, \alpha_2, \ldots, \alpha_l$ (if necessary), one may assume that $h_1, \ldots, h_s > 0$ and $h_{s+1} = \cdots = h_l = 0$, namely

$$H_1 = \textstyle\sum_{b=1}^s h_b Z_b, \quad h_b > 0. \tag{a.2}$$

For a $Y \in \mathfrak{t}_\mathbb{R}$ we set

$$\triangle_0(Y) := \{\beta \in \triangle \,|\, \beta(Y) = 0\}, \quad \triangle_\pm(Y) := \{\gamma \in \triangle \,|\, \pm\gamma(Y) > 0\}.$$

Now, (a.2) and $\alpha_a(Z_b) = \delta_{a,b}$ ensure that for $\alpha = \sum_{a=1}^l m_a\alpha_a \in \triangle$, it belongs to $\triangle_0(H_1)$ if and only if $\alpha = \sum_{c=s+1}^l m_c\alpha_c$. That implies $\alpha_{s+1}, \ldots, \alpha_l \in \triangle_0(H_1)$; and therefore for $X = \sum_{a=1}^l x_a Z_a \in \mathfrak{t}_\mathbb{R}$,

$$\beta(X) = 0 \text{ for all } \beta \in \triangle_0(H_1) \text{ if and only if } X = \textstyle\sum_{b=1}^s x_b Z_b. \tag{a.3}$$

For an $\alpha = \sum_{a=1}^l m_a\alpha_a \in \triangle$ we define a continuous function $\dot{\alpha} : \mathbb{R}^s \to \mathbb{R}$ by

$$\dot{\alpha}(x_1, \ldots, x_s) := \textstyle\sum_{b=1}^s x_b m_b \quad \text{for } (x_1, \ldots, x_s) \in \mathbb{R}^s,$$

and see that $\dot{\gamma}(h_1, \ldots, h_s) = \sum_{b=1}^s h_b n_b \overset{(a.2)}{=} \gamma(H_1) > 0$ for each $\gamma = \sum_{a=1}^l n_a\alpha_a \in \triangle_+(H_1)$. Consequently, for each $\gamma \in \triangle_+(H_1)$ there exists an $\epsilon_\gamma > 0$ such that

$$\dot{\gamma}(x_1, \ldots, x_s) > 0 \text{ for all } (x_1, \ldots, x_s) \in B_{\epsilon_\gamma}(h_1, \ldots, h_s)$$

because $\dot{\gamma}$ is continuous. Here $B_{\epsilon_\gamma}(h_1, \ldots, h_s)$ denotes the open ball in $\mathbb{R}^s$ with center $(h_1, \ldots, h_s)$ and radius ϵ_γ. Let us set $\epsilon_+ := \min\{\epsilon_\gamma \,|\, \gamma \in \triangle_+(H_1)\}$. Since $\triangle_+(H_1)$ is a finite set, one has $\epsilon_+ > 0$. Moreover, it turns out that

$$\dot{\gamma}(y_1, \ldots, y_s) > 0 \text{ for all } (\gamma, (y_1, \ldots, y_s)) \in \triangle_+(H_1) \times B_{\epsilon_+}(h_1, \ldots, h_s). \tag{a.4}$$

Similarly, there exists an $\epsilon_- > 0$ such that

$$\dot{\delta}(z_1, \ldots, z_s) < 0 \text{ for all } (\delta, (z_1, \ldots, z_s)) \in \triangle_-(H_1) \times B_{\epsilon_-}(h_1, \ldots, h_s). \tag{a.5}$$

Here $B_{\epsilon_+}(h_1, \ldots, h_s) \cap B_{\epsilon_-}(h_1, \ldots, h_s)$ is a non-empty open subset in $\mathbb{R}^s$. By the denseness of rational numbers, there exist $\bar{m}_b \in \mathbb{Z}$ and $\bar{n}_b \in \mathbb{N}$ satisfying

$$\Big(\frac{\bar{m}_1}{\bar{n}_1}, \ldots, \frac{\bar{m}_s}{\bar{n}_s}\Big) \in B_{\epsilon_+}(h_1, \ldots, h_s) \cap B_{\epsilon_-}(h_1, \ldots, h_s). \tag{a.6}$$

From them we construct an element $\bar{H} \in \mathfrak{t}_\mathbb{R} = \mathrm{span}_\mathbb{R}\{Z_a\}_{a=1}^l$ as follows:

$$\bar{H} := \bar{n}_1 \cdots \bar{n}_s \sum_{b=1}^s \frac{\bar{m}_b}{\bar{n}_b} Z_b = (\bar{m}_1\bar{n}_2 \cdots \bar{n}_s)Z_1 + \cdots + (\bar{n}_1 \cdots \bar{n}_{s-1}\bar{m}_s)Z_s. \tag{a.7}$$

With the setting above, the following four items hold for $\bar{H}$:

(i′) All the eigenvalues of $\mathrm{ad}\,\bar{H}$ in $\mathfrak{g}_\mathbb{C}$ are integer, since $\bar{H} \in \mathfrak{t}_\mathbb{R} \subset \mathfrak{t}_\mathbb{C}$, $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\triangle} \mathfrak{g}_\alpha$ and $\alpha(\bar{H}) = (\bar{m}_1\bar{n}_2 \cdots \bar{n}_s)m_1 + \cdots + (\bar{n}_1 \cdots \bar{n}_{s-1}\bar{m}_s)m_s \in \mathbb{Z}$ for each $\alpha = \sum_{a=1}^l m_a\alpha_a \in \triangle$.

(ii′) $\beta(\bar{H}) = 0$ for all $\beta \in \triangle_0(H_1)$, because of (a.3) and (a.7).

(iii′) $\gamma(\bar{H}) > 0$ for all $\gamma \in \triangle_+(H_1)$. Indeed; if $\gamma = \sum_{a=1}^l n_a\alpha_a$, then

$$\gamma(\bar{H}) \overset{(a.7)}{=} \bar{n}_1 \cdots \bar{n}_s \sum_{b=1}^s \frac{\bar{m}_b}{\bar{n}_b} n_b = \bar{n}_1 \cdots \bar{n}_s \dot{\gamma}\Big(\frac{\bar{m}_1}{\bar{n}_1}, \ldots, \frac{\bar{m}_s}{\bar{n}_s}\Big) > 0$$

by virtue of $\bar{n}_1 \cdots \bar{n}_s > 0$, (a.4) and (a.6).

(iv′) $\delta(\bar{H}) < 0$ for all $\delta \in \triangle_-(H_1)$.

These (ii′), (iii′) and (iv′) yield $\triangle_0(H_1) \subset \triangle_0(\bar{H})$, $\triangle_\pm(H_1) \subset \triangle_\pm(\bar{H})$. Therefore we deduce

$$\triangle_0(H_1) = \triangle_0(\bar{H}), \quad \triangle_+(H_1) = \triangle_+(\bar{H}), \quad \triangle_-(H_1) = \triangle_-(\bar{H}) \tag{a.8}$$

from $\triangle_0(H_1) \sqcup \triangle_+(H_1) \sqcup \triangle_-(H_1) = \triangle = \triangle_0(\bar{H}) \sqcup \triangle_+(\bar{H}) \sqcup \triangle_-(\bar{H})$ (disjoint union). Setting $\bar{T} := i\bar{H}$, one has $\bar{T} \in i\mathfrak{t}_\mathbb{R} = \mathfrak{t} \subset \mathfrak{g}$. This assures that $\bar{T}$ is an elliptic element of $\mathfrak{g}$. In addition, (i′), $T = iH_1$, $\bar{T} = i\bar{H}$ and (a.8) imply that

(i) all the eigenvalues of $\operatorname{ad} i\bar{T}$ in $\mathfrak{g}_{\mathbb{C}}$ are integer;

(ii) $\mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta\in\triangle_0(H_1)} \mathfrak{g}_\beta = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta\in\triangle_0(\bar{H})} \mathfrak{g}_\beta = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(\bar{T})$;

(iii) $\mathfrak{u}_+(T) = \bigoplus_{\gamma\in\triangle_+(H_1)} \mathfrak{g}_\gamma = \bigoplus_{\gamma\in\triangle_+(\bar{H})} \mathfrak{g}_\gamma = \mathfrak{u}_+(\bar{T})$;

(iv) $\mathfrak{u}_-(T) = \mathfrak{u}_-(\bar{T})$.

Accordingly Lemma 2.5 holds in case (a). Thus, the rest of proof is to confirm that in case (b).

Case (b): $\mathfrak{k}_{\mathbb{C}}$ is a complex semisimple Lie algebra and $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. We denote by Ξ the set of non-zero roots of $\mathfrak{k}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$ and obtain

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\xi\in\Xi} \mathfrak{k}_\xi,$$

where $\mathfrak{k}_\xi := \{K \in \mathfrak{k}_{\mathbb{C}} \,|\, \operatorname{ad} H(K) = \xi(H)K \text{ for all } H \in \mathfrak{t}_{\mathbb{C}}\}$ for $\xi \in \Xi$. Let us set $\mathfrak{t}_{\mathbb{R}} := \{H \in \mathfrak{t}_{\mathbb{C}} \,|\, \xi(H) \in \mathbb{R} \text{ for all } \xi \in \Xi\}$, fix a fundamental root system $\{\xi_a\}_{a=1}^n \subset \Xi$, and denote the dual basis of $\{\xi_a\}_{a=1}^n$ by $\{W_a\}_{a=1}^n$. Then it follows that

$$iT \in \mathfrak{t}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{W_a\}_{a=1}^n, \quad i\mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}.$$

Express a Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and define a representation ρ of $\mathfrak{k}_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ by $\rho : \mathfrak{k}_{\mathbb{C}} \to \operatorname{End}(\mathfrak{p}_{\mathbb{C}})$, $K \mapsto \operatorname{ad} K|_{\mathfrak{p}_{\mathbb{C}}}$, where $\mathfrak{p}_{\mathbb{C}}$ denotes the complex vector subspace of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{p}$. Denoting by Ω the set of weights of ρ relative to $\mathfrak{t}_{\mathbb{C}}$, we have

$$\mathfrak{p}_{\mathbb{C}} = \bigoplus_{\omega\in\Omega} \mathfrak{p}_\omega,$$

where $\mathfrak{p}_\omega := \{P \in \mathfrak{p}_{\mathbb{C}} \,|\, \operatorname{ad} H(P) = \omega(H)P \text{ for all } H \in \mathfrak{t}_{\mathbb{C}}\}$ for $\omega \in \Omega$. With the setting above, the following statement holds:

$$\text{for any } \alpha \in \Xi \sqcup \Omega, \text{ there exist unique rational numbers } q_a \text{ such that } \alpha = \textstyle\sum_{a=1}^n q_a\xi_a. \tag{b.1}$$

For an $X \in \mathfrak{t}_{\mathbb{R}}$ we set

$$\triangle_0(X) := \{\beta \in \Xi \sqcup \Omega \,|\, \beta(X) = 0\}, \quad \triangle_\pm(X) := \{\gamma \in \Xi \sqcup \Omega \,|\, \pm\gamma(X) > 0\}.$$

From now on, we put $H := -iT$, suppose that $\triangle_0(H)$ consists of m-elements $\beta_1, \ldots, \beta_m$, and express them as

$$\begin{cases} \beta_1 = q_{11}\xi_1 + \cdots + q_{1n}\xi_n, \\ \quad\vdots \\ \beta_m = q_{m1}\xi_1 + \cdots + q_{mn}\xi_n, \end{cases} \qquad q_{ba} \in \mathbb{Q} \quad (1 \le b \le m,\ 1 \le a \le n)$$

in accordance with (b.1). Denote by r the rank of the matrix $(q_{ba})_{1\le b\le m, 1\le a\le n}$ above. Let us study the following system of m linear homogeneous equations:

$$\begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{b.2}$$

Applying the elementary row or column operations to the coefficient matrix $(q_{ba})_{1\le b\le m, 1\le a\le n}$, we can obtain

$$\left(\begin{array}{ccc|ccc} 1 & & O & \tilde{q}_{1,r+1} & \cdots & \tilde{q}_{1n} \\ & \ddots & & \vdots & \ddots & \vdots \\ O & & 1 & \tilde{q}_{r,r+1} & \cdots & \tilde{q}_{rn} \\ \hline & O_{m-r,r} & & & O_{m-r,n-r} & \end{array}\right), \quad \tilde{q}_{jk} \in \mathbb{Q} \quad (1 \le j \le r,\ r+1 \le k \le n).$$

Changing the indexes of $x_1, x_2, \ldots, x_n$, one may assume that $x_j = -\sum_{k=r+1}^n \tilde{q}_{jk}t_k$ $(1 \le j \le r)$, $x_k = t_k$ $(r+1 \le k \le n)$ is the solution of (b.2), where $t_{r+1}, \ldots, t_n$ are indeterminate. Hence, by $\xi_a(W_c) = \delta_{a,c}$ and changing the indexes of $\xi_1, \xi_2, \ldots, \xi_n$, we assert the following statement: for $X = \sum_{a=1}^n x_a W_a \in \mathfrak{t}_{\mathbb{R}}$,

$$\begin{aligned} &\beta(X) = 0 \text{ for all } \beta \in \triangle_0(H) = \{\beta_b\}_{b=1}^m \text{ if and only if there exists a } (t_{r+1}, \ldots, t_n) \in \mathbb{R}^{n-r} \\ &\qquad\qquad \text{such that } x_j = -\textstyle\sum_{k=r+1}^n \tilde{q}_{jk}t_k \ (1 \le j \le r) \text{ and } x_k = t_k \ (r+1 \le k \le n). \end{aligned} \tag{b.3}$$

For this reason one elucidates that

$$h_j = -\sum_{k=r+1}^{n} \tilde{q}_{jk} h_k \quad (1 \leq j \leq r) \tag{b.4}$$

when the element $H \in \mathfrak{t}_{\mathbb{R}}$ is expressed as $H = \sum_{a=1}^{n} h_a W_a$. Now, we take an $\alpha = \sum_{a=1}^{n} q_a \xi_a \in \Xi \sqcup \Omega$ and define a continuous function $\ddot{\alpha} : \mathbb{R}^{n-r} \to \mathbb{R}$ by

$$\ddot{\alpha}(y_{r+1}, \ldots, y_n) := \sum_{k=r+1}^{n} (q_k - \sum_{j=1}^{r} \tilde{q}_{jk} q_j) y_k \quad \text{for } (y_{r+1}, \ldots, y_n) \in \mathbb{R}^{n-r}.$$

