Modules Whose Nonzero Endomorphisms Have E-small Kernels

Abdoul Djibril DIALLO\(^1\), Papa Cheikhou DIOP\(^2\), Mamadou BARRY\(^3\)

\(^1\) Département de Mathématiques et Informatiques, Université Cheikh Anta Diop, Dakar, Sénégal
\(^2\) Département de Mathématiques, Université de Thies, Thies, Sénégal
\(^3\) Département de Mathématiques et Informatiques, Université Cheikh Anta Diop, Dakar, Sénégal

Correspondence: Papa Cheikhou DIOP, Département de Mathématiques, UFR SET, Université de Thies, Thies, Sénégal.

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Abstract

Let \( R \) be a commutative ring and \( M \) an unital \( R \)-module. A submodule \( L \) of \( M \) is called essential submodule of \( M \), if \( L \cap K \neq \{0\} \) for any nonzero submodule \( K \) of \( M \). A submodule \( N \) of \( M \) is called e-small submodule of \( M \) if, for any essential submodule \( L \) of \( M \), \( N + L = M \) implies \( L = M \). An \( R \)-module \( M \) is called e-small quasi-Dedekind module if, for each \( f \in \text{End}_R(M) \), \( f \neq 0 \) implies \( \text{Ker} f \) is e-small in \( M \). In this paper we introduce the concept of e-small quasi-Dedekind modules as a generalisation of quasi-Dedekind modules, and give some of their properties and characterizations.

Keywords: Essential submodules, e-small submodules, e-small quasi-Dedekind modules

1. Introduction

Throughout all rings are associative, commutative with identity and all modules are unitary \( R \)-module. A submodule \( L \) of \( M \) is called essential submodule of \( M \), if \( L \cap K \neq \{0\} \) for any nonzero submodule \( K \) of \( M \). (Zhou, D. X. & all (2011)) introduce and study the concept of e-small submodules, where a submodule \( N \) of \( M \) is called e-small submodule of \( M \) if, for any essential submodule \( L \) of \( M \), \( N + L = M \) implies \( L = M \). An \( R \)-module \( M \) is called e-small quasi-Dedekind module if, for each \( f \in \text{End}_R(M) \), \( f \neq 0 \) implies \( \text{Ker} f \) is e-small in \( M \). (Mijbass, A. S. (1997)) introduced and studied the concept of quasi-Dedekind module. In this paper we introduce and study the concept of e-small quasi-Dedekind as a generalization of quasi-Dedekind module. Also, we investigate the basic properties and characterizations of e-small quasi-Dedekind module. Finally we study the relations between e-small quasi-Dedekind modules and some classes of modules.

The notation \( N \leq M \) means that \( N \) is a submodule of \( M \) and \( N \leq^0 M \) denotes that \( N \) is a direct summand of \( M \).

2. Preliminaries

Definition 1 Let \( M \) be an \( R \)-module and \( N \leq M \).

1. \( N \) is called essential submodule of \( M \) (\( N \leq_e M \), for short) if, \( N \cap K \neq \{0\} \) for any nonzero submodule \( K \) of \( M \).

2. \( N \) is called small submodule of \( M \) (\( N \ll_e M \), for short) if, for any submodule \( L \) of \( M \), \( N + L = M \) implies \( L = M \)

3. \( N \) is called e-small \( N \ll_e M \), (\( N \ll_e M \), for short) if, for any essential submodule \( L \) of \( M \), \( N + L = M \) implies \( L = M \).

Remark 1 Each small submodule is e-small submodule. But the converse is not true in general for example: \( N = \{0, 3\} \) is a submodule of \( \mathbb{Z}/6\mathbb{Z} \) as a \( \mathbb{Z} \)-module. \( N \) is e-small but \( N \) is not small.

Lemma 1 (Zhou, D. X. & all, Proposition 2.5)

1. Let \( N, K \) and \( L \) are submodules of an \( R \)-module \( M \) such that \( N \subseteq K \), if \( K \ll_e M \), then \( N \ll_e M \) and \( K/N \ll_e M/N \).

