New Types of Fuzzy Filter on Lattice Implication Algebras

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Abstract

Extending the belongs to (∈) relation and quasi-coincidence with (q) relation between fuzzy points and a fuzzy subsets, the concept of (α, β)-fuzzy filters and (α, β)-fuzzy filters of lattice implication algebras are introduced, where α,β ∈ {∈, ⊤, ⊥, α, β, α ∨ β}. Some equivalent characterizations of these generalized fuzzy filters are derived. Finally, the relations among these generalized fuzzy filters are investigated. Special attention to (α, β)-fuzzy filter and (α, β)-fuzzy filter are generalizations of (∈, ⊤)-fuzzy filter and (∈, ⊤)-fuzzy filter, respectively.

Keywords: Lattice implication algebras, Fuzzy filter, (α, β)-fuzzy filter, (α, β)-fuzzy filter

1. Introduction

Intelligent information processing is one important research direction in artificial intelligence. Information processing dealing with certain information is based on the classical logic. However, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, etc. Therefore, non-classical logic and implication logic, a great extension and development of classical logic, have always been a crucial direction in non-classical logic. In the field of many-valued logic, lattice-valued logic plays an important role for the following two aspects: One is that it extends the chain-type truth-valued field of some well known present logic to some relatively general lattice. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of human being’s thinking, judging and decision. Hence, lattice-valued logic is becoming an active research field which strongly influences the development of algebraic logic, computer science and artificial intelligence technology. In order to provide a reliable logical foundation for uncertain information processing theory, especially for the fuzziness, the incomparability in uncertain information in the reasoning, and, establish a logical system with truth value in a relatively general lattice. Combining algebraic lattice and implication algebra, Xu (Xu, 1993) proposed the concept of lattice implication algebras (LIA for short) and discussed some of its properties. Since then this logical algebra has been extensively investigated by several researchers (Y.Xu,1993, etc). Recently, Jun et al(Y.B.Jun,2007) introduced the concept of (∈, ⊤)-fuzzy implicative filter of a lattice implication algebra. Zhan et al(J.M.Zhan,2009) further to investigated this kind of fuzzy complicative filters.

The concept of fuzzy set was introduced by Zadeh(Zadeh, 1965). Rosenfeld inspired the fuzzification of algebraic structure and introduced the notion of fuzzy subgroup (Rosenfeld, 1971). The idea of fuzzy point and ‘belongingness’ and ‘quasi-coincidence’ with a fuzzy set were given by Pu and Liu(P.M.Pu, 1980). A new type of fuzzy subgroup (viz (∈, ⊤)-fuzzy subgroup) was introduced(S.K.Bhakat,1996). In fact, (∈, ⊤)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. The idea of fuzzy point and ‘belongingness’ and ‘quasi-coincidence’ with a fuzzy set have been applied some important algebraic system(S.K.Bhakat,1999, etc).

This paper, as a continuation of (Y.B.Jun,2007, J.M.Zhan,2009), we extend the concept of quasi-coincidence and further to investigate the (∈, ⊤)-fuzzy filters, proposing the concept of (∈, ⊤)-fuzzy filter and (∈, ⊤)-fuzzy filter. We investigate relations between (∈, ⊤)-fuzzy filters, (∈, ⊤)-fuzzy filters (∈, ⊤)-fuzzy filters, filters of . We establish characterizations of (∈, ⊤)-fuzzy filters and give some equivalent conditions of (∈, ⊤)-fuzzy filters. Of course, we can discuss (α, β)-fuzzy (implicative, ultra-, associative) filter in the same way. It will be of great use to provide theoretical foundation to design intelligent information processing systems.

In this paper, denote as a lattice implication algebra (L, ∨, ∧, →, O, I).
2. Preliminaries

Let $(L, \vee, \wedge, O, I)$ be a bounded lattice with an order-reversing involution $'$, the greatest element $I$ and the smallest element $O$, and $\rightarrow: L \times L \rightarrow L$ be a mapping. $\mathcal{L} = (L, \vee, \wedge, ', \rightarrow, O, I)$ is called a lattice implication algebra if the following conditions hold for any $x, y, z \in L$:

- $(I_1) \ x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
- $(I_2) \ x \rightarrow x = I$.
- $(I_3) \ x \rightarrow y = y' \rightarrow x'$.
- $(I_4) \ x \rightarrow y = y \rightarrow x = I$ implies $x = y$.
- $(I_5) \ (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.
- $(I_6) \ (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z)$.
- $(I_7) \ (x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z)$.