For any $\gamma = \sum_{a=1}^{n} p_a \xi_a \in \triangle_+(H)$, it follows from $H = \sum_{a=1}^{n} h_a W_a$ and $\xi_a(W_c) = \delta_{a,c}$ that

$$0 < \gamma(H) = \sum_{a=1}^{n} p_a h_a \overset{\text{(b.4)}}{=} \sum_{k=r+1}^{n} \Big(p_k - \sum_{j=1}^{r} \tilde{q}_{jk} p_j \Big) h_k = \ddot{\gamma}(h_{r+1}, \ldots, h_n).$$

Accordingly there exists an $\epsilon_+ > 0$ such that

$$\ddot{\gamma}(y_{r+1}, \ldots, y_n) > 0 \text{ for all } (\gamma, (y_{r+1}, \ldots, y_n)) \in \triangle_+(H) \times B_{\epsilon_+}(h_{r+1}, \ldots, h_n). \tag{b.5}$$

Similarly, there exists an $\epsilon_- > 0$ such that

$$\ddot{\delta}(z_{r+1}, \ldots, z_n) < 0 \text{ for all } (\delta, (z_{r+1}, \ldots, z_n)) \in \triangle_-(H) \times B_{\epsilon_-}(h_{r+1}, \ldots, h_n). \tag{b.6}$$

Here $B_{\epsilon_+}(h_{r+1}, \ldots, h_n) \cap B_{\epsilon_-}(h_{r+1}, \ldots, h_n)$ is a non-empty open subset in $\mathbb{R}^{n-r}$. So, the denseness of rational numbers provides us with $\tilde{q}_{r+1}, \ldots, \tilde{q}_n \in \mathbb{Q}$ which satisfy

$$(\tilde{q}_{r+1}, \ldots, \tilde{q}_n) \in B_{\epsilon_+}(h_{r+1}, \ldots, h_n) \cap B_{\epsilon_-}(h_{r+1}, \ldots, h_n). \tag{b.7}$$

By use of the rational numbers $\tilde{q}_{jk}$ and $\tilde{q}_k$, we define an element $\tilde{H} \in \mathfrak{t}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{W_a\}_{a=1}^{n}$ by

$$\tilde{H} := \Big(-\sum_{k=r+1}^{n} \tilde{q}_{1k} \tilde{q}_k \Big) W_1 + \cdots + \Big(-\sum_{k=r+1}^{n} \tilde{q}_{rk} \tilde{q}_k \Big) W_r + \tilde{q}_{r+1} W_{r+1} + \cdots + \tilde{q}_n W_n. \tag{b.8}$$

Then, it turns out that

(i′) all the eigenvalues of $\operatorname{ad} \tilde{H}$ in $\mathfrak{g}_{\mathbb{C}}$ are rational numbers, because of $\tilde{H} \in \mathfrak{t}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\xi \in \Xi} \mathfrak{k}_\xi \oplus \bigoplus_{\omega \in \Omega} \mathfrak{p}_\omega$ and $\alpha(\tilde{H}) \overset{\text{(b.8)}}{=} (-\sum_{k=r+1}^{n} \tilde{q}_{1k} \tilde{q}_k) q_1 + \cdots + (-\sum_{k=r+1}^{n} \tilde{q}_{rk} \tilde{q}_k) q_r + \tilde{q}_{r+1} q_{r+1} + \cdots + \tilde{q}_n q_n \in \mathbb{Q}$ for every $\alpha = \sum_{a=1}^{n} q_a \xi_a \in \Xi \sqcup \Omega$, cf. (b.1);

(ii′) $\beta(\tilde{H}) = 0$ for all $\beta \in \triangle_0(H)$, in terms of (b.3) and (b.8);

(iii′) $\gamma(\tilde{H}) > 0$ for all $\gamma \in \triangle_+(H)$ because if $\gamma = \sum_{a=1}^{n} p_a \xi_a$, then

$$\gamma(\tilde{H}) \overset{\text{(b.8)}}{=} -\sum_{j=1}^{r} \sum_{k=r+1}^{n} \tilde{q}_{jk} \tilde{q}_k p_j + \sum_{k=r+1}^{n} \tilde{q}_k p_k = \ddot{\gamma}(\tilde{q}_{r+1}, \ldots, \tilde{q}_n) > 0$$

comes from (b.5) and (b.7);

(iv′) $\delta(\tilde{H}) < 0$ for all $\delta \in \triangle_-(H)$.

These (ii′), (iii′) and (iv′) imply that

$$\triangle_0(H) = \triangle_0(\tilde{H}), \quad \triangle_+(H) = \triangle_+(\tilde{H}), \quad \triangle_-(H) = \triangle_-(\tilde{H}). \tag{b.9}$$

In view of (i′) we suppose that the set of the eigenvalues of $\operatorname{ad} \tilde{H}$ consists of $\tilde{m}_1/\tilde{n}_1, \ldots, \tilde{m}_u/\tilde{n}_u$, where $\tilde{m}_v \in \mathbb{Z}$ and $\tilde{n}_v \in \mathbb{N}$. In this case we set

$$\tilde{T} := i(\tilde{n}_1 \cdots \tilde{n}_u)\tilde{H}$$

and have $\tilde{T} \in i\mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \subset \mathfrak{g}$. Hence, $\tilde{T}$ is an elliptic element of $\mathfrak{g}$. Moreover, it follows from $\tilde{n}_1 \cdots \tilde{n}_u > 0$ that $\triangle_0(\tilde{H}) = \triangle_0(\tilde{n}_1 \cdots \tilde{n}_u \tilde{H})$ and $\triangle_\pm(\tilde{H}) = \triangle_\pm(\tilde{n}_1 \cdots \tilde{n}_u \tilde{H})$, so that

(i) all the eigenvalues of $\mathrm{ad}\, i\tilde{T}$ in $\mathfrak{g}_{\mathbb{C}}$ are integer, because the set of the eigenvalues of $\mathrm{ad}\, i\tilde{T}$ consists of $-(\tilde{m}_1\tilde{n}_2\cdots\tilde{n}_u)$, $\ldots$, $-(\tilde{n}_1\cdots\tilde{n}_{u-1}\tilde{m}_u)$;

(ii) $\mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta\in\triangle_0(H)\cap\Xi} \mathfrak{k}_\beta \oplus \bigoplus_{\beta\in\triangle_0(H)\cap\Omega} \mathfrak{p}_\beta = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta\in\triangle_0(\tilde{H})\cap\Xi} \mathfrak{k}_\beta \oplus \bigoplus_{\beta\in\triangle_0(\tilde{H})\cap\Omega} \mathfrak{p}_\beta = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(\tilde{T})$, since $T = iH$ and $\triangle_0(H) \overset{(b.9)}{=} \triangle_0(\tilde{H}) = \triangle_0(\tilde{n}_1\cdots\tilde{n}_u\tilde{H})$;

(iii) $\mathfrak{u}_+(T) = \bigoplus_{\gamma\in\triangle_+(H)\cap\Xi} \mathfrak{k}_\gamma \oplus \bigoplus_{\gamma\in\triangle_+(H)\cap\Omega} \mathfrak{p}_\gamma = \bigoplus_{\gamma\in\triangle_+(\tilde{H})\cap\Xi} \mathfrak{k}_\gamma \oplus \bigoplus_{\gamma\in\triangle_+(\tilde{H})\cap\Omega} \mathfrak{p}_\gamma = \mathfrak{u}_+(\tilde{T})$;

(iv) $\mathfrak{u}_-(T) = \mathfrak{u}_-(\tilde{T})$.

Consequently, Lemma 2.5 holds in case (b), also. □

Lemma 2.5 leads to

Proposition 2.6. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then for any elliptic element $T \in \mathfrak{g}$, there exists an elliptic element $T' \in \mathfrak{g}$ such that*

(i) *all the eigenvalues of* $\mathrm{ad}\, iT'$ *in* $\mathfrak{g}_{\mathbb{C}}$ *are integer,*

(ii′) $\mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T')$*,*

(iii) $\mathfrak{u}_+(T) = \mathfrak{u}_+(T')$, $\mathfrak{u}_-(T) = \mathfrak{u}_-(T')$.

Proof. Since $\mathfrak{g}$ is real semisimple, one can decompose it as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n,$$

where all $\mathfrak{g}_a$ are real simple ideals of $\mathfrak{g}$ ($1 \leq a \leq n$). Express the element T as $T = T_1 + T_2 + \cdots + T_n$ ($T_a \in \mathfrak{g}_a$). Then T_a is an elliptic element of $\mathfrak{g}_a$ for every $1 \leq a \leq n$. Denoting by $\mathfrak{g}_{a,\mathbb{C}}$ the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{g}_a$, we confirm that (1) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{1,\mathbb{C}} \oplus \mathfrak{g}_{2,\mathbb{C}} \oplus \cdots \oplus \mathfrak{g}_{n,\mathbb{C}}$, (2) each $\mathfrak{g}_{a,\mathbb{C}}$ is a complex simple or semisimple ideal of $\mathfrak{g}_{\mathbb{C}}$ and (3) $\mathfrak{g}_a$ is a real form of $\mathfrak{g}_{a,\mathbb{C}}$. Accordingly for each T_a ($1 \leq a \leq n$), Lemma 2.5 provides us with an elliptic element $T'_a \in \mathfrak{g}_a$ such that

(i) all the eigenvalues of $\mathrm{ad}\, iT'_a$ in $\mathfrak{g}_{a,\mathbb{C}}$ are integer,

(ii′) $\mathfrak{c}_{\mathfrak{g}_{a,\mathbb{C}}}(T_a) = \mathfrak{c}_{\mathfrak{g}_{a,\mathbb{C}}}(T'_a)$,

(iii) $\mathfrak{u}_{a,+}(T_a) = \mathfrak{u}_{a,+}(T'_a)$, $\mathfrak{u}_{a,-}(T_a) = \mathfrak{u}_{a,-}(T'_a)$,

where $\mathfrak{g}_{a,\lambda}(\bar{T}) := \{A \in \mathfrak{g}_{a,\mathbb{C}} \mid \mathrm{ad}\,\bar{T}(A) = i\lambda A\}$ for $\lambda \in \mathbb{R}$ and $\mathfrak{u}_{a,\pm}(\bar{T}) := \bigoplus_{\lambda>0} \mathfrak{g}_{a,\pm\lambda}(\bar{T})$. Setting $T' := T'_1 + T'_2 + \cdots + T'_n$, we get the conclusion. □

Now, let us demonstrate Theorem 2.3.

Proof of Theorem 2.3. By Proposition 2.6 there exists an elliptic element $T' \in \mathfrak{g}$ such that

(i) all the eigenvalues of $\mathrm{ad}\, iT'$ in $\mathfrak{g}_{\mathbb{C}}$ are integer,

(ii′) $\mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T')$,

(iii) $\mathfrak{u}_+(T) = \mathfrak{u}_+(T')$, $\mathfrak{u}_-(T) = \mathfrak{u}_-(T')$.

Hence the rest of proof is to conclude (ii) $C_G(T) = C_G(T')$. On the one hand; it is immediate from (ii′) that $\mathfrak{c}_{\mathfrak{g}}(T) = (\mathfrak{g} \cap \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T)) = (\mathfrak{g} \cap \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T')) = \mathfrak{c}_{\mathfrak{g}}(T')$. On the other hand; both $C_G(T)$ and $C_G(T')$ are connected, since G is connected semisimple and since both T and T' are elliptic elements of $\mathfrak{g}$ (e.g. Lemma 2 in Boumuki, 2013, p.9). Therefore (ii) follows. □

2.3 A Known Result about the Haar Measure

Let us recall a known result about the Haar measure.

Let G be a connected real semisimple Lie group, and let $\mathscr{B}$ denote the σ-algebra generated by the set of open subsets in G (namely, $\mathscr{B}$ is the Borel field on G). Since G is connected, it satisfies the second countability axiom. Hence, there uniquely exists an extended real-valued function μ on $\mathscr{B}$ (up to positive constant) having the following seven properties, which we call this μ the non-zero *Haar measure* on G:

(p1) $0 \leq \mu(A) \leq \infty$ for all $A \in \mathscr{B}$,

(p2) $\mu(B) > 0$ for each non-empty, open subset B in G,

(p3) $\mu(C) < \infty$ for each compact subset C in G,

(p4) If $A_n \in \mathscr{B}$ $(n = 1, 2, \dots)$ and $A_j \cap A_k = \emptyset$ $(j \neq k)$, then $\sum_{n=1}^{\infty} \mu(A_n) = \mu(\sum_{n=1}^{\infty} A_n)$,

(p5) $\mu(gA) = \mu(A)$ for all $(g, A) \in G \times \mathscr{B}$,

(p6) $\mu(Ag) = \mu(A)$ for all $(g, A) \in G \times \mathscr{B}$,

(p7) $\mu(A) = \inf\{\mu(U) : U \text{ is open in } G, A \subset U\}$ for every $A \in \mathscr{B}$.

cf. Haar, 1933; von Neumann, 1936.

Remark 2.7. Since G is a locally compact Hausdorff space and satisfies the second countability axiom, it follows from (p3) that μ is a regular Borel measure on $(G, \mathscr{B})$. cf. Proposition 7.2.3 in Cohn, 2013, p.190.

2.4 Elementary Facts about Complex Flag Manifolds, Homogeneous Holomorphic Line Bundles and So On

In this subsection we mainly review elementary facts about complex flag manifolds, homogeneous holomorphic line bundles and so on. Throughout this subsection, $G_{\mathbb{C}}$ is a connected complex semisimple Lie group, G is a connected closed subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$, and T is a non-zero elliptic element of $\mathfrak{g}$. Here we remark that the center $Z(G)$ of G is finite due to $Z(G) \subset Z(G_{\mathbb{C}})$.

