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Reproduction of Fuchs Relation for the Group $S_4^{SL_2}$ by Using Groebner Bases

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Abstract

In 1872, Lazarus Fuchs used a new tool which is The Invariant Theory to construct the minimal polynomial of an algebraic solution of a differential equation of second order. He expressed the coefficient of the equation in terms of the (semi-)invariants of its differential Galois group. In this paper we will give another method to obtain Fuchs Relation:

$$I = \frac{1}{w^2}\left(\frac{36}{5}a_0 S^2 - \frac{6}{5}SS'' + S'^2\right)$$

for the octahedral groupe $S_4^{SL_2}$ by using Groebner Basis; a tool which is introduced in 1965 nearly two century after Fuchs Relation.

Keywords: Differential Galois Theory, Algebraic Solutions, Groebner Basis, Invariant Theory, Representation Theory

1. Introduction

This section is a recall of some definitions in the theories that we will depond on in our study.

In the second section we will give an Algorithm to obtain Fuchs Relation and in the third section we will expose the codes Maple and Magma we have used.

In 1872, Lazarus Fuchs used a new tool which is the Invariant Theory to construct the minimal polynomial of an algebric solution of a differential equation of second order whose solutions space is generated by $\{y_1, y_2\}$ (Baldassarri, F. & Dwork, B., 1979; Kovacic,. J.. 1985). He expressed the coefficient of the equation in terms of the (semi-)invariants I, S and differential semi-invariant S', S'' of its differential Galois group . L. Fuchs found this relation:

$$I = \frac{1}{w^2}\left(\frac{36}{5}a_0 S^2 - \frac{6}{5}SS'' + S'^2\right)$$

such that w is the wronskian and a_0 is the coefficient in the ordinairy linear differential equation:

$$L(y) = y'' + a_0 y$$

over a differential field $(k,')$ whose field of constants C is algebraically closed of characteristic 0.

I and S are the invariant and the semi-invariant: $I = y_1^8 + 14y_1^4 y_2^4 + y_2^8$, $S = y_1^5 y_2 - y_1 y_2^5$ of the group $S_4^{SL_2}$ generated by the matrices A and B such that:

$$A = \frac{1}{2}\begin{pmatrix} i-1 & i-1 \\ i+1 & i-1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

(for more knowledge about Representation Theory refer to (Fulton, W. & Harris, J., 1991), (Weyl, H., 1946)).

2. Invariants Theory

Definition 1. Let V be a finite dimensional K -vector space and G be a linear subgroup of $GL(V)$. An ***invariant*** is a polynomial function $f \in K[V]$ which remains unchanged under the group action, i.e $f = f \circ g$ for all $g \in G$. If, for some $g \in G$, f and $f \circ g$ differ from each other only by a constant factor then the polynomial function f is called a ***semi-invariant (or a relative invariant)*** (for more knowledge about Invariant Theory refer to (Benson, D. J., 1993), (Cox, D. J., 1992), (Sturmfels,. B., 1993)).

Remark. The computation of Reynold Operator and Molien series which is needed to find invariants is implemented in different Programs such as Mathematica and Magma.

Differential Galois Theory

Definition 1. ***A differential field*** $(k,')$ is a field k together with a derivation $'$ in k. The set of all constants $C = \{a \in k,\ a' = 0\}$ is a subfield of $(k,')$.

Let C be a field algebraically closed and $(k,')$ a differential field be of characteristic 0. Consider the following ordinary homogeneous linear differential equation:

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad (a_i \in k) \qquad (1)$$

over $(k,')$ with a system $\{y_1,\ldots,y_n\}$ of fundamental solutions. By extending the derivation $'$ to a system of fundamental solutions and by adjunction of these solutions and their derivatives to $(k,')$ in a way the field of constants does not change, one gets $K = k\langle y_1,\ldots,y_n\rangle$, the so-called *Picard-Vessiot* extension (PVE) of $L(y)=0$. With these assumptions, the PVE of $L(y)=0$ always exists and is unique up to differential isomorphisms. This extension plays the same role for a differential equation as a splitting field for a polynomial equation.

The set of all automorphisms of K, which fix $(k,')$ elementwise and commute with the derivation in K, is a group**,** ***the differential Galois group*** $\mathfrak{G}(K/k) = \mathfrak{G}(L)$ of $L(y)=0$.