In this paper, denote $\mathcal{L}$ as a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, O, I)$.

We list some basic properties of lattice implication algebras. It is useful to develop these topics in other sections.

Let $\mathcal{L}$ be a lattice implication algebra. Then for any $x, y, z \in L$, the following conclusions hold:

1. if $I \rightarrow x = I$, then $x = I$.
2. $I \rightarrow x = x$ and $x \rightarrow O = x'$.
3. $O \rightarrow x = I$ and $x \rightarrow I = I$.
4. $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = I$.
5. $z \leq y$ if and only if $x \rightarrow z \leq x \rightarrow (x \rightarrow z)$.

A non-empty subset $F$ of a lattice implication algebra $\mathcal{L}$ is called a filter of $\mathcal{L}$ if it satisfies

- $(F1) \ I \in F$.
- $(F2) \ (\forall x \in F)(\forall y \in L)(x \rightarrow y \in F \Rightarrow y \in F)$.

A fuzzy subset of a nonempty set $X$ is defined as a mapping from $X$ to $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers.

A fuzzy subset $A$ of $\mathcal{L}$ is said to be a fuzzy filter if

- $(F3) (\forall x \in L)(A(I) \geq A(x))$.
- $(F4) (\forall x, y \in L)(A(y) \geq \min(A(x), A(x \rightarrow y)))$.

A level set of a fuzzy set $A$ in $\mathcal{L}$ is the set $U(A; \alpha) := \{x \in L | A(x) \geq \alpha\}, \alpha \in [0, 1]$. A fuzzy set $A$ of a lattice implication algebra $\mathcal{L}$ of the form

$$A(y) = \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. A fuzzy point $x_t$ is said to belong to (resp. be quasi-coincident with) a fuzzy set $A$, written as $x_t \in A$ (resp. $x_t q A$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_t \in A$ or (resp. and) $x_t q A$, then we write $x_t \in q A$. The symbol $\in \bigcup q$ means $\in \bigcup q$ doesn’t hold. Using the notion of ‘ belongingness ($\in$)’ and ‘quasi-coincidence ($q$)’ of fuzzy point with fuzzy subsets, the concept of $(\alpha, \beta)$-fuzzy subsemigroup, where $\alpha$ and $\beta$ are any one of $[e, e \in \bigcup q, e \in \wedge q]$, was introduced in (S.K.Bhakat, 1996). It is worthy to note that the most viable generalization of Rosenfeld’s fuzzy subgroup is the notion of $(e, e \in \bigcup q)$-fuzzy subgroup. The detailed research with $(e, e \in \bigcup q)$-fuzzy subgroup has been considered in (S.K.Bhakat, 1996).
3. $(\alpha, \beta)$-fuzzy filters

In this section, we first extend to the concept of quasi-coincidence. In what follows, we let $h, \delta \in [0, 1]$ be such that $h < \delta$ and $r \in (h, 1]$. For a fuzzy point $x_r$ and a fuzzy subset $A$ on $\mathcal{L}$, we say that

1. $x_r \in_h A$ if $A(x) \geq r > h$. 
2. $x_r \in_{\alpha} A$ if $A(x) + r > 2\delta$. 
3. $x_r \in_h \vee_{\alpha} A$ if $x_r \in_h A$ or $x_r \in_{\alpha} A$. 
4. $x_r \in_{\alpha} \wedge_{\alpha} A$ if $x_r \in_h A$ and $x_r \in_{\alpha} A$. 
5. $x_r \in_{\alpha} A$ doesn’t hold for $a \in \{\in_h, q_{\delta}, \in_h \vee_{\delta}, \in_h \wedge_{\delta}\}$.