2.4.1 Root Systems and Iwasawa Decompositions

Since $\mathfrak{g}$ is semisimple and T is elliptic, there exists a Cartan involution θ of $\mathfrak{g}$ which satisfies $\theta(T) = T$. From $\mathfrak{k} := \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}$, we construct a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad T \in \mathfrak{k}.$$

Take a maximal torus $\mathfrak{t}$ in $\mathfrak{k}$ containing T. Then, $\mathfrak{h} := \{X \in \mathfrak{g} \,|\, [X, Y] = 0 \text{ for all } Y \in \mathfrak{t}\}$ satisfies $\theta(\mathfrak{h}) \subset \mathfrak{h}$, and

$$\mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{h}), \quad \mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}, \quad T \in \mathfrak{t}.$$

So, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with $T \in \mathfrak{h}$. Let $\triangle$ denote the set of non-zero roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$, and let $\triangle^+$ denote the set of positive roots in $\triangle$, where $\mathfrak{h}_{\mathbb{C}}$ denotes the complex vector subspace of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{h}$ and we define positivity by means of the lexicographic linear ordering with respect to a real basis

$$-iT = A_1, A_2, \dots, A_p \tag{2.8}$$

of $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{t} \oplus (\mathfrak{p} \cap \mathfrak{h})$. Denote by $\mathfrak{g}_\alpha$ the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for $\alpha \in \triangle$, and set

$$\mathfrak{n}_{\pm} := \bigoplus_{\beta \in \triangle^+} \mathfrak{g}_{\pm\beta}, \quad \mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}.$$

In this setting, one has $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \triangle} \mathfrak{g}_\alpha = \mathfrak{n}_+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_-$ and

Lemma 2.9. *Let $H_{\mathbb{R}}$, $N_{\pm}$ and G_u be the connected Lie subgroups of $G_{\mathbb{C}}$ corresponding to the subalgebras $\mathfrak{h}_{\mathbb{R}}$, $\mathfrak{n}_{\pm}$ and $\mathfrak{g}_u$ of $\mathfrak{g}_{\mathbb{C}}$, respectively. Then*

(1) *$H_{\mathbb{R}}$ is a simply connected, closed abelian subgroup of $G_{\mathbb{C}}$, and $\exp : \mathfrak{h}_{\mathbb{R}} \to H_{\mathbb{R}}$ is a real analytic diffeomorphism.*

(2) *N_s is a simply connected, complex closed nilpotent subgroup of $G_{\mathbb{C}}$, and* $\exp : \mathfrak{n}_s \to N_s$ *is a biholomorphism, for each $s = \pm$.*

(3) *G_u is a maximal compact subgroup of $G_{\mathbb{C}}$.*

(4) *The product mapping $G_u \times H_{\mathbb{R}} \times N_s \ni (k, a, n) \mapsto kan \in G_{\mathbb{C}}$ is a real analytic diffeomorphism for each $s = \pm$.*

Proof. Note that (i) $\mathfrak{g}_u$ is a maximal compact subalgebra of $\mathfrak{g}_{\mathbb{C}}$, (ii) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$, (iii) $\mathfrak{h}_{\mathbb{R}}$ is a maximal abelian subspace of $i\mathfrak{g}_u$ and (iv) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_u \oplus \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{n}_s$ is an Iwasawa decomposition of $\mathfrak{g}_{\mathbb{C}}$ ($s = \pm$), where we here regard $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{n}_s$ as real Lie algebras, respectively.

(1) By Theorem 6.46 in Knapp, 2004, p.374, and its proof, $H_{\mathbb{R}}$ is a simply connected, closed abelian subgroup of $G_{\mathbb{C}}$. Theorem 1.127 in Knapp, p.107, implies that $\exp : \mathfrak{h}_{\mathbb{R}} \to H_{\mathbb{R}}$ is a real analytic diffeomorphism, since $H_{\mathbb{R}}$ is simply connected and nilpotent.

(2) One can conclude (2) by arguments similar to those above, where we remark that $\mathfrak{n}_s$ is a complex nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

(3) cf. Theorem 6.31-(g) in Knapp, p.362.

(4) cf. Theorem 6.46 in Knapp, p.374, again. □

2.4.2 Complex Flag Manifolds and Flag Domains

In addition to the notation in Paragraph 2.4.1 we set

$$
\begin{aligned}
& L := C_G(T), \quad L_{\mathbb{C}} := C_{G_{\mathbb{C}}}(T), \\
& \mathfrak{g}_\lambda := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\} \quad \text{for } \lambda \in \mathbb{R}, \\
& \mathfrak{u}_\pm := \bigoplus_{\lambda>0} \mathfrak{g}_{\pm\lambda}, \quad U_\pm := \exp \mathfrak{u}_\pm, \quad Q_- := N_{G_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-).
\end{aligned}
\tag{2.10}
$$

Note that $\mathfrak{u}_\pm \subset \mathfrak{n}_\pm$ comes from (2.8). Taking Lemma 2.9 into account, one can show

Proposition 2.11.

(1) *U_s is a simply connected, complex closed nilpotent subgroup of $G_{\mathbb{C}}$, and* $\exp : \mathfrak{u}_s \to U_s$ *is a biholomorphism, for each $s = \pm$.*

(2) *Q_- is a connected, complex closed parabolic subgroup of $G_{\mathbb{C}}$, $\mathfrak{q}_- = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-$, and $Q_- = L_{\mathbb{C}} \ltimes U_-$ (semi-direct).*

(3) *L is a connected, closed subgroup of G, and the homogeneous space G/L is simply connected.*

(4) *The product mapping $U_+ \times Q_- \ni (u, q) \mapsto uq \in G_{\mathbb{C}}$ is a holomorphic embedding whose image is a domain in $G_{\mathbb{C}}$.*

Proof. (1) follows by $\mathfrak{u}_s \subset \mathfrak{n}_s$ and Lemma 2.9-(2).

(2) Remark that (i) $\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_-$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$, (ii) $\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_- \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-$ and (iii) $\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-$ is a complex parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ whose Levi factor (resp. unipotent radical) is $\mathfrak{l}_{\mathbb{C}}$ (resp. $\mathfrak{u}_-$). We refer to Warner, 1972, p.53 for the rest of proof.

(3) e.g. Lemma 2 in Boumuki, 2013, p.9, and the proof of Proposition 2-(ii) in Boumuki, p.11.

(4) By Proposition 1.2.4.10 in Warner, p.77, and its proof, one sees that U_+Q_- is open in $G_{\mathbb{C}}$, and that the intersection $U_+ \cap Q_-$ consists of the unit element $e \in G_{\mathbb{C}}$ only. It follows from (1) and (2) that U_+Q_- is connected. □

Proposition 2.11 leads to

Corollary 2.12.

(a) *$\iota : G/L \to G_{\mathbb{C}}/Q_-$, $gL \mapsto gQ_-$, is a G-equivariant, real analytic embedding whose image is a simply connected domain in $G_{\mathbb{C}}/Q_-$.*

(b) *GQ_- is a domain in $G_{\mathbb{C}}$.*

Proof. (a) By virtue of Proposition 2.11-(3) and $\dim_{\mathbb{R}} G/L = \dim_{\mathbb{R}} \mathfrak{u}_+ = \dim_{\mathbb{R}} G_{\mathbb{C}}/Q_-$, it suffices to confirm

$$L = G \cap Q_-. \tag{2.13}$$

We are going to confirm $G \cap Q_- \subset L$ only, because $L \subset G \cap Q_-$ is clear. Take an arbitrary $x \in G \cap Q_-$. Then, by $x \in Q_-$ and Proposition 2.11-(1), (2) there exists a unique $(l, Z) \in L_{\mathbb{C}} \times \mathfrak{u}_-$ such that

$$x = l \exp Z.$$

We want to show $\exp Z = e$. Let $\bar{\sigma}$ denote the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. On the one hand, $x \in G$, $T \in \mathfrak{l} = \mathfrak{g} \cap \mathfrak{l}_{\mathbb{C}}$, $L_{\mathbb{C}} = C_{G_{\mathbb{C}}}(T)$ and $Q_- = N_{G_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-)$ yield

$$\mathfrak{g} \ni \operatorname{Ad} x^{-1}(T) = (\operatorname{Ad} \exp(-Z) l^{-1})T = (\operatorname{Ad} \exp(-Z))T \in \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-.$$

On the other hand, $\operatorname{Ad} x^{-1}(T) \in \mathfrak{g}$ implies that $\bar{\sigma}(\operatorname{Ad} x^{-1}(T)) = \operatorname{Ad} x^{-1}(T)$, so that $(\operatorname{Ad} \exp(-Z))T = \bar{\sigma}((\operatorname{Ad} \exp(-Z))T) \in \bar{\sigma}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-) \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_+$. Consequently we assert that

$$(\operatorname{Ad} \exp(-Z))T \in (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_-) \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_+) = \mathfrak{l}_{\mathbb{C}}.$$

Therefore $\mathfrak{l}_{\mathbb{C}} \ni -T + (\operatorname{Ad} \exp(-Z))T = \sum_{n=1}^{\infty} (1/n!)(-\operatorname{ad} Z)^n T \in \mathfrak{u}_-$, and hence

$$(\operatorname{Ad} \exp(-Z))T = T.$$

This implies $\exp Z \in L_{\mathbb{C}} \cap U_- = \{e\}$. From $\exp Z = e$ we conclude $x = l \exp Z = l \in G \cap L_{\mathbb{C}} = L$, and $G \cap Q_- \subset L$. So, (2.13) holds.

(b) We denote by $\pi_{\mathbb{C}}$ the projection from $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/Q_-$. It is immediate from (a) that $GQ_- = \pi_{\mathbb{C}}^{-1}(\iota(G/L))$ is an open subset in $G_{\mathbb{C}}$. Moreover, GQ_- is connected because the product mapping $G \times Q_- \ni (g, q) \mapsto gq \in GQ_-$ is surjective continuous and both G and Q_- are connected. □

Remark 2.14.

(1) In general, there exist several kinds of invariant complex structures on the elliptic orbit G/L. In this paper we deal with the complex structure on G/L induced by $\iota : G/L \to G_{\mathbb{C}}/Q_-$, $gL \mapsto gQ_-$.

(2) If G is compact, then $\iota : G/L \hookrightarrow G_{\mathbb{C}}/Q_-$ is surjective and we may identify G/L with $G_{\mathbb{C}}/Q_-$.

2.4.3 Homogeneous Holomorphic Line Bundles over Complex Flag Manifolds

We continue to use the notation in Paragraphs 2.4.1 and 2.4.2.

Let $\chi : Q_- \to \mathbb{C}^* = GL(1, \mathbb{C})$, $q \mapsto \chi(q)$, be a holomorphic homomorphism. For (x, u), $(y, v) \in G_{\mathbb{C}} \times \mathbb{C}$ we say that they are *equivalent*, if there exists a $q \in Q_-$ satisfying

$$y = xq, \quad v = \chi^{-1}(q)u.$$

Denote by $G_{\mathbb{C}} \times_{\chi} \mathbb{C}$ the set of the equivalence classes on $G_{\mathbb{C}} \times \mathbb{C}$, and define a mapping $\mathrm{Pr}_{\mathbb{C}} : G_{\mathbb{C}} \times_{\chi} \mathbb{C} \to G_{\mathbb{C}}/Q_-$ as follows:

$$\mathrm{Pr}_{\mathbb{C}} : [(x, u)] \mapsto \pi_{\mathbb{C}}(x) \quad \text{for } [(x, u)] \in G_{\mathbb{C}} \times_{\chi} \mathbb{C}.$$

Then, $G_{\mathbb{C}} \times_{\chi} \mathbb{C}$ becomes a fiber bundle associated to the principal fiber bundle $\pi_{\mathbb{C}} : G_{\mathbb{C}} \to G_{\mathbb{C}}/Q_-$ with standard fiber $\mathbb{C}$ and structure group Q_-, which we call it the *homogeneous holomorphic line bundle over* $G_{\mathbb{C}}/Q_-$ *associated with* χ. Here we refer to Kobayashi-Nomizu, 1963, p.54–55 for the holomorphic structure on $G_{\mathbb{C}} \times_{\chi} \mathbb{C}$ above.

Remark 2.15. It holds that (S) $|\chi(\ell)| = 1$ for all $\ell \in L$, if G has a compact Cartan subgroup. Indeed; let $\mathfrak{l}_{\mathrm{ss}}$ and $\mathfrak{l}_{\mathrm{z}}$ denote the semisimple part and the center of the reductive Lie algebra $\mathfrak{l}$, respectively. Then any element $X \in \mathfrak{l}$ can be uniquely expressed as $X = X_{\mathrm{ss}} + X_{\mathrm{z}}$ ($X_{\mathrm{ss}} \in \mathfrak{l}_{\mathrm{ss}}$, $X_{\mathrm{z}} \in \mathfrak{l}_{\mathrm{z}}$); and we deduce

$$\chi_*(X) = \chi_*(X_{\mathrm{z}})$$

from $\chi_*(\mathfrak{l}_{\mathrm{ss}}) = \chi_*([\mathfrak{l}, \mathfrak{l}]) \subset [\chi_*(\mathfrak{l}), \chi_*(\mathfrak{l})] \subset \{0\}$, where χ_* denotes the differential homomorphism of $\chi : Q_- \to \mathbb{C}^*$. Besides, it follows from $T \in \mathfrak{h}$ that $\mathfrak{h} \subset \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{l}$, so that $\mathfrak{l}_{\mathrm{z}} \subset \mathfrak{h}$. Therefore $\mathfrak{l}_{\mathrm{z}} \subset \mathfrak{h} = \mathfrak{t}$ whenever G has a compact Cartan subgroup. Consequently, if G has a compact Cartan subgroup, then $\chi_*(X)$ must be a purely imaginary for any $X \in \mathfrak{l}$ because $\chi_*(\mathfrak{l}) = \chi_*(\mathfrak{l}_{\mathrm{z}}) \subset \chi_*(\mathfrak{t})$, and then $|\chi(\ell)| = 1$ for all $\ell \in L$ because L is connected.

Now, let $\Gamma(G_{\mathbb{C}} \times_\chi \mathbb{C})$ denote the complex vector space of holomorphic cross-sections of the bundle $G_{\mathbb{C}} \times_\chi \mathbb{C}$. To any cross-section s of $G_{\mathbb{C}} \times_\chi \mathbb{C}$ one can associate a function $f_s : G_{\mathbb{C}} \to \mathbb{C}$ which satisfies

$$s(\pi_{\mathbb{C}}(x)) = [(x, f_s(x))] \quad \text{for all } x \in G_{\mathbb{C}}.$$

Under this correspondence, holomorphic cross-sections go to holomorphic functions, and one may assume that

$$\Gamma(G_{\mathbb{C}} \times_\chi \mathbb{C}) = \left\{ f : G_{\mathbb{C}} \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ f \text{ is holomorphic,} \\ (2)\ f(xq) = \chi(q)^{-1} f(x) \text{ for all } (x,q) \in G_{\mathbb{C}} \times Q_- \end{array} \right\}. \tag{2.16}$$

2.4.4 A Representation ϱ of G on $\mathcal{V}$ and a Topology for $\mathcal{V}$

The notation below is the same as in Paragraphs 2.4.1, 2.4.2 and 2.4.3.