Definition 2. Let $y_1,\cdots,y_n$ be elements in a differential field $(k,')$. The ***Wronskian*** of these elements is the matrix:

$$w(y_1,\ldots,y_n) = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1^{(2)} & y_2^{(2)} & \cdots & y_n^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

(for more knowledge about *Differential Galois theory* refer to (Magid,. A., 1994), (Kolchin, E. R., 1948)).

Theorem(Kolchin, E. R., 1948) A differential equation $L(y)=0$ with coefficients in k has Only solutions which are algebraic over k if and only if $\mathfrak{G}(L)$ is a finite groupe.

Groebner Basis

Definition 1. A ***monomial ordering*** $\succ$ on $k[x_1,\ \ldots, x_n]$ is any relation $\succ$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, any relation $\succ$ on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying :

i. $\succ$ is a total (or linear) ordering on $\mathbb{Z}^n_{\geq 0}$.

ii. If $\alpha \succ \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma \succ \beta + \gamma$.

iii. $\succ$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$.

Definition 2. ((Cox, D. J., 1992), p.77). Fix a monomial order. A finite subset $G = \{g_1,\ \ldots, g_t\}$ of an ideal I is said to be a Groebner basis if

$$\langle \mathrm{LT}(g_1),\ \ldots, \mathrm{LT}(g_t)\rangle = \langle \mathrm{LT}(I)\rangle$$

Equivalently, but more informally, a set $\{g_1,\ \ldots, g_t\} \subset I \subset k[x_1,\ \ldots, x_n]$ is a ***Groebner basis*** of I if and only if the leading term of any element of I is divisible by one of the $\mathrm{LT}(g_i)$.

Corollary ((Cox, D. J., 1992), p.77). Fix a monomial order. Then every ideal $I \subset k[x_1,\ \ldots, x_n]$ other than $\{0\}$ has a Groebner basis. Further more, any Groebner basis for an ideal I is a basis of I.

Definition 3. Given $I = \langle f_1,\ldots, f_s\rangle \subset k[x_1,\ldots,x_n]$, the $l-th$ **elimination ideal** I_l is the ideal of $k[x_{l+1},\ldots, x_n]$ defined by:

$$I_l = I \cap k\left[x_{l+1},\dots, x_n\right].$$

Thus, I_l consists of all consequences of $f_1 = \cdots = f_s = 0$ which eliminate the variables $x_1,\dots, x_l$.

Theorem (The Elimination Theorem). (**Theorem2** (Cox, D. J., 1992), p.116)Let $I \subset k\left[x_1, \dots, x_n\right]$ be an ideal and let G be a Groebner basis of I with respect to lex order where $x_1 \succ x_2 \dots \succ x_n$. Then, for every $0 \le l \le n$, the set:

$$G_l = G \cap k\left[x_{l+1}, \dots, x_n\right]$$

is a Groebner basis of the $l-th$ elimination ideal I_l.

Remark. an ordinairy linear differential equation over a differential field $(k,')$ whose field of constants C is algebraically closed of characteristic 0:

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y,\ \ a_{n-1},\ \dots,\ a_0 \in k \qquad (1)$$

can be transformed by the variable transformation $y = z \cdot \exp^{-\int \frac{a_1}{2}dx}$ to an equation of the form:

$$L_{SL}(y) = y^{(n)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y \qquad (2)$$

(see procedure transfereL, III), without losing the Liouvillian character of the solutions (cf. (Kolchin, E. R., 1948), p.184). Indeed, the differential Galois group of (2) is unimodulair (cf. (Kaplansky, I., 1957), p.41), so we will deal with the form (2)(for more knowledge about Liouvillian solutions of second order homogeneous Linear Differential Equations refer to (Fakler, W., 1996)).

2. Compute L. Fuchs Relation by using Groebner Bases

In this section, we compute the relation of L. Fuchs by using Groebner Bases (for more knowledge about Groebner Basis refer to (Adams, W., & Loustaunau, P., 1994), (Cox, D. J., 1992)).