2. If \( K \ll_e M \) and \( f : M \rightarrow M' \) is a homomorphism, then \( f(K) \ll_e M' \). In particular, if \( K \ll_e M \subseteq M' \), then \( K \ll_e M' \).

3. Assume that \( K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M \) and \( M = M_1 \oplus M_2 \), then \( K_1 \oplus K_2 \ll_e M_1 \oplus M_2 \) if and only \( K_1 \ll_e M_1 \) and \( K_2 \ll_e M_2 \).

Lemma 2 (Aidi, S. H. & all, (2015), Lemma 2.8)

Let \( M \) be an \( R \)-module, let \( K \leq N \leq M \) be submodules of \( M \). If \( K \ll_e M \) and \( N \leq M \), then \( K \ll_e N \).
3. Some Properties Related to E-small Quasi-Dedekind Modules

Definition 2 An R-module M is called e-small quasi-Dedekind if for all f ∈ EndR(M), f ≠ 0 implies Ker f ≪e M.

Example 1 Every semi-simple module is e-small quasi-Dedekind.

Remark 2 It is clear that every quasi-Dedekind R-module is an e-small quasi-Dedekind R-module. But the converse is not true in general, for example $\mathbb{Z}$ as a $\mathbb{Z}$-module is e-small quasi-Dedekind but it is not quasi-Dedekind.

Remark 3

1. The epimorphic image of e-small quasi-Dedekind module is not necessary e-small quasi-Dedekind; for example $\mathbb{Z}$ as a $\mathbb{Z}$-module is e-small quasi-Dedekind. Let $\pi : \mathbb{Z} → \mathbb{Z}/12\mathbb{Z}$, where $\pi$ is the natural projection. $\mathbb{Z}/12\mathbb{Z}$ as a $\mathbb{Z}$-module is not e-small quasi-Dedekind.

2. The direct sum of e-small quasi-Dedekind modules is not necessarily an e-small quasi-Dedekind module; for example each of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as $\mathbb{Z}$-module is e-small quasi-Dedekind. But $\mathbb{Z}/4\mathbb{Z} ⊕ \mathbb{Z}/3\mathbb{Z} ≅ \mathbb{Z}/12\mathbb{Z}$ is not an e-small quasi-Dedekind $\mathbb{Z}$-module, since $\mathbb{Z}/12\mathbb{Z}$ is not e-small quasi-Dedekind.

Proposition 1 Let $M_1, M_2$ be R-modules such that $M_1 \cong M_2$. Then $M_1$ is an e-small quasi-Dedekind R-module if and only if $M_2$ is an e-small quasi-Dedekind R-module.

Proof. $\Rightarrow$ Let $f ∈ End_R(M_2), f \neq 0$. Since $M_1 \cong M_2$, there exists an isomorphism $g : M_1 → M_2$ and $g^{-1} : M_2 → M_1$. Let $h = g^{-1} ∘ f ∘ g ∈ End_R(M_1)$. It is clear that $h \neq 0$ and so $g(\ker h) ≪_e M_2$ by lemma 1. On the other hand we have $g(\ker h) = \ker f$. Hence $\ker f ≪_e M_2$. So $M$ is e-small quasi-Dedekind.

$\Leftarrow$ The proof of the converse is similarly.

Proposition 2 Every direct summand of an e-small quasi-Dedekind module is an e-small quasi-Dedekind module.

Proof. Let $M = N ⊕ K$ such that $M$ is an e-small quasi-Dedekind R-module.

Let $f : K → K, f \neq 0$. We have $h = i ∘ f ∘ p ∈ End_R(M), h \neq 0$, where $p$ is the natural projection and $i$ is the inclusion mapping. Hence $\ker h ≪_e M$. But $\ker f ≪_e \ker h$, so by lemma 1, $\ker f ≪_e M$. On the other hand $\ker f ≤ K$ implies $\ker f ≪_e K$ by lemma 2. Thus $K$ is an e-small quasi-Dedekind R-module.