Let us assume that G/L is a domain in $G_{\mathbb{C}}/Q_-$ via $\iota : G/L \to G_{\mathbb{C}}/Q_-$, $gL \mapsto gQ_-$, and denote by $\iota^\sharp(G_{\mathbb{C}} \times_\chi \mathbb{C})$ the restriction of the bundle $G_{\mathbb{C}} \times_\chi \mathbb{C}$ to the domain $G/L \subset G_{\mathbb{C}}/Q_-$. In this case, (2.16) tells us that

$$\mathcal{V} := \left\{ \psi : GQ_- \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ \psi \text{ is holomorphic,} \\ (2)\ \psi(xq) = \chi(q)^{-1} \psi(x) \text{ for all } (x,q) \in GQ_- \times Q_- \end{array} \right\} \tag{2.17}$$

is the complex vector space of holomorphic cross-sections of the bundle $\iota^\sharp(G_{\mathbb{C}} \times_\chi \mathbb{C})$. We want to set a representation of G on $\mathcal{V}$ and define a topology for $\mathcal{V}$. In order to do so, we first treat a complex vector space C defined by

$$C := \{\xi : GQ_- \to \mathbb{C} \,|\, \xi \text{ is continuous}\}.$$

On the one hand, we define an algebraic representation ϱ of G on C by

$$(\varrho(g)\xi)(x) := \xi(g^{-1}x) \quad \text{for } (g,\xi) \in G \times C \text{ and } x \in GQ_-. \tag{2.18}$$

On the other hand, we define a metric on C in the following way. For a non-empty compact subset $E \subset GQ_-$ and $\xi_1, \xi_2 \in C$ we put

$$d_E(\xi_1, \xi_2) := \sup\{|\xi_1(a) - \xi_2(a)| : a \in E\}. \tag{2.19}$$

Since $G_{\mathbb{C}}$ is connected, it satisfies the second countability axiom. So, GQ_- is a locally compact Hausdorff space and satisfies the same axiom because GQ_- is open in $G_{\mathbb{C}}$. Therefore there exist non-empty open subsets $W_n \subset GQ_-$ such that

- (d1) $GQ_- = \bigcup_{n=1}^\infty W_n$ (countable union),
- (d2) the closure $\overline{W_n}$ in GQ_- is compact for each $n \in \mathbb{N}$.

Then we define $E_n := \overline{W_n}$ for $n \in \mathbb{N}$, and moreover define

$$d(\xi_1, \xi_2) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{d_{E_n}(\xi_1, \xi_2)}{1 + d_{E_n}(\xi_1, \xi_2)} \tag{2.20}$$

for $\xi_1, \xi_2 \in C$. This d is called the *Fréchet metric*.

Proposition 2.21. *With respect to the Fréchet metric d in* (2.20)*, the following six items hold*:

- (i) *d is a metric on C.*
- (ii) *The metric topology for (C, d) coincides with the topology of uniform convergence on compacts sets.*
- (iii) *The metric space (C, d) is complete.*
- (iv) *The metric topology for (C, d) coincides with the locally convex topology determined by a countable number of seminorms $\{p_n\}_{n\in\mathbb{N}}$, where $p_n(\xi) := d_{E_n}(\xi, 0)$ for $n \in \mathbb{N}$, $\xi \in C$.*
- (v) *Both mappings $C \times C \ni (\xi_1, \xi_2) \mapsto \xi_1 + \xi_2 \in C$ and $\mathbb{C} \times C \ni (\alpha, \xi) \mapsto \alpha\xi \in C$ are continuous, with respect to the locally convex topology in* (iv).
- (vi) *$G \times C \ni (g, \xi) \mapsto \varrho(g)\xi \in C$ is a continuous mapping, with respect to the topology of uniform convergence on compacts sets in* (ii).

Proof. (i) Take any $\xi_1, \xi_2 \in C$. It follows from (2.19) and (2.20) that $0 \leq d_E(\xi_1, \xi_2), d(\xi_1, \xi_2)$. Besides,

$$d(\xi_1, \xi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\xi_1, \xi_2)}{1 + d_{E_n}(\xi_1, \xi_2)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Hence d is a non-negative function on $C \times C$. Needless to say, $d(\xi_1, \xi_2) = d(\xi_2, \xi_1)$; and $d(\xi_1, \xi_2) = 0$ in case of $\xi_1 = \xi_2$. Now, for $\xi_1', \xi_2' \in C$ we suppose that $d(\xi_1', \xi_2') = 0$. Then for each $k \in \mathbb{N}$, one has

$$0 \leq \frac{1}{2^k} \frac{d_{E_k}(\xi_1', \xi_2')}{1 + d_{E_k}(\xi_1', \xi_2')} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\xi_1', \xi_2')}{1 + d_{E_n}(\xi_1', \xi_2')} \overset{(2.20)}{=} d(\xi_1', \xi_2') = 0.$$

This implies that $d_{E_k}(\xi_1', \xi_2') = 0$, so that $\xi_1' = \xi_2'$ on E_k for all $k \in \mathbb{N}$. Therefore $\xi_1' = \xi_2'$ on the whole GQ_- in terms of (d1) and $E_n = \overline{W_n}$. Hence one can assert (i), if we conclude that

$$d(\xi_1, \xi_3) \leq d(\xi_1, \xi_2) + d(\xi_2, \xi_3) \text{ for all } \xi_1, \xi_2, \xi_3 \in C.$$

For any $\xi_1, \xi_2, \xi_3 \in C$, it is immediate from (2.19) that $d_{E_n}(\xi_1, \xi_3) \leq d_{E_n}(\xi_1, \xi_2) + d_{E_n}(\xi_2, \xi_3)$ for all $n \in \mathbb{N}$. From this and $0 \leq d_{E_n}(\xi_i, \xi_j)$ we obtain

$$\begin{aligned}
\frac{d_{E_n}(\xi_1, \xi_3)}{1 + d_{E_n}(\xi_1, \xi_3)} &\leq \frac{d_{E_n}(\xi_1, \xi_2) + d_{E_n}(\xi_2, \xi_3)}{1 + d_{E_n}(\xi_1, \xi_2) + d_{E_n}(\xi_2, \xi_3)} \\
&= \frac{d_{E_n}(\xi_1, \xi_2)}{1 + d_{E_n}(\xi_1, \xi_2) + d_{E_n}(\xi_2, \xi_3)} + \frac{d_{E_n}(\xi_2, \xi_3)}{1 + d_{E_n}(\xi_1, \xi_2) + d_{E_n}(\xi_2, \xi_3)} \\
&\leq \frac{d_{E_n}(\xi_1, \xi_2)}{1 + d_{E_n}(\xi_1, \xi_2)} + \frac{d_{E_n}(\xi_2, \xi_3)}{1 + d_{E_n}(\xi_2, \xi_3)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Accordingly

$$\begin{aligned}
d(\xi_1, \xi_3) &\overset{(2.20)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\xi_1, \xi_3)}{1 + d_{E_n}(\xi_1, \xi_3)} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\xi_1, \xi_2)}{1 + d_{E_n}(\xi_1, \xi_2)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\xi_2, \xi_3)}{1 + d_{E_n}(\xi_2, \xi_3)} \overset{(2.20)}{=} d(\xi_1, \xi_2) + d(\xi_2, \xi_3).
\end{aligned}$$

(ii) First, let us demonstrate that the metric topology $\mathscr{D}_d$ for (C, d) is coarser than the topology $\mathscr{D}_{\text{cu}}$ of uniform convergence on compacts sets, namely $\mathscr{D}_d \subset \mathscr{D}_{\text{cu}}$. For any $\xi_0 \in C$ and $\epsilon > 0$, we set $O_d := \{\xi \in C \,|\, d(\xi, \xi_0) < \epsilon\}$. Let ξ be an arbitrary element of O_d, and $r := d(\xi, \xi_0)$. Since $\epsilon - r > 0$ there exists an $m \in \mathbb{N}$ satisfying

$$1/2^m < (\epsilon - r)/2.$$

By use of m, ϵ and r we put

$$E := \textstyle\bigcup_{j=1}^m E_j, \quad \delta := (\epsilon - r)/(2m).$$

Then, E becomes a non-empty compact subset in GQ_- and $\delta > 0$. Moreover, (2.19) yields

$$d_{E_1}(\xi_1, \xi_2) + \cdots + d_{E_m}(\xi_1, \xi_2) \leq m d_E(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in C$. Hence for any $\eta \in C$ with $d_E(\eta, \xi) < \delta$, we have

$$\begin{aligned}
d(\eta, \xi) &\overset{(2.20)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\eta, \xi)}{1 + d_{E_n}(\eta, \xi)} = \sum_{j=1}^{m} \frac{1}{2^j} \frac{d_{E_j}(\eta, \xi)}{1 + d_{E_j}(\eta, \xi)} + \sum_{k=m+1}^{\infty} \frac{1}{2^k} \frac{d_{E_k}(\eta, \xi)}{1 + d_{E_k}(\eta, \xi)} \\
&\leq \sum_{j=1}^{m} d_{E_j}(\eta, \xi) + \sum_{k=m+1}^{\infty} \frac{1}{2^k} = \sum_{j=1}^{m} d_{E_j}(\eta, \xi) + \frac{1}{2^m} \leq m d_E(\eta, \xi) + \frac{1}{2^m} \\
&< m\delta + \frac{1}{2^m} < \epsilon - r.
\end{aligned}$$

This and $d(\eta, \xi_0) \leq d(\eta, \xi) + d(\xi, \xi_0) = d(\eta, \xi) + r$ provide $\xi \in \{\eta \in C \,|\, d_E(\eta, \xi) < \delta\} \subset O_d$, and thus $\mathscr{D}_d \subset \mathscr{D}_{\text{cu}}$.

Next, let us prove that the converse inclusion $\mathscr{D}_{\mathrm{cu}} \subset \mathscr{D}_d$ also holds. For any $\xi'_0 \in C$, $\epsilon' > 0$ and non-empty compact subset $E' \subset GQ_-$, we set $O_{\mathrm{cu}} := \{\xi' \in C \,|\, d_{E'}(\xi', \xi'_0) < \epsilon'\}$, fix any element $\xi' \in O_{\mathrm{cu}}$ and put $r' := d_{E'}(\xi', \xi'_0)$. Since (d1), $E_n = \overline{W_n}$ and E' is compact, there exist $n(1), \ldots, n(k) \in \mathbb{N}$ such that $n(1) < \cdots < n(k)$ and $E' \subset \bigcup_{i=1}^{k} E_{n(i)}$. Then it follows from (2.19) that

$$d_{E'}(\xi_1, \xi_2) \leq d_{E_{n(1)}}(\xi_1, \xi_2) + \cdots + d_{E_{n(k)}}(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in C$. Setting

$$\delta' := \frac{1}{2^{n(k)}} \frac{((\epsilon' - r')/k)}{1 + ((\epsilon' - r')/k)},$$

we deduce $\delta' > 0$. In addition; if $\eta' \in C$ satisfies $d(\eta', \xi') < \delta'$, then

$$\frac{1}{2^{n(i)}} \frac{d_{E_{n(i)}}(\eta', \xi')}{1 + d_{E_{n(i)}}(\eta', \xi')} \overset{(2.20)}{\leq} d(\eta', \xi') < \delta' = \frac{1}{2^{n(k)}} \frac{((\epsilon' - r')/k)}{1 + ((\epsilon' - r')/k)} \leq \frac{1}{2^{n(i)}} \frac{((\epsilon' - r')/k)}{1 + ((\epsilon' - r')/k)}$$

and $d_{E_{n(i)}}(\eta', \xi') < (\epsilon' - r')/k$ for all $1 \leq i \leq k$. Consequently, if $\eta' \in C$ satisfies $d(\eta', \xi') < \delta'$, then $d_{E'}(\eta', \xi') \leq \sum_{i=1}^{k} d_{E_{n(i)}}(\eta', \xi') < \epsilon' - r'$. This and $d_{E'}(\eta', \xi'_0) \leq d_{E'}(\eta', \xi') + d_{E'}(\xi', \xi'_0) = d_{E'}(\eta', \xi') + r'$ give us $\xi' \in \{\eta' \in C \,|\, d(\eta', \xi') < \delta'\} \subset O_{\mathrm{cu}}$, and so $\mathscr{D}_{\mathrm{cu}} \subset \mathscr{D}_d$.

(iii) follows by (ii).

(iv) Similar to the proof of (ii).

(v) is immediate from (iv). Needless to say, C is a Fréchet space due to (i) through (v).

(vi) The rest of proof is to demonstrate the item (vi). C is a barreled space. So, one can conclude this item, if we prove the following items (1) and (2):

(1) For a given $\xi_0 \in C$, the mapping $G \ni g \mapsto \varrho(g)\xi_0 \in C$ is continuous at the point e.

(2) For a given $g_0 \in G$, the mapping $C \ni \xi \mapsto \varrho(g_0)\xi \in C$ is continuous.

cf. Proposition 13.2 in Alain, 1983, p.128. Let us verify (1) and (2) from now on.

(1) Fix any non-empty compact subset $E \subset GQ_-$ and $\epsilon > 0$. We will show that there exists an open neighborhood V of $e \in G$ satisfying $d_E(\varrho(h)\xi_0, \xi_0) < \epsilon$ for all $h \in V$. By use of ξ_0, let us define a continuous function $\mathtt{f} : G \times GQ_- \to \mathbb{C}$ by $\mathtt{f}(g, x) := \xi_0(g^{-1}x)$ for $(g, x) \in G \times GQ_-$. Then for each $y \in E$, there exist open subsets $V_y \subset G$ and $U'_y \subset GQ_-$ satisfying $e \in V_y$, $y \in U'_y$ and

$$\text{(b1) } |\xi_0(h^{-1}z') - \xi_0(e^{-1}y)| = |\mathtt{f}(h, z') - \mathtt{f}(e, y)| < \epsilon/4$$

for all $(h, z') \in V_y \times U'_y$, since $\mathtt{f}$ is continuous at (e, y). Moreover, there exists an open neighborhood U_y of $y \in GQ_-$ such that

$$\text{(b2) } U_y \subset U'_y, \quad \text{(b3) } |\xi_0(y) - \xi_0(z)| < \epsilon/4 \text{ for all } z \in U_y$$

because U'_y is an open neighborhood of $y \in GQ_-$ and $\xi_0 : GQ_- \to \mathbb{C}$ is continuous at y. Now, E is compact and $E \subset \bigcup_{y \in E} U_y$. Hence, there exist finite elements $y_1, \ldots y_k \in E$ such that $E \subset \bigcup_{j=1}^{k} U_{y_j}$. Setting $V := \bigcap_{j=1}^{k} V_{y_j}$, we see that V becomes an open neighborhood of $e \in G$. Furthermore, for an arbitrary $(h, a) \in V \times E$, there exists $1 \leq i \leq k$ such that $a \in U_{y_i}$; and besides $h \in \bigcap_{j=1}^{k} V_{y_j} \subset V_{y_i}$. Hence (b1), (b2) and (b3) provide

$$|\xi_0(h^{-1}a) - \xi_0(a)| \leq |\xi_0(h^{-1}a) - \xi_0(e^{-1}y_i)| + |\xi_0(y_i) - \xi_0(a)| < \frac{\epsilon}{4} + \frac{\epsilon}{4}.$$

This, together with (2.18) and (2.19), assures that $d_E(\varrho(h)\xi_0, \xi_0) \leq \epsilon/2 < \epsilon$ for all $h \in V$. Accordingly (1) follows.