Let $L(y) = y'' + a_0 y$ be an ordinairy linear differential equation over a differential field $(k,')$ whose field of constants C is algebraically closed of characteristic 0, with differential Galois group $\mathfrak{G} = S_4^{SL_2}$. L. Fuchs has found this relation:

$$I = \frac{1}{w^2}\left(\frac{36}{5}a_0 S^2 - \frac{6}{5}SS'' + S'^2\right)$$

Which relates (semi-)invariants S, I, differential semi-invariants S' and S'', the wronskian w and the coefficient a_0. In fact, we will prove in this paper that we can obtain this relation by using Groebner Bases, in particular the elimination ideal ((Cox, D. J., 1992), p.115).

Before giving The Algorithm, we will recall this proposition from (Cox, D. J., 1992):

Proposition1 (Cox, D. J., 1992), Proposition 7.3.3**.**Suppose that $f_1,\dots, f_m \in k\left[x_1,\dots, x_n\right]$ are given. Fix a monomial order in $k\left[x_1,\dots, x_n, y_1, \dots, y_m\right]$ where any monomial involving one of $x_1,\dots, x_n$ is greater than all monomials in $k\left[y_1, \dots, y_m\right]$. Let G be a Groebner basis of the ideal $\langle f_1 - y_1, \dots, f_m - y_m\rangle \subset k\left[x_1,\dots, x_n, y_1, \dots, y_m\right]$. Given $f \in k\left[x_1,\dots, x_n\right]$, let $g = \overline{f}^G$ be the remainder of f on division by G. Then:

(i) $f \in k\left[f_1, \dots, f_m\right]$ if and only if $g \in k\left[y_1, \dots, y_m\right]$.

(ii) If $f \in k\left[f_1, \dots, f_m\right]$, then $f = g\left(f_1, \dots, f_m\right)$ is an expression of f as a polynomial in $f_1, \dots, f_m$

The Algorithm is as follow:

1. Take the ring $R = F\left[y_1, \dots, y_{10}\right]$ with the lex order, Such that $F = F_0(x)$, F_0 is the cyclotomic field 120th.
2. Eliminate the variable y_i, for $i = \overline{1,4}$, from the ideal $I = \langle S, I, S', S'', w\rangle$ that is to calculate the Groebner basis for the ideal $I = \langle S, I, S', S'', w\rangle \cap F\left[y_5,\dots, y_{10}\right]$ (**Proposition1**, (Fuchs Relation, III).

In this section we provide two codes in Magma and in Maple.

We can test the Algorithm in section two on Magma:

```
// Fuchs Relation
Q:=RationalField();
L<a>:=CyclotomicField(8*5*3);
s2 :=a^(15)-a^(45);
i :=a^(30);
M :=GeneralLinearGroup(2,L);
U :=M![(1+i)/s2,0,0,(1-i)/s2];
S :=M ![(i-1)/2,(i-1)/2,(i+1)/2,(-i-1)/2];
s4 :=MatrixGroup<2,L|S,U>;
P<x1,x2,x3,x4,r,f1,f2,f3,f4,f5>:=PolynomialRing(L,10);
S :=x1^5*x2-x1*x2^5;
I :=x1^8+14*x1^4*x2^4+x2^8;
ID:=InvariantDifferentie(S);
S1 :=ID[1];
S2 :=ID[2];
W :=x1*x3-x2*x4;
id :=ideal<P|f1-S,f2-I,f3-S1,f4-S2,f5-W>;
EliminationIdeal(id,{r,f1,f2,f3,f4,f5});
```

Code Maple

transfereL is a procedure to transform an ordinairy linear differential equation over a differential field k whose field of constants C is algebraically closed of characteristic 0 from the form:

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y,\ a_{n-1}, \cdots, a_0 \in k \quad (1)$$

to the form:

$$L_{SL}(y) = y^{(n)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y, \quad (2)$$

```
with(DEtools):_Envdiffopdomain:=[Dx,x]:
transfereL:=proc(E,n)
local a,b; global x,y,L;
L:=de2diffop(E,y(x),[Dx,x]);
a(x):=coeff(L,Dx,n-1);
b:=simplify(exp(-int(a(x),x)/n));
L:=diffop2de(L,y(x));
L:=eval(subs(y(x)=y(x)*b,L));
L:=de2diffop(L,y(x),[Dx,x]);
L:=collect(L*(1/coeff(L,Dx,n)),Dx);
L:=diffop2de(L,y(x));
return(L);
end:
```

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