Proposition 3 Let $M$ be an R-module and let $N, L ≤ M$ with $M = N + L$ and $M/N ⊕ N$ e-small quasi-Dedekind. Then $M/N$ and $M/L$ are e-small quasi-Dedekind R-modules.

Proof. Since $M/N ⊕ N$, by definition 1, $M/N$ is e-small quasi-Dedekind. But $N/L ≤ M/N$ and $L/N ⊕ N$. Then there exists $\phi : M/N → M/N$ such that $\phi ≫_e M$. Then by lemma 1, $\ker \phi ≪_e M$ and so $M$ is e-small quasi-Dedekind.

Definition 3 An R-module M is called N-e-small quasi-Dedekind if, for every $0 ≠ φ ∈ \text{Hom}_R(M, N), \ker φ ≪_e M$.

In view of the above definition, an R-module M is e-small quasi-Dedekind if and only if $M$ is $M$-e-small quasi-Dedekind.

Theorem 1 Let $M_1$ and $M_2$ be two R-modules and let $M = M_1 ⊕ M_2$. If $M$ is e-small quasi-Dedekind, then $M_1$ is $M_1$-e-small quasi-Dedekind for all $i, j = 1, 2$.

Proof. Suppose that $M = M_1 ⊕ M_2$ is e-small quasi-Dedekind. Then, by proposition 2, $M_1$ and $M_2$ are e-small quasi-Dedekind. Thus $M_1$ is $M_1$-e-small quasi-Dedekind and $M_2$ is $M_2$-e-small quasi-Dedekind. Let $0 ≠ f ∈ \text{Hom}_R(M_1, M_2)$. Then $h = i ∘ f ∘ p ∈ \text{End}_R(M)$, where $p$ is the natural projection and $i$ is the inclusion mapping. It is clear that $h ≠ 0$. So $\ker h ≪_e M_1 ⊕ M_2$, because $M = M_1 ⊕ M_2$ is e-small quasi-Dedekind. On the other hand, we may assume that $\ker f ≫_e [0] ≤ \ker h$. Thus $\ker f ≫_e M_1 ⊕ M_2$. Then by lemma 1, $\ker f ≪_e M_1$ and so $M_1$ is $M_1$-e-small quasi-Dedekind.

Theorem 2 Let $M$ be an R-module. Then $M$ is e-small quasi-Dedekind if and only if $\text{Hom}(M/N, M) = [0]$, for all $N ≪_e M$.

Proof. $\Rightarrow$ Suppose on the contrary that there exists $N ≪_e M$ such that $\text{Hom}(M/N, M) ≠ [0]$. Then there exists $\phi : M/N → M, \phi ≠ 0$. Hence $\phi ∘ π ∈ \text{End}_R(M)$, where $π$ is the canonical projection. It is clear that $\phi ∘ π ≠ 0$ and so $\ker(\phi ∘ π) ≪_e M$. Since $N ⊆ \ker(\phi ∘ π), N ≪_e M$ by lemma 1. This is a contradiction.

$\Leftarrow$ Suppose on the contrary that there exists $f ∈ \text{End}_R(M), f ≠ 0$ such that $\ker f ≪_e M$. Define $g : M/\ker f → M$ by $g(m + \ker f) = f(m)$ for all $m ∈ M$. It is clear that $g ≠ 0$. So $\text{Hom}(M/\ker f, M) ≠ [0]$ which is a contradiction.

Remark 4 If $M$ is an e-small quasi-Dedekind R-module, and $N ≤ M$. Then it is not necessary that $M/N$ is an e-small quasi-Dedekind R-module; for example the $\mathbb{Z}$-module $M = \mathbb{Z}$ is e-small quasi-Dedekind. Let $N = 12\mathbb{Z} ≤ \mathbb{Z}$, then
$M/N = \mathbb{Z}/12\mathbb{Z}$ is not an e-small quasi-Dedekind R-module.