(2) Let us demonstrate that $\varrho(g_0) : C \to C$, $\xi \mapsto \varrho(g_0)\xi$, is uniformly continuous. For any non-empty compact subset $E \subset GQ_-$ and $\epsilon > 0$, we set $E' := g_0^{-1}E$ and $\delta := \epsilon$. Then, E' is a non-empty compact subset in GQ_- and $\delta > 0$. Moreover, (2.18) and (2.19) imply that for any $\xi_1, \xi_2 \in C$ with $d_{E'}(\xi_1, \xi_2) < \delta$,

$$\begin{aligned} d_E(\varrho(g_0)\xi_1, \varrho(g_0)\xi_2) &= \sup\{|\xi_1(g_0^{-1}a) - \xi_2(g_0^{-1}a)| : a \in E\} \\ &= \sup\{|\xi_1(b) - \xi_2(b)| : b \in g_0^{-1}E\} = d_{E'}(\xi_1, \xi_2) < \delta = \epsilon. \end{aligned}$$

Consequently $\varrho(g_0) : C \to C$ is uniformly continuous. □

In view of the lemma below, one can naturally set a representation of G on $\mathcal{V}$ and define a topology for $\mathcal{V}$.

Lemma 2.22. *$\mathcal{V}$ is a closed $\varrho(G)$-invariant complex vector subspace of C. Here, the topology for C is induced by the Fréchet metric d in* (2.20).

Proof. From (2.17) and (2.18) it is obvious that $\mathcal{V}$ is a $\varrho(G)$-invariant complex vector subspace of C. Let us confirm that $\mathcal{V}$ is closed in C. Let $\{\psi_n\}_{n=1}^{\infty}$ be an arbitrary Cauchy sequence in $(\mathcal{V}, d)$. Then Proposition 2.21-(iii) and $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{V} \subset C$ enable us to obtain a unique $\xi \in C$ such that $\lim_{n\to\infty} d(\psi_n, \xi) = 0$. If this ξ belongs to $\mathcal{V}$, then $\mathcal{V}$ is closed in C. Therefore we devote ourselves to showing $\xi \in \mathcal{V}$ hereafter. Our first aim is to verify that

$$\xi \text{ is holomorphic on } GQ_-.$$

For any $x \in GQ_-$ we take a holomorphic coordinate neighborhood (D, ϕ) of x such that (a1) $z^i(\phi(x)) = 0$ for all $1 \leq i \leq N = \dim_{\mathbb{C}} GQ_-$ and (a2) ϕ is a homeomorphism of D onto an open subset in $\mathbb{C}^N$ defined by $|z^1| < R, \dots, |z^N| < R$ for some $R > 0$. Since ξ is continuous, it follows from Morera's theorem, $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{V}$, $\lim_{n\to\infty} d(\psi_n, \xi) = 0$ and Proposition 2.21-(ii) that $\xi = \xi(z^1, \dots, z^N)$ is holomorphic with respect to each variable z^i. Accordingly ξ is holomorphic on D, since ξ is continuous. Hence we have accomplished the first aim. Our next aim is to conclude that

$$\xi(xq) = \chi(q)^{-1}\xi(x) \text{ for all } (x, q) \in GQ_- \times Q_-.$$

Let us use proof by contradiction. Suppose that there exists a $(y, r) \in GQ_- \times Q_-$ satisfying $\chi(r)\xi(yr) \neq \xi(y)$. Since GQ_- is locally compact, there exist compact subsets $E, E' \subset GQ_-$ such that

$$yr \in E, \quad y \in E',$$

respectively. By the supposition, $\delta := |\chi(r)\xi(yr) - \xi(y)|$ must be positive. Therefore, in terms of $\lim_{n\to\infty} d(\psi_n, \xi) = 0$ and Proposition 2.21-(ii), there exists an $M \in \mathbb{N}$ such that for every $m \geq M$

$$d_E(\xi, \psi_m) < \delta/(2|\chi(r)|), \quad d_{E'}(\xi, \psi_m) < \delta/2,$$

where we remark that $|\chi(r)| > 0$ follows from $\chi(r) \in \mathbb{C}^*$. Then it turns out that

$$\begin{aligned}
\delta &= |\chi(r)\xi(yr) - \xi(y)| \leq |\chi(r)\xi(yr) - \psi_M(y)| + |\psi_M(y) - \xi(y)| \\
&= |\chi(r)\xi(yr) - \chi(r)\psi_M(yr)| + |\psi_M(y) - \xi(y)| \qquad (\because \psi_M \in \mathcal{V}, (2.17)\text{-}(2)) \\
&\leq |\chi(r)| d_E(\xi, \psi_M) + d_{E'}(\psi_M, \xi) < \delta.
\end{aligned}$$

This is a contradiction. Accordingly $\xi(xq) = \chi(q)^{-1}\xi(x)$ for all $(x, q) \in GQ_- \times Q_-$. Thus we conclude $\xi \in \mathcal{V}$. □

From Proposition 2.21 and Lemma 2.22 we deduce

Corollary 2.23. *With respect to the Fréchet metric d on $\mathcal{V}$ in* (2.20)*, the following four items hold*:

1. *$(\mathcal{V}, d)$ is a complete metric space.*
2. *The metric topology for $(\mathcal{V}, d)$ coincides with the topology of uniform convergence on compacts sets*; *and besides it also coincides with the locally convex topology determined by a countable number of seminorms $\{p_n\}_{n\in\mathbb{N}}$, where $p_n(\psi) := d_{E_n}(\psi, 0)$ for $n \in \mathbb{N}$, $\psi \in \mathcal{V}$.*
3. *Both mappings $\mathcal{V} \times \mathcal{V} \ni (\psi_1, \psi_2) \mapsto \psi_1 + \psi_2 \in \mathcal{V}$ and $\mathbb{C} \times \mathcal{V} \ni (\alpha, \psi) \mapsto \alpha\psi \in \mathcal{V}$ are continuous.*
4. *$G \times \mathcal{V} \ni (g, \psi) \mapsto \varrho(g)\psi \in \mathcal{V}$ is a continuous mapping.*

We end Section 2 with proving

Lemma 2.24. $\Delta_x : (\mathcal{V}, d) \to \mathbb{C}$, *$\psi \mapsto \psi(x)$, is a continuous linear function for every $x \in GQ_-$.*

Proof. (linear) is obvious.

(continuous) Take an arbitrary $\epsilon > 0$ and $\psi_0 \in \mathcal{V}$. By $x \in GQ_-$ and (d1) there exists an $m \in \mathbb{N}$ such that $x \in E_m$. By use of this m we set $\delta := \dfrac{\epsilon}{2^m(1+\epsilon)}$. Then, $\delta > 0$ holds. For any $\psi \in \mathcal{V}$ with $d(\psi, \psi_0) < \delta$, we obtain

$$\frac{1}{2^m}\frac{d_{E_m}(\psi, \psi_0)}{1 + d_{E_m}(\psi, \psi_0)} \leq d(\psi, \psi_0) < \delta = \frac{\epsilon}{2^m(1+\epsilon)}$$

from (2.20). This gives $d_{E_m}(\psi, \psi_0) < \epsilon$, and

$$|\Delta_x(\psi) - \Delta_x(\psi_0)| = |\psi(x) - \psi_0(x)| \le d_{E_m}(\psi, \psi_0) < \epsilon$$

follows by $x \in E_m$ and (2.19). Hence, Δ_x is continuous at the point $\psi_0 \in \mathcal{V}$. □

3. Proof of the Main Result

The main purpose of this section is to complete the proof of Theorem 1.2. Throughout Section 3 we utilize the same notation as in Subsection 2.4.

3.1 Key Propositions in Proving Theorem 1.2

To prove Theorem 1.2 we need Propositions 3.1 and 3.9 below.

Proposition 3.1. *Suppose that* (S) $|\chi(\ell)| = 1$ *for all* $\ell \in L$. *In this case* $F_1 : GQ_- \to \mathbb{R}^*$, $gq \mapsto |\chi(q)|$, *is a continuous function.*

Proof. (well-defined) If $g_1, g_2 \in G$, $q_1, q_2 \in Q_-$ and $g_1q_1 = g_2q_2$, then (2.13) enables us to get an $\ell \in L$ such that $g_2 = g_1\ell^{-1}$, $q_2 = \ell q_1$. Therefore the supposition (S) assures that $|\chi(q_2)| = |\chi(\ell q_1)| = |\chi(\ell)\chi(q_1)| = |\chi(q_1)|$.

(continuous) We consider the following four kinds of continuous mappings:

$F_2 : Q_- \to \mathbb{R}^*$, $q \mapsto |\chi(q)|$;
$\pi_3 : Q_- \to L\backslash Q_-$, $q \mapsto Lq$ (the right coset space);
$F_4 : L\backslash Q_- \to G\backslash G_{\mathbb{C}}$, $Lq \mapsto Gq$;
$\pi_5 : G_{\mathbb{C}} \to G\backslash G_{\mathbb{C}}$, $x \mapsto Gx$.

$$\begin{array}{ccc} \mathbb{R}^* \xleftarrow{F_2} Q_- & & G_{\mathbb{C}} \\ \downarrow {\scriptstyle \pi_3} & & \downarrow {\scriptstyle \pi_5} \\ L\backslash Q_- & \xrightarrow{F_4} & G\backslash G_{\mathbb{C}} \end{array}$$

It is obvious that

$$\begin{aligned} \dim_{\mathbb{R}} L\backslash Q_- &= \dim_{\mathbb{R}} \mathfrak{q}_- - \dim_{\mathbb{R}} \mathfrak{l} = \dim_{\mathbb{R}} \mathfrak{l}_{\mathbb{C}} + \dim_{\mathbb{R}} \mathfrak{u}_- - \dim_{\mathbb{R}} \mathfrak{l} \\ &= \dim_{\mathbb{R}} \mathfrak{u}_- + \dim_{\mathbb{R}} \mathfrak{l} = \dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{R}} G\backslash G_{\mathbb{C}}, \end{aligned}$$

and hence it follows from (2.13) that

$$F_4 : L\backslash Q_- \to G\backslash G_{\mathbb{C}} \text{ is an open mapping.} \tag{3.2}$$

Now, let us demonstrate that $F_1 : GQ_- \to \mathbb{R}^*$ is continuous. Take any open subset I in $\mathbb{R}^*$. On the one hand; since F_2 is continuous and π_3 is an open mapping, $\pi_3(F_2^{-1}(I))$ is an open subset in $L\backslash Q_-$. Thus, since (3.2) and π_5 is continuous, we conclude that $\pi_5^{-1}(F_4(\pi_3(F_2^{-1}(I))))$ is an open subset in $G_{\mathbb{C}}$. That, together with $\pi_5^{-1}(F_4(\pi_3(F_2^{-1}(I)))) \subset GQ_-$, implies the following:

$$\pi_5^{-1}(F_4(\pi_3(F_2^{-1}(I)))) \text{ is an open subset in } GQ_-.$$

On the other hand; a direct computation yields $F_1^{-1}(I) = \pi_5^{-1}(F_4(\pi_3(F_2^{-1}(I))))$. Accordingly $F_1^{-1}(I)$ is open in GQ_-, and hence F_1 is continuous. □

Let us set

$$\hat{\psi}_\lambda(x) := \psi((\exp \lambda T)x \exp(-\lambda T)) \quad \text{for } (\psi, \lambda) \in \mathcal{V} \times [0, 2\pi] \text{ and } x \in GQ_-, \tag{3.3}$$

and clarify some properties of $\hat{\psi}_\lambda$ from now on.

Lemma 3.4.

(i) $\hat{\psi}_\lambda \in \mathcal{V}$ *for all* $(\psi, \lambda) \in \mathcal{V} \times [0, 2\pi]$.

(ii) *For each* $\psi \in \mathcal{V}$, *the mapping* $[0, 2\pi] \ni \lambda \mapsto \hat{\psi}_\lambda \in \mathcal{V}$ *is continuous with respect to the Fréchet metric* d *on* $\mathcal{V}$ *in* (2.20).

(iii) $\int_0^{2\pi} \hat{\psi}_\lambda d\lambda \in \mathcal{V}$ *for each* $\psi \in \mathcal{V}$. *Furthermore,* $\Big(\int_0^{2\pi} \hat{\psi}_\lambda d\lambda\Big)(x) = \int_0^{2\pi} \hat{\psi}_\lambda(x) d\lambda$ *for all* $(\psi, x) \in \mathcal{V} \times GQ_-$.

Proof. (i) Since $\chi(\exp\lambda T) \in \mathbb{C}^* \subset \mathbb{C}$ and $\psi \in \mathcal{V}$, the scalar multiple $\chi(\exp\lambda T)\psi$ belongs to the complex vector space $\mathcal{V}$. Moreover, $\varrho(\exp(-\lambda T))(\chi(\exp\lambda T)\psi) \in \mathcal{V}$ because $\mathcal{V}$ is $\varrho(G)$-invariant. Thus $\hat{\psi}_\lambda \overset{(3.3)}{=} \varrho(\exp(-\lambda T))(\chi(\exp\lambda T)\psi) \in \mathcal{V}$.

(ii) The mapping $[0, 2\pi] \ni \lambda \mapsto \hat{\psi}_\lambda \in \mathcal{V}$ is continuous, since it is composed of the following three continuous mappings (cf. Corollary 2.23-(3), (4)):

$$\begin{array}{llll} [0,2\pi] & \to G \times \mathbb{C} \times \mathcal{V} & \to G \times \mathcal{V} & \\ \lambda & \mapsto (\exp(-\lambda T), \chi(\exp\lambda T), \psi) & \mapsto (\exp(-\lambda T), \chi(\exp\lambda T)\psi) & \\ & & \to \mathcal{V} & \\ & & \mapsto \varrho(\exp(-\lambda T))(\chi(\exp\lambda T)\psi) = \hat{\psi}_\lambda. & \end{array}$$

(iii) $\int_0^{2\pi} \hat{\psi}_\lambda d\lambda \in \mathcal{V}$ follows by (ii) and $(\mathcal{V}, d)$ being a complete, topological vector space. Besides, we conclude by Lemma 2.24 that $\Big(\int_0^{2\pi} \hat{\psi}_\lambda d\lambda\Big)(x) = \int_0^{2\pi} \hat{\psi}_\lambda(x) d\lambda$. □

We want to first prove the following three lemmas, and afterwards deduce Proposition 3.9 from them:

Lemma 3.5.