**Definition 1** A fuzzy subset $A$ on $\mathcal{L}$ is called an $(\alpha, \beta)$-fuzzy filter, if it satisfies, for any $x, y \in L, t, r \in (h, 1]$ and $h < \delta$:

1. $x_r \in_{\alpha} A$ implies $I_{\beta} A$,  
2. if $x_r \in_{\alpha} A$ and $(x \rightarrow y)_{\alpha} A$, then $y_{\min(t, r)} \in_{\beta} A$, where $\alpha, \beta \in \{\in_h, q_{\delta}, \in_h \vee_{\delta}, \in_h \wedge_{\delta}\}$ but $a \notin \in_h \wedge_{\delta}$.

In Definition 1, the case $a = \in_h \wedge_{\delta}$ can be omitted. Since for a fuzzy subset $A$ such that $A(x) < \delta$ for any $x \in L$, and $x_r \in_{\alpha} A$, we have $A(x) \geq r > h$ and $A(x) + r > 2\delta$. Thus $2A(x) = A(x) + A(x) \geq A(x) + r > 2\delta$ and so $A(x) > \delta$. Hence $(x_r)_{x_r} \in_h \wedge_{\delta} A = \emptyset$. This explains why $\alpha, \in_h \wedge_{\delta}$ should be omitted in Definition 1.

It isn’t difficult to see, any $(\alpha, \beta)$-fuzzy filter on $\mathcal{L}$ must be an $(\alpha, \in_h \vee_{\delta})$-fuzzy filter on $\mathcal{L}$. Hence the $(\alpha, \in_h \vee_{\delta})$-fuzzy filter plays a central role in the theory of $(\alpha, \beta)$-fuzzy filter. So we only need to study the $(\alpha, \in_h \vee_{\delta})$-fuzzy filter.

In Definition 1, we taking $h = 0$ and $\delta = 0.5$, an $(\in_h, \in_h \vee_{\delta})$-fuzzy filters will be an $(e, \in \vee_{\delta})$-fuzzy filter. So $(\in_h, \in_h \vee_{\delta})$-fuzzy filter is a generalization of $(e, \in \vee_{\delta})$-fuzzy filter in (J.M.Zhan, 2009).

**Example 1** Let $L = \{0, a, b, c, d, I\}$, the Hasse diagram of $L$ be defined as Figure 1 and its implication operator $\rightarrow$ and negation operator $\neg$ be defined as Table 1. Then $\mathcal{L} = (L, \wedge, \vee, \rightarrow, O, I)$ is a lattice implication algebra.

1. (1) we define a fuzzy subset $A$ of $\mathcal{L}$
   
   $A(x) = \begin{cases} 
   0.7 & x \in \{I, b, c\}, \\
   0.2 & x \in \{O, d, a\}.
   \end{cases}$

   It is routine to verify that $A$ is an $(\in_{0.3}, \in_{0.3})$-fuzzy filter.

   2. (2) we define a fuzzy subset $B$ of $\mathcal{L}$
   
   $B(x) = \begin{cases} 
   0.6 & x = I, \\
   0.7 & x \in \{b, c\}, \\
   0.2 & x \in \{O, d, a\}.
   \end{cases}$

   It is routine to verify that $B$ is an $(\in_{0.3}, \in_{0.3} \vee_{\delta})$-fuzzy filter.

**Theorem 1** Let $h, \delta \in [0, 1], h < \delta$ and $A$ be a fuzzy subset of $\mathcal{L}$. Then $A$ is an $(\in_h, \in_h \vee_{\delta})$-fuzzy filter if and only $A$ satisfies following two conditions:

1. $(\forall x \in L)(\max(A(I), h) \geq \min(A(x), \delta))$, 
2. $(\forall x, y \in L)(\max(A(y), h) \geq \min(A(x), A(x \rightarrow y), \delta))$.

**Proof** $(F5) \Rightarrow (1)$ Assume there exist $x \in L$ such that $\max(A(I), h) < \min(A(x), \delta) = r$, then $A(x) \geq r > h$ and $r \leq \delta$, hence $x_r \in_h A$. It follows that $I_r$, $\in_h \vee_{\delta} A$ by $(F5)$, we have $A(I) \geq r > h$ or $A(I) + r > 2\delta$. Since $\max(A(I), h) < r$, it follows that $A(I) \geq r + r < 2r \leq 2\delta$, contradiction. Therefore, $(1)$ is valid.