**Definition 4** Let M be an R-module, put $Z(M) = \{m \in M : \text{ann}_R(m) \leq \leq R\}$. M is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. M is called Goldie torsion if $Z(M)$.

**Proposition 4** Let M be an e-small quasi-Dedekind R-module such that $M/U$ is semi-simple nonsingular for all $U \not\subseteq M$. Then $M/N$ is an e-small quasi-Dedekind R-module, for all $N \leq M$.

Proof. Let $K/N \not\subseteq M/N$. So by lemma 1, $K \not\subseteq M$. Suppose that $\text{Hom}((M/N)/(K/N), M/N) \neq \{0\}$. Since $\text{Hom}((M/N)/(K/N), (M/N) \equiv \text{Hom}(M/K, M/N)$, there exists $f : M/K \rightarrow M/N$ such that $f \neq 0$. Since $M/K$ is semi-simple nonsingular, so by (Lam T.Y(1999), Exer. 12A.), there exists $g : M/K \rightarrow M$ such that $\pi \circ g = f$, where $\pi$ is the canonical projection. Hence $\pi \circ g(M/K) = f(M/K) \neq 0$, so $g \neq 0$. But $g \in \text{Hom}(M/K, M)$ and $K \not\subseteq M$. Thus $\text{Hom}(M/K, M) \neq \{0\}$. Finally $K$ and $K \not\subseteq M$, which is a contradiction. Thus $M/N$ is an e-small quasi-Dedekind R-module.

**Proposition 5** Let M be an R-module. The following statements are equivalent:

1. $M$ is e-small quasi-Dedekind.

2. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N = \text{Im}g$ for some $g \in \text{End}_R(M)$.

3. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $\text{Ker}f + N = M$, then there exists a unique complete set $(g_i)$ of orthogonal idempotents in $\text{End}_R(M)$ and $N_1 \leq M$ with $N = Mg_1$ and $N_1 = Mg_1$.

Proof. 1) $\Rightarrow$ 2) Suppose that $M$ is $\delta$-small quasi-Dedekind. Let $0 \neq f \in \text{End}_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. For any complement $L = N$ in $M$, we have $N \leq L \leq M$. It is clear that $N + L + \text{Ker}f = M$. So $N \leq L = M$, because $M$ is e-small quasi-Dedekind. So there exists $g^2 = g \in \text{End}_R(M)$ with $N = \text{Im}g$.

2) $\Rightarrow$ 3) Let $0 \neq f \in \text{End}_R(M)$. Suppose that there exists $N \leq M$ such that $\text{Ker}f + N = M$. Then $f(N) = f(M)$, as $f(\text{Ker}f) = 0$. So, by 2) and (Anderson, F.W., & all (1973), Corollary 5.8 and Corollary 6.20), the result is obtained.

3) $\Rightarrow$ 1) Let $0 \neq f \in \text{End}_R(M)$. Let $\text{Ker}f + N = M$ where $N \leq M$. By 3) and (Anderson, F.W., & all (1973), Corollary 6.20), $N \leq M$. Thus $M = N$ and so $M$ is e-small quasi-Dedekind.

Recall that a submodule $N$ of an R-module $M$ is called fully invariant if $f(N) \subseteq N$ for any $f \in \text{End}_R(M)$. An idempotent $e$ in a ring $R$ is called left semicentral if $xe = exe$ for each $x \in R$.

$N$ is a closed submodule of $M$ if $N$ has no proper essential extension inside $M$.

**Proposition 6** Let $M$ be an R-module. Then the following conditions are equivalent:

1. $M$ is e-small quasi-Dedekind.

2. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N$ is closed in $M$.

Proof. 1) $\Rightarrow$ 2) By proposition 5, $N \leq M$. So $N$ is closed.

2) $\Rightarrow$ 1) Let $0 \neq f \in \text{End}_R(M)$. Let $\text{Ker}f + N = M$ where $N \leq M$. Then $f(N) = f(M)$. By 2) $N$ is closed in $M$. Thus $M = N$ and so $M$ is e-small quasi-Dedekind.