(i) $(U_+ \cap GQ_-)Q_- = U_+Q_- \cap GQ_-$.

(ii) $(U_+ \cap GQ_-)_e$ *is a domain in* U_+, *where* $(U_+ \cap GQ_-)_e$ *denotes the connected component of* $U_+ \cap GQ_-$ *containing* e.

(iii) $(U_+ \cap GQ_-)_eQ_-$ *is an open subset in* GQ_- *containing* e.

(iv) *If* $\psi_1, \psi_2 \in \mathcal{V}$ *and* $\psi_1 = \psi_2$ *on* $(U_+ \cap GQ_-)_e$, *then* $\psi_1 = \psi_2$ *on the whole* GQ_-.

Proof. (i) Let us only confirm $U_+Q_- \cap GQ_- \subset (U_+ \cap GQ_-)Q_-$, since $(U_+ \cap GQ_-)Q_- \subset U_+Q_- \cap GQ_-$ is trivial. For any $x \in U_+Q_- \cap GQ_-$, there exists a unique $(u, q) \in U_+ \times Q_-$ such that $x = uq$ because of $x \in U_+Q_-$ and Proposition 2.11-(4). Hence one has $U_+ \ni u = xq^{-1} \in GQ_-Q_- \subset GQ_-$, and so $u \in U_+ \cap GQ_-$. Therefore $x = uq \in (U_+ \cap GQ_-)Q_-$, and $U_+Q_- \cap GQ_- \subset (U_+ \cap GQ_-)Q_-$.

(ii) Corollary 2.12-(b) implies that $U_+ \cap GQ_-$ is open in U_+, so that $U_+ \cap GQ_-$ is locally connected. For this reason $(U_+ \cap GQ_-)_e$ is open in $U_+ \cap GQ_-$. Accordingly $(U_+ \cap GQ_-)_e$ is an open subset in U_+. Needless to say, $(U_+ \cap GQ_-)_e$ is connected.

(iii) Since (ii) and the product mapping $U_+ \times Q_- \to U_+Q_-$ is an open mapping, $(U_+ \cap GQ_-)_eQ_-$ is open in U_+Q_-. This and Proposition 2.11-(4) assure that $(U_+ \cap GQ_-)_eQ_-$ is open in $G_\mathbb{C}$. Thus, we conclude (iii) by $(U_+ \cap GQ_-)_eQ_- \subset (U_+ \cap GQ_-)Q_- \overset{(i)}{=} U_+Q_- \cap GQ_- \subset GQ_-$.

(iv) For $\psi_1, \psi_2 \in \mathcal{V}$ we suppose that $\psi_1 = \psi_2$ on $(U_+ \cap GQ_-)_e$. Then it follows from $\psi_1, \psi_2 \in \mathcal{V}$ and (2.17)-(2) that $\psi_1 = \psi_2$ on $(U_+ \cap GQ_-)_eQ_-$. Therefore $\psi_1 = \psi_2$ on the whole GQ_-, because (iii), Corollary 2.12-(b) and the theorem of identity for the holomorphic functions ψ_1, ψ_2. □

Lemma 3.6. *For any holomorphic function* $h : GQ_- \to \mathbb{C}$, *the restriction* $h|_{(U_+ \cap GQ_-)_e}$ *is holomorphic on* $(U_+ \cap GQ_-)_e$.

Proof. The inclusion $\iota : U_+ \to G_\mathbb{C}$ is holomorphic; and $(U_+ \cap GQ_-)_e$ and GQ_- are open subsets in U_+ and $G_\mathbb{C}$, respectively. In addition, $\iota((U_+ \cap GQ_-)_e) \subset GQ_-$. These imply that $\iota : (U_+ \cap GQ_-)_e \to GQ_-$ is holomorphic, so that $h \circ \iota = h|_{(U_+ \cap GQ_-)_e}$ is holomorphic. □

Lemma 3.7. *For each* $\psi \in \mathcal{V}$, $\int_0^{2\pi} \hat{\psi}_\lambda d\lambda$ *is the constant function with the value* $2\pi\psi(e)$ *on* $(U_+ \cap GQ_-)_e$.

Proof. Theorem 2.3 allows us to choose a finite subset $\{n, \dots, m\} \subset \mathbb{N}$ such that

$$\mathfrak{u}_+ \overset{(2.10)}{=} \bigoplus_{\lambda > 0} \mathfrak{g}_\lambda = \mathfrak{g}_n \oplus \cdots \oplus \mathfrak{g}_m.$$

Let us fix complex bases $\{Z^n_a\}_{a=1}^{k_n}, \dots, \{Z^m_a\}_{a=1}^{k_m}$ of $\mathfrak{g}_n, \dots, \mathfrak{g}_m$, respectively, and denote by $z^1_n, \dots, z^{k_n}_n, \dots, z^1_m, \dots, z^{k_m}_m$ the canonical coordinates of the second kind associated with the basis $\{Z^n_a\}_{a=1}^{k_n} \cup \cdots \cup \{Z^m_a\}_{a=1}^{k_m}$ of $\mathfrak{u}_+$. Here, there exists an open subset $O \subset U_+$ such that $e \in O$ and

$$O \ni (\exp z^1_n Z^n_1) \cdots (\exp z^{k_n}_n Z^n_{k_n}) \cdots (\exp z^1_m Z^m_1) \cdots (\exp z^{k_m}_m Z^m_{k_m}) \mapsto (z^1_n, \dots, z^{k_n}_n, \dots, z^1_m, \dots, z^{k_m}_m) \in \mathbb{C}^{k_n + \cdots + k_m}.$$

By Lemma 3.6 $\psi|_{(U_+ \cap GQ_-)_e}$ is holomorphic. Therefore Lemma 3.5-(ii) provides us with an $R > 0$ such that the following conditions (c1) and (c2) hold for

$$\mathcal{P} := \{u \in O : |z^a_b(u)| < R,\ 1 \leq a \leq k_b = \dim_{\mathbb{C}} \mathfrak{g}_b,\ b = n, \dots, m\} :$$

(c1) $\mathcal{P}$ is an open subset in $(U_+ \cap GQ_-)_e$ containing e,

(c2) On $\mathcal{P}$ we can express $\psi|_{(U_+ \cap GQ_-)_e}$ as

$$\psi(z^1_n, \dots, z^{k_n}_n, \dots, z^1_m, \dots, z^{k_m}_m) = \sum_{l^1_n, \dots, l^{k_n}_n, \dots, l^1_m, \dots, l^{k_m}_m \geq 0} \beta_{l^1_n \cdots l^{k_n}_n \cdots l^1_m \cdots l^{k_m}_m} (z^1_n)^{l^1_n} \cdots (z^{k_n}_n)^{l^{k_n}_n} \cdots (z^1_m)^{l^1_m} \cdots (z^{k_m}_m)^{l^{k_m}_m}$$

(the Taylor expansion of $\psi|_{(U_+ \cap GQ_-)_e}$ at $e = (0, \dots, 0, \dots, 0, \dots, 0)$).

If we suppose that

$$\int_0^{2\pi} \hat{\psi}_\lambda d\lambda = 2\pi\psi(e) \text{ on } \mathcal{P}, \tag{3.8}$$

then one can get the conclusion. Indeed; if (3.8) holds, then it follows from (c1), the theorem of identity and Lemma 3.5-(ii) that $\int_0^{2\pi} \hat{\psi}_\lambda d\lambda = 2\pi\psi(e)$ on $(U_+ \cap GQ_-)_e$. Consequently, the rest of proof is to confirm (3.8). Let

$$u = (\exp \alpha^1_n Z^n_1) \cdots (\exp \alpha^{k_n}_n Z^n_{k_n}) \cdots (\exp \alpha^1_m Z^m_1) \cdots (\exp \alpha^{k_m}_m Z^m_{k_m})$$

be any element of $\mathcal{P}$. From $Z^b_a \in \mathfrak{g}_b = \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = ibA\}$ we obtain

$$(\exp \lambda T)(\exp \alpha^a_b Z^b_a) \exp(-\lambda T) = \exp(\alpha^a_b e^{i\lambda b} Z^b_a)$$

for all $1 \leq a \leq k_b$, $b = n, \dots, m$ and $\lambda \in [0, 2\pi]$. Therefore

$$\begin{aligned}
\Big(\int_0^{2\pi} \hat{\psi}_\lambda d\lambda\Big)(u) &= \int_0^{2\pi} \hat{\psi}_\lambda(u) d\lambda \qquad (\because \text{Lemma 3.4-(iii)}) \\
&= \int_0^{2\pi} \hat{\psi}_\lambda(\alpha^1_n, \dots, \alpha^{k_n}_n, \dots, \alpha^1_m, \dots, \alpha^{k_m}_m) d\lambda \\
&= \int_0^{2\pi} \psi(\alpha^1_n e^{i\lambda n}, \dots, \alpha^{k_n}_n e^{i\lambda n}, \dots, \alpha^1_m e^{i\lambda m}, \dots, \alpha^{k_m}_m e^{i\lambda m}) d\lambda \qquad (\because (3.3)) \\
&\overset{(c2)}{=} \int_0^{2\pi} \sum_{l^1_n, \dots, l^{k_n}_n, \dots, l^1_m, \dots, l^{k_m}_m \geq 0} \beta_{l^1_n \cdots l^{k_n}_n \cdots l^1_m \cdots l^{k_m}_m} e^{i\lambda\left(n(l^1_n + \cdots + l^{k_n}_n) + \cdots + m(l^1_m + \cdots + l^{k_m}_m)\right)} (\alpha^1_n)^{l^1_n} \cdots (\alpha^{k_n}_n)^{l^{k_n}_n} \cdots (\alpha^1_m)^{l^1_m} \cdots (\alpha^{k_m}_m)^{l^{k_m}_m} d\lambda \\
&= 2\pi\beta_{0\cdots0\cdots0\cdots0} = 2\pi\psi(e).
\end{aligned}$$

□

Now, let us demonstrate

Proposition 3.9. *Suppose that* (A′) *there exists a* $\varphi_{\max} \in \mathcal{V}$ *satisfying* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$. *Then,* $2\pi\psi(e)\varphi_{\max} = \int_0^{2\pi} \hat{\psi}_\lambda d\lambda$ *for all* $\psi \in \mathcal{V}$.

Proof. Lemma 3.7 implies that for all $u \in (U_+ \cap GQ_-)_e$,

$$(2\pi\psi(e)\varphi_{\max})(u) = 2\pi\psi(e) = \Big(\int_0^{2\pi} \hat{\psi}_\lambda d\lambda\Big)(u).$$

This, combined with Lemma 3.5-(iv) and $2\pi\psi(e)\varphi_{\max}, \int_0^{2\pi} \hat{\psi}_\lambda d\lambda \in \mathcal{V}$, enables one to conclude that $2\pi\psi(e)\varphi_{\max} = \int_0^{2\pi} \hat{\psi}_\lambda d\lambda$ on the whole GQ_-. □

We end this subsection with showing two lemmas.

Lemma 3.10. *The following items* (A′) *and* (B′) *are equivalent*:

(A′) *There exists a* $\varphi_{\max} \in \mathcal{V}$ *satisfying* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$.

(B′) $\mathcal{V} \neq \{0\}$.

Proof. (A′)⇒(B′). Obvious.

(B′)⇒(A′). Suppose that $\mathcal{V} \neq \{0\}$. By $\mathcal{V} \neq \{0\}$, there exists a $\psi_a \in \mathcal{V}$ such that $\psi_a \neq 0$. Since $0 \neq \psi_a \in \mathcal{V}$ and (2.17)-(2), we can select a $g \in G$ satisfying $\psi_a(g) \neq 0$. Now, let $\psi_b := \varrho(g^{-1})\psi_a$. Then, it is immediate from $\varrho(g^{-1})\psi_a \in \mathcal{V}$ and (2.18) that

$$\psi_b \in \mathcal{V}, \quad \psi_b(e) \neq 0.$$

Accordingly, Lemma 3.4-(iii) yields $\int_0^{2\pi} (\hat{\psi}_b)_\lambda d\lambda \in \mathcal{V}$. Setting

$$\varphi_{\max} := \frac{1}{2\pi\psi_b(e)} \int_0^{2\pi} (\hat{\psi}_b)_\lambda d\lambda,$$

we deduce by Lemma 3.7 that $\varphi_{\max} \in \mathcal{V}$ and $\varphi_{\max}(u) = 1$ for all $u \in (U_+ \cap GQ_-)_e$. □

Lemma 3.11. *Let* $\varphi_{\max}$ *be an element of* $\mathcal{V}$ *such that* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$. *Then,* $\varphi_{\max}(\ell x \ell^{-1}) = \varphi_{\max}(x)$ *for all* $(\ell, x) \in L \times GQ_-$.

Proof. Fix any $(\ell, x) \in L \times GQ_-$, and remark that

$$\varrho(\ell^{-1})(\chi(\ell)\varphi_{\max}) \in \mathcal{V}.$$

Since $\ell(U_+ \cap GQ_-)\ell^{-1} \subset U_+ \cap GQ_-$, we see that for any $u \in (U_+ \cap GQ_-)_e$, $\ell u \ell^{-1} \in (U_+ \cap GQ_-)_e$ and hence

$$(\varrho(\ell^{-1})(\chi(\ell)\varphi_{\max}))(u) = \varphi_{\max}(\ell u \ell^{-1}) = 1 = \varphi_{\max}(u).$$

Consequently Lemma 3.5-(iv) allows us to assert that $\varrho(\ell^{-1})(\chi(\ell)\varphi_{\max}) = \varphi_{\max}$ on the whole GQ_-, and so $\varphi_{\max}(\ell x \ell^{-1}) = (\varrho(\ell^{-1})(\chi(\ell)\varphi_{\max}))(x) = \varphi_{\max}(x)$. □

3.2 The Entrance of $\mathcal{H}$

First, we set

$$\langle \psi_1, \psi_2 \rangle := \int_G \psi_1(g)\overline{\psi_2(g)}d\mu(g), \quad \|\psi\| := \sqrt{\langle \psi, \psi \rangle} \quad \text{for } \psi_1, \psi_2, \psi \in \mathcal{V}, \tag{3.12}$$

$$\mathcal{H} := \{\phi \in \mathcal{V} : \|\phi\| < \infty\}, \tag{3.13}$$

where μ denotes the non-zero Haar measure on G (recall Subsection 2.3 for the arguments below). With this setting, we assert

Proposition 3.14.