$(1) \Rightarrow (F5)$ Assume $(1)$ holds and $(F5)$ doesn’t hold, then there exist $y \in L$ such that $y_r \in_h A$, but $(y_r \in_h \vee_{\delta} A$, that is $A(I) < r$ and $A(I) + r \leq 2\delta$, it follows that $A(I) < \delta$, hence $A(I) < \min(\delta, r)$. Since $A(y) \geq r > h$, we have

$\max(A(I), h) \geq \min(A(y), \delta) > \min(\delta, r)$.

It follows that $\min(\delta, r) > h$ for $\delta, r > h$, therefore $A(I) > \min(\delta, r)$, contradiction. Hence $(F5)$ holds.

$(F6) \Rightarrow (2)$ Assume that there exist $x, y \in L$ such that $\max(A(y), h) < \min(A(x), A(x \rightarrow y)) = r$, then $A(x) \geq r > h$, $A(x \rightarrow y) \geq r > h, \delta \geq r, A(y) < r$. Therefore $x_r \in_h A$ and $(x \rightarrow y)_r \in_h A$. It follows that $y_r \in_h \vee_{\delta} A$ by $(F6)$, that is $A(y) \geq r > h$ or $A(y) + r > 2\delta$. Since $A(y) < r$, we must have $A(y) + r < r + r = 2r \leq 2\delta$, which contradicts with $A(y) + r > 2\delta$. Hence $(2)$ holds.

$(2) \Rightarrow (F6)$ Assume that there exist $x, y \in L$ such that $x_r \in_h A, (x \rightarrow y)_r \in_h A$, but $y_{\min(t, r)} \in_h \vee_{\delta} A$, then $A(x) \geq r > h, A(x \rightarrow y) \geq r > h$ but $A(y) < \min(t, r)$ and $A(y) + \min(t, r) \leq 2\delta$. Hence $A(y) < \delta$. It follows that $A(y) < \min(\delta, t, r)$. We
have $\max[A(y), h] \geq \min[A(x), A(x \to y), \delta] \geq \min[t, r, \delta]$, it follows that $\min[t, r, \delta] > \max[A(y), h] \geq \min[A(x), A(x \to y), \delta] \geq \min[t, r, \delta]$, contradiction. Therefore (2) is valid.

**Remark** From Theorem 1, the Definition of $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter coincident with the concept of fuzzy filter with a threshold $(h, \delta)$ in a lattice implication algebra (J.M. Zhan 2009).

**Theorem 2** Let $A$ be an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$ and $1 + h = 2\delta$, then $U(A; h) = U(A; h^*)$ is a filter of $\mathcal{L}$, where $h < \delta$ and $U(A; h^*) = \{x \in L | A(x) > h\}$.

**Proof** Let $A$ be an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter. Since $x_{A(x)} \in A$, we have $I_{A(x)} e_h \lor q_{h} A$, that is $A(I) \geq A(x) > h$ or $A(I) + A(x) > 2\delta$. Hence $A(I) > h$ or $A(I) > 2\delta - A(x) > 2\delta - 1 = h$, we must have $I \in U(A; h^*)$.

Let $x, y \in U(A; h^*)$, then $A(x) > h$ and $A(x \to y) > h$. Since $A$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter, we have $\max[A(y), h] \geq \min[A(x), A(x \to y), \delta] \geq \min[\delta, \delta] = \delta$ for any $x, y \in L$ by Theorem 1. Hence $A(y) > h$, that is $y \in U(A; h^*)$. It follows that $U(A; h^*)$ is a filter of $\mathcal{L}$.

Let $S$ be a non-empty subset of $\mathcal{L}$, we define a fuzzy subset $A^h_S$ as follows: $A^h_S(x) \geq \delta$ if and only if $x \in S$, $A^h_S(x) = h$ if and only if $x \notin S$.

**Theorem 3** Let $S$ be a non-empty subset of $\mathcal{L}$ and $h, \delta \in [0, 1], h < \delta, 1 + h = 2\delta$. Then $S$ is a filter of $\mathcal{L}$ if and only if $A^h_S$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$.