**Proposition 7** Let $M$ be an R-module such that for any $N \leq M$, $Z_2(M) \leq N$. Then the following statements are equivalent:

1. $M$ is e-small quasi-Dedekind.

2. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $\text{Ker}f + N = M$, then $M/N$ is nonsingular.

3. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $\text{Ker}f + N = M$, then $N$ is closed in $M$.

Proof. 1) $\Rightarrow$ 2) Let $0 \neq f \in \text{End}_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. By proposition 6, $N$ is closed in $M$. By our assumption,
Z_2(M) \subseteq N. So by (Asgari, S. & all(2012), Proposition 1.2), M/N is nonsingular.  
2) \Rightarrow 3). Let 0 \neq f \in \text{End}_R(M). Suppose that there exists N \leq M with Ker f + N = M. Then f(N) = f(M). By 2), M/N is nonsingular. Hence by (Asgari, S. & all(2012), Proposition 1.2), N is closed in M.  
3) \Rightarrow 1) Follows from proposition 6.

**Proposition 8** Let M be an e-small quasi-Dedekind R-module such that for any nonzero f \in \text{End}_R(M), there exists N \leq M with f(N) = f(M). Then the following conditions are equivalent:

1. For any fully invariant submodule L \leq M, N + L is fully invariant in M if and only if g_1 \text{End}(M)N \subseteq L for some g_1^2 = g_1 \in \text{End}_R(M).

2. N is fully invariant in M if only if there exists g^2 = g \in \text{End}_R(M) with g semicentral.

3. There exists N' \leq M such that N is a fully invariant submodule of M if and only if Hom_R(N, N') = \{0\}.

4. There exists an epimorphism g \in Hom_R(M, N) and a monomorphism h \in Hom_R(N, M) such that M = \text{Ker}g \oplus \text{Im}h.

5. N = eE(M) \cap M for some e^2 = e \in \text{End}_R(E(M)).

**Proof.** Since M is e-small quasi-Dedekind, by proposition 5, N \subseteq M. Thus N = gM for some g^2 = g \in \text{End}_R(M). It is clear that g_1 = (1 - g) is an idempotent in \text{End}_R(M).

1. \Rightarrow) Let x \in g_1 \text{End}(M)g and m \in M. Then xm = xgm \in gM + L = N + L. On the other hand we have xm = g_1 xm \in g_1L \leq L.  
   (=) Let s \in \text{End}_R(M). Then s(gM) = (gs + g_1s)gM \subseteq gM + g_1 \text{End}_R(M)gM \subseteq gM + L.

2. \Rightarrow) Let s \in \text{End}_R(M) and m \in M. By 1), N = gM for some g^2 = g \in \text{End}_R(M). Then sgm = gm' for some m' \in M. It follows that sgm = g^2m' = gm' = smg. Hence g is semicentral.  
   (=) Let s \in \text{End}_R(M) and m \in M. Thus sgm = sgm \in gM and so N is fully invariant in M.

3. Since N \subseteq M, there exists N' \leq M with N = N \oplus N'. In this case, it is is well know that N is a fully invariant submodule of M if and only if Hom_R(N, N') = \{0\}.

4. Since N \subseteq M, there exists an epimorphism g \in Hom_R(M, N) and a homomorphism h \in Hom_R(N, M) such that g \circ h = 1_M. It follows that h is a monomorphism. It is clear that hg is an idempotent and so M = \text{Ker}g \oplus \text{Im}h.

5. Since N \subseteq M, N is a closed submodule of M. Since N \subseteq M \subseteq E(M), E(M) contains a copy of E(N). Thus N \subseteq E(N) \cap M implies that N = E(N) \cap M. Since E(N) is injective, E(N) = eE(M) for some e^2 = e \in \text{End}_R(E(M)). So the result is obtained.

**Proposition 9** Let M be an e-small quasi-Dedekind R-module such that for any nonzero f \in \text{End}_R(M), there exists N \leq M with f(N) = f(M). Let L be a fully invariant submodule of M such that L \leq N. Then the following assertions are verified:

1. g_1 \text{End}_R(M)N \subseteq Z(M) for some g_1^2 = g_1 \in \text{End}_R(M).