(1) $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ *is a complex pre-Hilbert space.*

(2) $\langle \varrho(g)\phi_1, \varrho(g)\phi_2 \rangle = \langle \phi_1, \phi_2 \rangle$ *for all* $g \in G$ *and* $\phi_1, \phi_2 \in \mathcal{H}$.

(3) *Both mappings* $\mathcal{H} \times \mathcal{H} \ni (\phi_1, \phi_2) \mapsto \phi_1 + \phi_2 \in \mathcal{H}$ *and* $\mathbb{C} \times \mathcal{H} \ni (\alpha, \phi) \mapsto \alpha\phi \in \mathcal{H}$ *are continuous, with respect to the norm* $\|\cdot\|$ *in* (3.12).

(4) $G \times \mathcal{H} \ni (g, \phi) \mapsto \varrho(g)\phi \in \mathcal{H}$ *is a continuous mapping, with respect to the norm* $\|\cdot\|$ *in* (3.12).

Proof. (1) One can deduce (1) by arguments similar to those in showing that $\mathscr{L}^2(G)$ is a complex vector space and $\langle f_1, f_2\rangle$ is a Hermitian inner product of f_1 and $f_2 \in \mathscr{L}^2(G)$, except for the property "$\langle f, f\rangle = 0$ implies $f = 0$." Here

$$\mathscr{L}^2(G) := \{f : G \to \mathbb{C} \,|\, f \text{ is } \mathscr{B}\text{-measurable and } \|f\| < \infty\}$$

and we apply the same notation as in (3.12) to the elements of $\mathscr{L}^2(G)$. Let us check that $\|\phi\| = 0$ ($\phi \in \mathcal{H}$) implies $\phi = 0$. For $\phi \in \mathcal{H}$ we suppose that $\|\phi\| = 0$. Then, $\phi = 0$ (a.e.) on G, and therefore $\phi = 0$ on G because $\phi : G \to \mathbb{C}$ is continuous and (p2) in Subsection 2.3. From $\phi = 0$ on G, $\phi \in \mathcal{V}$ and (2.17)-(2) we see that $\phi = 0$ on the whole GQ_-.

(2) From $\phi_i \in \mathcal{H} \subset \mathcal{V}$, we obtain $\varrho(g)\phi_i \in \mathcal{V}$. (2) follows by (p5) in Subsection 2.3, (2.18) and (3.12). Incidentally, (2) ensures that $\varrho(g)\phi \in \mathcal{H}$ for all $(g, \phi) \in G \times \mathcal{H}$.

(3) comes from (1).

(4) We will deduce (4) from the following item $(*)$:

$(*)$ For a given $\phi_0 \in \mathcal{H}$, the mapping $G \ni g \mapsto \varrho(g)\phi_0 \in \mathcal{H}$ is continuous at the point e.

First, let us confirm the item above.

$(*)$ Take any $\epsilon > 0$. We will conclude that there exists an open neighborhood V of $e \in G$ satisfying $\|\varrho(h)\phi_0 - \phi_0\| < \epsilon$ for all $h \in V$. Since μ is a regular Borel measure on $(G, \mathscr{B})$ (cf. Remark 2.7), Proposition 7.4.3 in Cohn, p.207 implies that $\mathscr{C}_0(G) := \{f_0 : G \to \mathbb{C} \,|\, f_0 \text{ is a continuous function whose support is compact}\}$ is dense in $\mathscr{L}^2(G)$. Hence by $\phi_0|_G \in \mathscr{L}^2(G)$ there exists a continuous function $f_0 : G \to \mathbb{C}$ such that

(a1) $\operatorname{supp} f_0$ is a non-empty compact subset in G, (a2) $\|\phi_0 - f_0\| < \epsilon/3$.

Since G is a locally compact Hausdorff space, there exists an open neighborhood V_1 of $e \in G$ whose closure $\overline{V_1}$ in G is compact. Here, (a1) implies that $\overline{V_1} \operatorname{supp} f_0$ is a compact subset in G. This and (p1), (p3) in Subsection 2.3 assure $0 \le \mu(\overline{V_1} \operatorname{supp} f_0) < \infty$. Thus $\delta := 1 + \mu(\overline{V_1} \operatorname{supp} f_0)$ satisfies

$$0 < \delta < \infty.$$

Now, it follows from (a1) that f_0 is uniformly continuous on G, so that for $\epsilon/(3\sqrt{\delta}) > 0$ there exists an open neighborhood V of $e \in G$ such that (a3) $V = V^{-1}$, (a4) $V \subset V_1$ and (a5) $|f_0(x) - f_0(y)| < \epsilon/(3\sqrt{\delta})$ for every $x, y \in G$ with $xy^{-1} \in V$. Then, any $(h, g) \in V \times G$ satisfies $(h^{-1}g)g^{-1} = h^{-1} \in V$, and hence

$$\text{(a6) } |f_0(h^{-1}g) - f_0(g)| < \epsilon/(3\sqrt{\delta}).$$

If $g \notin \overline{V_1} \operatorname{supp} f_0$, then $g \notin \operatorname{supp} f_0$, and $f_0(g) = 0$. If $g \notin \overline{V_1} \operatorname{supp} f_0$ and $h \in V$, then $h^{-1}g \notin \operatorname{supp} f_0$, and $f_0(h^{-1}g) = 0$. Consequently $h \in V$ implies

$$\begin{aligned}
\|\varrho(h)f_0 - f_0\|^2 &\overset{(3.12)}{=} \int_G |f_0(h^{-1}g) - f_0(g)|^2 d\mu(g) \\
&= \int_{\overline{V_1} \operatorname{supp} f_0} |f_0(h^{-1}g) - f_0(g)|^2 d\mu(g) + \int_{G - \overline{V_1} \operatorname{supp} f_0} |f_0(h^{-1}g) - f_0(g)|^2 d\mu(g) \\
&= \int_{\overline{V_1} \operatorname{supp} f_0} |f_0(h^{-1}g) - f_0(g)|^2 d\mu(g) \\
&\overset{(a6)}{\le} \int_{\overline{V_1} \operatorname{supp} f_0} \frac{\epsilon^2}{9\delta} d\mu(g) = \frac{\epsilon^2}{9\delta} \cdot \mu(\overline{V_1} \operatorname{supp} f_0) = \frac{\epsilon^2}{9} \frac{\mu(\overline{V_1} \operatorname{supp} f_0)}{1 + \mu(\overline{V_1} \operatorname{supp} f_0)} < \frac{\epsilon^2}{9}.
\end{aligned}$$

Hence we assert that for any $h \in V$,

$$\begin{aligned}
\|\varrho(h)\phi_0 - \phi_0\| &\le \|\varrho(h)\phi_0 - \varrho(h)f_0\| + \|\varrho(h)f_0 - f_0\| + \|f_0 - \phi_0\| \\
&= \|\varrho(h)f_0 - f_0\| + 2\|f_0 - \phi_0\| \qquad \text{(cf. (2))} \\
&< \epsilon \qquad (\because \text{(a2)}).
\end{aligned}$$

This completes the proof of $(*)$.

Now, we are in a position to prove (4). Fix any $\epsilon > 0$ and $(g_0, \phi_0) \in G \times \mathcal{H}$. Since $(*)$ and $\phi_0' := \varrho(g_0)\phi_0$ belongs to $\mathcal{H}$, there exists an open neighborhood V of $e \in G$ satisfying

$$\|\varrho(h)\phi_0' - \phi_0'\| < \epsilon/2$$

for all $h \in V$. We get an open neighborhood U of $g_0 \in G$ by setting $U := Vg_0$. Then, for any $g \in U$ and $\phi \in \mathcal{H}$ with $\|\phi - \phi_0\| < \epsilon/2$, one obtains an $h' \in V$ such that $g = h'g_0$, and therefore

$$\begin{aligned}\|\varrho(g)\phi - \varrho(g_0)\phi_0\| &\leq \|\varrho(g)\phi - \varrho(g)\phi_0\| + \|\varrho(g)\phi_0 - \varrho(g_0)\phi_0\| \\ &\overset{(2)}{=} \|\phi - \phi_0\| + \|\varrho(h')\phi'_0 - \phi'_0\| < \epsilon.\end{aligned}$$

Consequently, the mapping $G \times \mathcal{H} \ni (g, \phi) \mapsto \varrho(g)\phi \in \mathcal{H}$ is continuous. □

3.3 The Completeness of $\mathcal{H}$

The notation in Subsections 3.1 and 3.2 remains valid. Our aim in this subsection is to establish Proposition 3.18. First, let us prepare three lemmas for the aim.

Lemma 3.15. *Suppose that* (A′) *there exists a* $\varphi_{\max} \in \mathcal{V}$ *satisfying* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$. *Then,* $|\psi(e)|\|\varphi_{\max}\| \leq \|\psi\|$ *for each* $\psi \in \mathcal{V}$.

Proof. By a direct computation we have

$$\begin{aligned}
&2\pi|\psi(e)|^2\|\varphi_{\max}\|^2 \\
&= 2\pi \int_G |\psi(e)\varphi_{\max}(g)|^2 d\mu(g) \qquad (\because (3.12)) \\
&= \frac{1}{2\pi}\int_G \Big|\int_0^{2\pi} \hat{\psi}_\lambda(g)d\lambda\Big|^2 d\mu(g) \qquad (\because \psi \in \mathcal{V}, \text{(A}'\text{), Proposition 3.9}) \\
&\leq \int_G \Big(\int_0^{2\pi} |\hat{\psi}_\lambda(g)|^2 d\lambda\Big) d\mu(g) \qquad (\because \text{the Schwarz inequality}) \\
&= \int_G \Big(\lim_{n\to\infty} \frac{2\pi}{n}\sum_{k=1}^{n} \big|\hat{\psi}_{\frac{2\pi k}{n}}(g)\big|^2\Big) d\mu(g) \\
&\leq \varliminf_{n\to\infty} \int_G \frac{2\pi}{n}\sum_{k=1}^{n} \big|\hat{\psi}_{\frac{2\pi k}{n}}(g)\big|^2 d\mu(g) \qquad (\because \text{the Fatou lemma}) \\
&= \varliminf_{n\to\infty} \frac{2\pi}{n}\sum_{k=1}^{n} \int_G \big|\hat{\psi}_{\frac{2\pi k}{n}}(g)\big|^2 d\mu(g) \\
&= \varliminf_{n\to\infty} \frac{2\pi}{n}\sum_{k=1}^{n} \int_G \big|\psi(g)\big|^2 d\mu(g) \qquad (\because \text{(p5), (p6) in Subsection 2.3, (3.3)}) \\
&= \varliminf_{n\to\infty} 2\pi \int_G \big|\psi(g)\big|^2 d\mu(g) = 2\pi \int_G \big|\psi(g)\big|^2 d\mu(g) \overset{(3.12)}{=} 2\pi\|\psi\|^2.
\end{aligned}$$

Here, we note that $\displaystyle\int_0^{2\pi} |\hat{\psi}_\lambda(g)|^2 d\lambda = \lim_{n\to\infty} \frac{2\pi}{n}\sum_{k=1}^{n} \big|\hat{\psi}_{\frac{2\pi k}{n}}(g)\big|^2$. □

Lemma 3.16. *The following items* (A) *and* (B) *are equivalent*:

(A) *There exists a* $\varphi_{\max} \in \mathcal{H}$ *such that* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$.

(B) $\mathcal{H} \neq \{0\}$.

Proof. (A)⇒(B). Obvious.

(B)⇒(A). Suppose that $\mathcal{H} \neq \{0\}$, and fix any $\phi_a \in \mathcal{H} - \{0\}$. On the one hand, there exists a $\varphi_{\max} \in \mathcal{V}$ such that $\varphi_{\max}(u) = 1$ for all $u \in (U_+ \cap GQ_-)_e$ by virtue of Lemma 3.10 and $\{0\} \neq \mathcal{H} \subset \mathcal{V}$. On the other hand, since $0 \neq \phi_a \in \mathcal{H} \subset \mathcal{V}$ and (2.17)-(2) there exists a $g \in G$ such that $\phi_a(g) \neq 0$. Thus, it follows from $\phi_b := \varrho(g^{-1})\phi_a$ and (2.18) that $\phi_b \in \mathcal{H}$ and $\phi_b(e) \neq 0$. Consequently Lemma 3.15 implies that $|\phi_b(e)|\|\varphi_{\max}\| \leq \|\phi_b\| < \infty$. This, together with $\phi_b(e) \neq 0$, shows that $\|\varphi_{\max}\| < \infty$, so that $\varphi_{\max}$ belongs to $\mathcal{H}$. □

Lemma 3.17. *Suppose that*

(S) $|\chi(\ell)| = 1$ *for all* $\ell \in L$,

(A) *there exists a* $\varphi_{\max} \in \mathcal{H}$ *such that* $\varphi_{\max}(u) = 1$ *for all* $u \in (U_+ \cap GQ_-)_e$.

Then for any non-empty compact subset $E \subset GQ_-$, *there exists a* $c_E > 0$ *such that*

$$\sup\{|\phi(a)| : a \in E\} \leq c_E \|\phi\| \quad \textit{for all } \phi \in \mathcal{H}.$$

Proof. Since E is compact in GQ_-, Proposition 3.1 and (S) provide us with an $m_E > 0$ such that

$$m_E \leq |\chi(q)| \quad \text{for all } gq \in E \subset GQ_-.$$

Noting $\varphi_{\max} \neq 0$ and $\|\varphi_{\max}\| < \infty$, we set

$$c_E := \frac{1}{m_E \|\varphi_{\max}\|}.$$

Then, $c_E > 0$ holds. For any $a = gq \in E$ ($\subset GQ_-$) and $\phi \in \mathcal{H}$, we obtain

$$\begin{aligned}
|\phi(a)| = |\phi(gq)| &= |\chi(q)^{-1}||(\varrho(g^{-1})\phi)(e)| && (\because \phi \in \mathcal{H} \subset \mathcal{V}, (2.18), (2.17)\text{-}(2)) \\
&\leq |\chi(q)^{-1}| \frac{\|\varrho(g^{-1})\phi\|}{\|\varphi_{\max}\|} && (\because \text{(A), Lemma 3.15}, \varrho(g^{-1})\phi \in \mathcal{V}) \\
&= |\chi(q)^{-1}| \frac{\|\phi\|}{\|\varphi_{\max}\|} && (\because \text{Proposition 3.14-(2)}) \\
&\leq c_E \|\phi\|.
\end{aligned}$$

□

Now, we are in a position to prove

Proposition 3.18. *Suppose that* (S) $|\chi(\ell)| = 1$ *for all* $\ell \in L$. *Then* $(\mathcal{H}, \langle \cdot , \cdot \rangle)$ *is a complex Hilbert space.*

Proof. Because of Proposition 3.14-(1), it suffices to confirm that $(\mathcal{H}, \| \cdot \|)$ is complete. That is trivial in case of $\mathcal{H} = \{0\}$. For this reason we investigate the case where $\mathcal{H} \neq \{0\}$ henceforth. Since $\mathcal{H} \neq \{0\}$ and Lemma 3.16, there exists a $\varphi_{\max} \in \mathcal{H}$ such that $\varphi_{\max}(u) = 1$ for all $u \in (U_+ \cap GQ_-)_e$. This and (S) permit us to use Lemma 3.17. Now, let $\{\phi_n\}_{n=1}^{\infty}$ be an arbitrary Cauchy sequence in $(\mathcal{H}, \| \cdot \|)$, namely

$$\lim_{n,m\to\infty} \|\phi_n - \phi_m\| = 0.$$

We want to first show that $\{\phi_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(\mathcal{V}, d)$. Here d is the Fréchet metric in (2.20). For any non-empty compact subset $E \subset GQ_-$ there exists a $c_E > 0$ such that

$$d_E(\phi_n, \phi_m) \overset{(2.19)}{=} \sup\{|\phi_n(a) - \phi_m(a)| : a \in E\} \leq c_E \|\phi_n - \phi_m\| \quad \text{for all } n, m \in \mathbb{N}$$

by Lemma 3.17. This, $\lim_{n,m\to\infty} \|\phi_n - \phi_m\| = 0$ and Corollary 2.23-(2) enable us to conclude that $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{V}, d)$. Since $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{V}, d)$ and since $(\mathcal{V}, d)$ is complete, there exists a unique $\psi \in \mathcal{V}$ such that $\lim_{n\to\infty} d(\phi_n, \psi) = 0$. For this $\psi \in \mathcal{V}$ one can demonstrate

$$\psi \in \mathcal{H} \text{ (namely, } \|\psi\| < \infty), \lim_{n\to\infty} \|\phi_n - \psi\| = 0$$

by arguments similar to those in the proof of Proposition 6, Weil, 1971, p.59–60. Therefore $(\mathcal{H}, \| \cdot \|)$ is complete. □

For $\phi \in \mathcal{H} \subset \mathcal{V}$, we have only computed the integral $\int_0^{2\pi} \hat{\phi}_\lambda d\lambda$ with respect to the Fréchet metric d on $\mathcal{V}$ so far. In the last subsection, we will need to compute the integral $\int_0^{2\pi} \hat{\phi}_\lambda d\lambda$ with respect to the norm $\| \cdot \|$ on $\mathcal{H}$. For this reason we prepare

Lemma 3.19.

(i) *$\hat{\phi}_\lambda \in \mathcal{H}$ for all $(\phi, \lambda) \in \mathcal{H} \times [0, 2\pi]$.*

(ii) *For each $\phi \in \mathcal{H}$, the mapping $[0, 2\pi] \ni \lambda \mapsto \hat{\phi}_\lambda \in \mathcal{H}$ is continuous with respect to $\|\cdot\|$.*

If $|\chi(\ell)| = 1$ for all $\ell \in L$, then the following items (iii) *and* (iv) *hold for every $\phi \in \mathcal{H}$:*

(iii) *The integral $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda$ with respect to $\|\cdot\|$ belongs to $\mathcal{H}$.*

(iv) *The integral $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda$ with respect to d is equal to that with respect to $\|\cdot\|$.*

Here d is the Fréchet metric on $\mathcal{V}$ in (2.20)*, and $\|\cdot\|$ is the norm on $\mathcal{H}$ in* (3.12).

Proof. (i), (ii), (iii) cf. the proof of Lemma 3.4, Proposition 3.14-(3), (4), Proposition 3.18.

(iv) It is trivial in case of $\phi = 0$. We suppose that $\phi \neq 0$ hereafter. Let us dare to denote by $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d$ (resp. $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}$) its integral with respect to the Fréchet metric d (resp. to the norm $\|\cdot\|$). By the definition of integral we have

$$d\Big(\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d, \frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}\Big) \to 0, \quad \Big\|\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|} - \frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}\Big\| \to 0 \quad (n \to \infty),$$

where we remark that both $(\mathcal{V}, d)$ and $(\mathcal{H}, \|\cdot\|)$ are complete, topological vector spaces. Now, let E be any non-empty compact subset in GQ_-. By virtue of $0 \neq \phi \in \mathcal{H}$ and Lemmas 3.16 and 3.17, there exists a $c_E > 0$ which satisfies

$$\sup\{|\phi_1(a)| : a \in E\} \leq c_E\|\phi_1\| \quad \text{for all } \phi_1 \in \mathcal{H}.$$

Consequently it follows that

$$\begin{aligned}
& d_E\Big(\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d, \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}\Big) \\
& \leq d_E\Big(\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d, \frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}\Big) + d_E\Big(\frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}, \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}\Big) \\
& \overset{(2.19)}{=} d_E\Big(\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d, \frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}\Big) + \sup\Big\{\Big|\Big(\frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}} - \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}\Big)(a)\Big| : a \in E\Big\} \\
& \leq d_E\Big(\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d, \frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}}\Big) + c_E\Big\|\frac{2\pi}{n}\sum_{k=1}^{n} \hat{\phi}_{\frac{2\pi k}{n}} - \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}\Big\| \to 0 \quad (n \to \infty).
\end{aligned}$$

Hence $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d = \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}$ on E. Consequently $\displaystyle\int_0^{2\pi} \hat{\phi}_\lambda d\lambda_d = \int_0^{2\pi} \hat{\phi}_\lambda d\lambda_{\|\cdot\|}$ on the whole GQ_-, because of (d1), (d2) in Paragraph 2.4.4. □

3.4 The Irreducibility of $\varrho : G \to GL(\mathcal{H})$

We utilize the notation in Subsections 3.1, 3.2 and 3.3. We prove Proposition 3.20 first and accomplish the main purpose afterwards.

Proposition 3.20. *Suppose that* (S) *$|\chi(\ell)| = 1$ for all $\ell \in L$. Then,*

(1) *$\mathcal{H} = (\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a separable complex Hilbert space,*

(2) *ϱ is an irreducible unitary representation of G on $\mathcal{H}$.*

Proof. Assume that $\mathcal{H} \neq \{0\}$ (otherwise our assertions are trivial). From Proposition 3.18 and (S) it follows that $\mathcal{H}$ is a complex Hilbert space. Hence Proposition 3.14-(2), (4) imply that ϱ is a unitary representation of G on $\mathcal{H}$. Consequently, it suffices to prove that

$$(1')\ \mathcal{H} \text{ is separable,} \quad (2')\ \varrho : G \to GL(\mathcal{H}) \text{ is irreducible.}$$

However, $(1')$ is immediate from $(2')$, since G satisfies the second countability axiom. For this reason we prove $(2')$ only. Now, let $\mathcal{H}_1$ be any closed $\varrho(G)$-invariant complex vector subspace of $\mathcal{H}$ with $\mathcal{H}_1 \neq \{0\}$, and let $\mathcal{H}_2$ denote the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$ with respect to $\langle\cdot,\cdot\rangle$. Then, $\mathcal{H}_2$ is also a closed $\varrho(G)$-invariant complex vector subspace of $\mathcal{H}$ and

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

If $\mathcal{H}_2 = \{0\}$, then we can conclude $(2')$. Let us prove $\mathcal{H}_2 = \{0\}$ by reductive absurdity. Suppose that $\mathcal{H}_2 \neq \{0\}$. Then for each $j = 1, 2$, there exists a non-zero $\phi_a^j \in \mathcal{H}_j$ by virtue of $\mathcal{H}_j \neq \{0\}$. From $0 \neq \phi_a^j \in \mathcal{H}_j \subset \mathcal{V}$ and (2.17)-(2) we obtain a $g_j \in G$ such that $\phi_a^j(g_j) \neq 0$. Setting $\phi_b^j := \varrho(g_j^{-1})\phi_a^j$ one can assert that

$$\phi_b^j \in \mathcal{H}_j, \quad \phi_b^j(e) \neq 0$$

because $\mathcal{H}_j$ is $\varrho(G)$-invariant. For any $0 \leq \lambda \leq 2\pi$, it turns out that $\chi(\exp \lambda T) \in \mathbb{C}$, so that the scalar multiple $\chi(\exp \lambda T)\phi_b^j$ belongs to the complex vector space $\mathcal{H}_j$. Therefore

$$(\hat{\phi}_b^j)_\lambda \overset{(3.3)}{=} \varrho(\exp(-\lambda T))(\chi(\exp \lambda T)\phi_b^j) \in \mathcal{H}_j \text{ for all } \lambda \in [0, 2\pi]$$

follows by $\mathcal{H}_j$ being $\varrho(G)$-invariant. This assures $\int_0^{2\pi} (\hat{\phi}_b^j)_\lambda d\lambda \in \mathcal{H}_j$ since $\mathcal{H}_j$ is complete. Hence for each $j = 1, 2$, we have $\phi_c^j := \dfrac{1}{2\pi\phi_b^j(e)} \int_0^{2\pi} (\hat{\phi}_b^j)_\lambda d\lambda \in \mathcal{H}_j$; and Lemmas 3.7 and 3.19-(iv) imply that $\phi_c^j(u) = 1$ for all $u \in (U_+ \cap GQ_-)_e$. Consequently Lemma 3.5-(iv) yields $\mathcal{H}_1 \ni \phi_c^1 = \phi_c^2 \in \mathcal{H}_2$, and so $0 \neq \phi_c^1 \in \mathcal{H}_1 \cap \mathcal{H}_2$. This contradicts $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. Thus $\mathcal{H}_2 = \{0\}$ holds. □

Now, we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By virtue of Proposition 3.20 it suffices to demonstrate that in case of $\mathcal{H} \neq \{0\}$, the following two items hold:

(I) There exists a unique $\varphi_{\max} \in \mathcal{H}$ such that $\varphi_{\max}(u) = 1$ for all $u \in (U_+ \cap GQ_-)_e$.

(II) There exists a non-zero $\phi \in \mathcal{H}$ satisfying $\int_G |\langle\varrho(g)\phi, \phi\rangle|^2 d\mu(g) = \|\phi\|^6$ $(< \infty)$.

Let us show (I) and (II) from now on.

(I) On the one hand, Lemma 3.16 and $\mathcal{H} \neq \{0\}$ assure the existence of $\varphi_{\max}$. On the other hand, Lemma 3.5-(iv) assures the uniqueness of $\varphi_{\max}$. Therefore (I) holds.

(II) From (3.12) it suffices to conclude that

$$\langle\varrho(g)\varphi_{\max}, \varphi_{\max}\rangle = \varphi_{\max}(g^{-1})\|\varphi_{\max}\|^2 \quad \text{for every } g \in G. \tag{3.21}$$

By a direct computation one has

$$\begin{aligned}
&\langle\varrho(g)\varphi_{\max}, \varphi_{\max}\rangle \\
&= \int_G (\varrho(g)\varphi_{\max})(a) \cdot \overline{\varphi_{\max}(a)} d\mu(a) \qquad (\because (3.12)) \\
&= \int_G (\varrho(g)\varphi_{\max})((\exp \lambda T)a \exp(-\lambda T)) \cdot \overline{\varphi_{\max}((\exp \lambda T)a \exp(-\lambda T))} d\mu(a) \\
&\qquad (\because \text{(p5), (p6) in Subsection 2.3}) \\
&= \int_G (\varrho(g)\varphi_{\max})((\exp \lambda T)a \exp(-\lambda T)) \cdot \overline{\varphi_{\max}(a)} d\mu(a) \qquad (\because \text{(I), Lemma 3.11}) \\
&= \int_G (\widehat{\varrho(g)\varphi_{\max}})_\lambda(a) \cdot \overline{\varphi_{\max}(a)} d\mu(a) \qquad (\because (3.3))
\end{aligned} \tag{3.22}$$

for all $\lambda \in [0, 2\pi]$. This provides us with

$$\begin{aligned}
&2\pi\varphi_{\max}(g^{-1})\|\varphi_{\max}\|^2 \\
&= \int_G 2\pi\varphi_{\max}(g^{-1})\varphi_{\max}(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \qquad (\because (3.12)) \\
&= \int_G 2\pi(\varrho(g)\varphi_{\max})(e)\varphi_{\max}(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \\
&= \int_G \Big(\int_0^{2\pi}(\widehat{\varrho(g)\varphi_{\max}})_\lambda d\lambda\Big)(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \\
&\qquad (\because \varrho(g)\varphi_{\max} \in \mathcal{H} \subset \mathcal{V}, \text{ (I), Proposition 3.9}) \\
&= \int_G \Big(\lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}(\widehat{\varrho(g)\varphi_{\max}})_{\frac{2\pi k}{n}}\Big)(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \\
&\Big(\text{where } \lim_{n\to\infty}\Big\|\int_0^{2\pi}(\widehat{\varrho(g)\varphi_{\max}})_\lambda d\lambda - \frac{2\pi}{n}\sum_{k=1}^{n}(\widehat{\varrho(g)\varphi_{\max}})_{\frac{2\pi k}{n}}\Big\| = 0 \text{ due to Lemma 3.19-(iv)}\Big) \\
&= \lim_{n\to\infty}\int_G \Big(\frac{2\pi}{n}\sum_{k=1}^{n}(\widehat{\varrho(g)\varphi_{\max}})_{\frac{2\pi k}{n}}\Big)(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \\
&\qquad (\because \mathcal{H} \ni \phi \mapsto \langle\phi, \varphi_{\max}\rangle \in \mathbb{C} \text{ is a continuous function, (3.12)}) \\
&= \lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\int_G (\widehat{\varrho(g)\varphi_{\max}})_{\frac{2\pi k}{n}}(a)\cdot\overline{\varphi_{\max}(a)}d\mu(a) \\
&= \lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\langle\varrho(g)\varphi_{\max}, \varphi_{\max}\rangle \qquad (\because (3.22)) \\
&= 2\pi\langle\varrho(g)\varphi_{\max}, \varphi_{\max}\rangle.
\end{aligned}$$

Hence (3.21) follows. □

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