2. N + Z(M) is fully invariant in M.

3. If Z(M) \subseteq N, then N is fully invariant. Moreover, if L \leq K, then K \subseteq N. In particular, Z_2(M) \subseteq N.

**Proof.**

1. We have N = gM for some g^2 = g \in \text{End}_R(M). Then g_1 = (1 - g) is an idempotent in \text{End}_R(M). Let m \in M. Then gml \subseteq L for some I \subseteq R. Thus g_1 \text{End}_R(M)gml \subseteq N \cap g_1 M = \{0\}. It follows that g_1 \text{End}_R(M)N \subseteq Z(M).

2. Result from (1) and proposition 8 (1).

3. By (2), N is a fully invariant submodule of M. Let k \in K. Thus kl \subseteq L for some I \subseteq R. So g_1k \in Z(M) \subseteq N. On the other hand k = gk + g_1k \in N. Thus K \subseteq N.

**Proposition 10** Let M be a nonsingular R-module. Then the following conditions are equivalent:
1. $M$ is e-small quasi-Dedekind.

2. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N \leq E M$.

3. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $\text{Ker}(f + N) = M$ and $((N + Z_2(M))/Z_2(M)) \leq_e M/Z_2(M)$, then $M = N$.

**Proof.**

1) $\Rightarrow$ 2) Let $0 \neq f \in \text{End}_R(M)$. Suppose that there exists $N \leq M$ such that $f(N) = f(M)$. Then the result follows directly from proposition 5.

2) $\Rightarrow$ 3) Let $0 \neq f \in \text{End}_R(M)$. Suppose that there exists $N \leq M$ with $\text{Ker}(f + N) = M$. Then $f(N) = f(M)$ and by 2), there exists $L \leq M$ such that $M = N \oplus L$. By hypothesis, $((N + Z_2(M))/Z_2(M)) \leq_e M/Z_2(M)$. Thus by (Asgari, S. & all(2012), Proposition 1.1), $M/N$ is Goldie torsion. On the other hand $M = N \oplus L$ implies that $M/N \cong L$ is Goldie torsion. It follows that $L = \{0\}$. This implies that $M = N$.

3) $\Rightarrow$ 1) Let $M$ an $R$-module and a nonzero $f \in \text{End}_R(M)$ such that $\text{Ker}(f + N) = M$, where $N \leq_e M$. We have $M/N$ is singular and so Goldie torsion. Thus by 3) and (Asgari, S. & all(2012), Proposition 1.1), $M = N$. So $M$ is e-small quasi-Dedekind.

**Definition 5**

1. An $R$-module $M$ is called prime if $\text{Ann}_R(M) = \text{Ann}_R(N)$ for each $0 \neq N \leq M$.

2. An $R$-module $M$ is called faithful if $\text{Ann}_R(M) = \{0\}$.

**Proposition 11** Let $M$ be a prime faithful $R$-module. Then the following conditions are equivalent:

1. $M$ is e-small quasi-Dedekind.

2. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $f(N) = f(M)$, then $N \leq E M$.

3. For any nonzero $f \in \text{End}_R(M)$, if there exists $N \leq M$ such that $\text{Ker}(f + N) = M$ and $(N + Z_2(M))/Z_2(M) \leq_e M/Z_2(M)$, then $M = N$.

**Proof.** Suppose that $M$ is prime. Then $\text{Ann}_R(M)$ is a prime ideal of $R$. Also $M$ is a torsion-free module $\overline{R}$-module, where $\overline{R} = R/\text{Ann}_R(M)$. We have $\overline{R} = R/\overline{\text{Ann}_R(M)} \cong R$, because $M$ is faithful. Hence $M$ is a torsion-free module over an integral domain. Thus by (Lam, T. Y. (1999), P.247), $M$ is nonsingular. Thus the result follows from proposition 10 and (Asgari, S. & all(2012), Proposition 1.1).

**Remark 5** Let $N \leq M$ and $f \in \text{End}_R(M), f \neq 0$. If $f(N) \leq_e f(M)$, then it is not necessarily that $N \leq_e M$ for example: let $\mathbb{Z}/12\mathbb{Z}$ as $\mathbb{Z}$-module, and let $N =< 3 > \leq \mathbb{Z}/12\mathbb{Z}$. Let $f = 4\overline{x} \in \text{End}_\mathbb{Z}(\mathbb{Z}/12\mathbb{Z})$. It is clear that $f \neq 0$ and $f(N) = f(< 3 >) = \{0\} \leq_e f(\mathbb{Z}/12\mathbb{Z}) = f(M)$, but $< 3 > \not\leq_e \mathbb{Z}/12\mathbb{Z}$.

**Proposition 12** Let $M$ be an e-small quasi-Dedekind nonsingular $R$-module. Let a nonzero $f \in \text{End}_R(M)$ such that for each $N \leq M$, $f(M)/f(N)$ is singular. If $f(N) \leq_e f(M)$, then $N \leq_e M$.

**Proof.** Let $N + K = M$ where $K \leq_e M$. Then $f(N) + f(K) = f(M)$. Since $M$ is nonsingular, $f(M)$ is nonsingular. By hypothesis, $f(M)/f(K)$ is singular, so $f(K) \leq_e f(M)$. Since $f(N) \leq_e f(M)$, $f(K) = f(M)$. It follows that $M = \text{Ker}(f + K)$. Thus $K = M$ and so $N \leq_e M$.

**Corollary 1** Let $M$ be an e-small quasi-Dedekind nonsingular $R$-module and a nonzero surjective $f \in \text{End}_R(M)$. Suppose that for each $N \leq M$, $M/f(N)$ is singular. Then $N \leq_e M$ if and only if $f(N) \leq_e M$.

**Proof.** Suppose that $N \leq_e M$. Then by lemma 1, $f(N) \leq_e M$. The converse follows directly from proposition 12.

**Definition 6**

1. A submodule $N$ of an $R$-module $M$ is called $\delta$-small ($N \leq_{\delta} M$, for short) if whenever $N + L = M$ and $M/L$ is singular then $L = M$.

2. An $R$-module $M$ is called $\delta$-Hollow if every proper submodule of $M$ is $\delta$-small in $M$.
3. A pair \((P, f)\) is a \(\delta\)-projective cover of an \(R\)-module \(M\), if \(P\) is a projective \(R\)-module and \(f : P \to M\) is an epimorphism and \(\text{Ker} f \triangleleft_{\delta} P\).

**Remark 6**

1. Every \(\delta\)-small submodule is e-small but not conversely (see (Zhou, D. X. & all (2000), P. 1052).

2. Obviously, every \(\delta\)-hollow is e-small quasi-Dedekind but not conversely; for example \(\mathbb{Q}\) as \(\mathbb{Z}\)-module is e-small quasi-Dedekind, but it is not \(\delta\)-hollow.

**Proposition 13** Let \((P, f)\) is a \(\delta\)-projective cover of an \(R\)-module \(M\) such that \(P\) is \(\delta\)-Hollow. Then \(M\) is e-small quasi-Dedekind.

**Proof.** We have \(f : P \to M\) is an epimorphism. Let \(g \in \text{End}_R(M)\) such that \(g \not= 0\). \(f^{-1}(\text{Ker} g)\) is a proper submodule of \(P\). Suppose that \(P\) is \(\delta\)-Hollow, then \(f^{-1}(\text{Ker} g) \triangleleft_{\delta} P\) and by (Zhou, Y.Q. (2000), Lemma 1.3), \(f(f^{-1}(\text{Ker} g)) \triangleleft_{\delta} M\). But \(f(f^{-1}(\text{Ker} g)) = \text{Ker} f\), then \(\text{Ker} f \triangleleft_{e} M\). Hence \(M\) is e-small quasi-Dedekind.

**Definition 7** A ring \(R\) is called \(\delta\)-semiperfect if every simple \(R\)-module has projective \(\delta\)-cover.

Recall that a module is \(\delta\)-semiperfect if for any endomorphism \(f \in \text{End}_R(M)\), \(f(M) \trianglelefteq_{e} M\).

**Proposition 14** Let \(R\) be an artinian principal ideal ring such that every projective \(R\)-module is \(\delta\)-Hollow. Then every \(\delta\)-small quasi-Dedekind.

**Proof.** Let \(M\) be a \(\delta\)-small quasi-Dedekind \(R\)-module. Since \(R\) is an artinian principal ideal ring, then by (Barry, M., & all (2010), Theorem 3.8), \(M\) is finitely generated. Thus by (Zhou, Y.Q. (2000), Theorem 3.6), \(M\) has a projective \(\delta\)-cover (\(P, f\)) because \(R\) is \(\delta\)-semiperfect. By hypothesis, \(P\) is \(\delta\)-Hollow. So by proposition 13, \(M\) is e-small quasi-Dedekind.

**Definition 8** An \(R\)-module \(M\) is called coretractable if for any proper submodule \(N\) of \(M\), there exists a nonzero homomorphism \(f : M \to M\) with \(f(N) = \{0\}\), that is \(\text{Hom}_R(M/N, M) \not= \{0\}\).

**Proposition 15** Let \(M\) be a coretractable \(R\)-module such that for any \(0 \not= f \in E, \text{ann}_E(\text{Ker} f) \leq_{e} E\) where \(E = \text{End}_R(M)\). Then \(M\) is e-small quasi-Dedekind.

**Proof.** Let \(f \in E\) such that \(f \not= 0\). Let \(K\) be a proper essential submodule \(K\) of \(M\). There exists \(0 \not= g \in E\) with \(g(K) = \{0\}\). Since \(\text{ann}_E(\text{Ker} f) \leq_{e} E\), there exists \(h \in E\) such that \(0 \not= hg \in \text{ann}_E(\text{Ker} f)\). Therefore, \(hg(\text{Ker} f + L) = \{0\}\) and hence \(\text{Ker} f + K \not= M\). Thus \(\text{Ker} f \leq_{e} M\) and so \(M\) is e-small quasi-Dedekind.

**Proposition 16** Let \(M\) be a nonzero coretractable \(R\)-module. If \(E\) is uniform as \(E\)-module, then \(M\) is e-small quasi-Dedekind.

**Proof.** Let \(f \in E\) such that \(f \not= 0\). Suppose that \(E\) is uniform. Since \(\text{ann}_E(\text{Ker} f) \not= \{0\}\), it is an essential ideal of \(E\). By proposition 15, \(\text{Ker} f \leq_{e} M\) and so \(M\) is small quasi-Dedekind.

**Definition 9** Let \(R\) be a ring.

1. An element \(x \in R\) is left quasi-regular in case \(1-x\) has a left inverse in \(R\). Similarly \(x \in R\) is right quasi-regular in case \(1-x\) has a right inverse in \(R\).

2. An ideal \(I\) of \(R\) is left quasi-regular in case each element of \(I\) is left quasi-regular.

**Proposition 17** Let \(R\) be a ring such that every proper ideal of \(R\) is quasi-regular. Then \(R\) is an e-small quasi-Dedekind \(R\)-module.

**Proof.** Let \(f \in \text{End}(R)\) such that \(f \not= 0\). Let \(R = \text{Ker} f + J\), with \(J \leq_{e} R\). Then there exists \(x \in \text{Ker} f\) and \(j \in J\) with \(1=x+j\). So \(j=1-x\) is invertible whence \(1 \in J\) and \(J = R\). Thus \(\text{Ker} f \leq_{e} R\) and so \(R\) is e-small quasi-Dedekind.

**Remark 7** Let \(R\) be a nonzero ring such that any ideal in \(R\) is free as an \(R\)-module. Then \(R\) is an e-small quasi-Dedekind ring.
References


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