**Proof** Let $S$ be a filter of $\mathcal{L}$ and $x \in S$, then $A^h_S(x) \geq t \geq h$. It follows that $A^h_S(x) \geq \delta$, then $x \in S$. Since $I \in S$, we have $A^h_S(I) \geq \delta > h$. If $I \leq \delta$, then $A^h_S(I) \geq \delta > t > h$, that is $I \in e_h A^h_S$. If $I > \delta$, then $A^h_S(I) + I > \delta + \delta = 2\delta$, that is $I \in q_{h} A^h_S$. Therefore, in any case, we have $I \in e_h \lor q_{h} A^h_S$.

Let $x, y \in S$, then $A^h_S(x) \geq t \geq h$, $A^h_S(x \to y) > h$. Hence $A^h_S(x) \geq \delta, A^h_S(x \to y) \geq \delta$, that is $x \in S, x \to y \in S$.

It follows that $y \in S$ for $S$ is filter of $\mathcal{L}$. That is $A^h_S(y) \geq \delta$. If $\min[t, r] \leq \delta$, then $A^h_S(y) \geq \delta \geq \min[t, r] > h$, it follows that $y_{\min[t, r]} \in e_h A^h_S$. If $\min[t, r] > \delta$, then $A^h_S(y) + \min[t, r] > h$, we have $y_{\min[t, r]} \lor q_{h} A^h_S$. Therefore, $A^h_S$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$.

Conversely, Assume that $A^h_S$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$. For any $x \in S$, we have $A^h_S(x) \geq \delta > h$, that is $x \in U(A^h_S; h^*)$, so $S \subseteq U(A^h_S; h^*)$. If $x \in U(A^h_S; h^*)$, then $A^h_S(x) > h$, that is $A^h_S \geq \delta$. It follows that $x \in S$. Hence $U(A^h_S; h^*) \subseteq S$. Therefore, $S$ is a filter by Theorem 2.

From the Proof of Theorem 3, it is easy to obtain the corollary 4.

**Corollary 4** Let $h, \delta \in [0, 1], h < \delta$ and $S$ be a filter of $\mathcal{L}$, then $A^h_S$ is an $(q_{h}, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$.

The following propositions are obvious, so the proofs are omitted.

**Proposition 5** Let $h, \delta \in [0, 1], h < \delta$ and $A$ be a fuzzy subset of $\mathcal{L}$. If $A$ is an $(e_h \lor q_{h}, e_h \lor q_{h})$-fuzzy filter of $\mathcal{L}$, then $A$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$.

The following Example show that the converse of Proposition 5 doesn’t hold in general.

**Example 2** In Example 1, we define a fuzzy subset $A$ of $\mathcal{L}$

$$A(x) = \begin{cases} 0.6 & x = I, \\ 0.7 & x \in [b, c], \\ 0.3 & x = d, \\ 0.2 & x \in \{O, a\}. \end{cases}$$

It is routine to verify that $A$ is an $(e_{0.3}, e_{0.3} \lor q_{0.6}\lor q_{0.6})$-fuzzy filter. But $A$ isn’t an $(e_{0.3}, e_{0.3} \lor q_{0.6})$-fuzzy filter, since $d_{0.92} e_{0.3} \lor q_{0.6} A$ and $(d \to a)_{0.7} e_{0.3} \lor q_{0.6} A$, but $a_{0.92} e_{0.3} \lor q_{0.6} A$.

Combining Theorem 1 and Proposition 5, we have the following corollary:

**Corollary 6** Any $(e_h \lor q_{h}, e_h \lor q_{h})$-fuzzy filter of $\mathcal{L}$ satisfies the following conditions:

1. $(\forall x \in L)\max[A(I), h] \geq \min[A(x), \delta]$.
2. $(\forall x, y \in L)\max[A(y), h] \geq \min[A(x), A(x \to y), \delta]$.

**Proposition 7** Let $h < \delta$ and $A$ be a fuzzy subset of $\mathcal{L}$. If $A$ is an $(e_h, e_h \lor q_{h}\lor q_{h})$-fuzzy filter of $\mathcal{L}$, then $A$ is an $(e_h, e_h \lor q_{h})$-fuzzy filter of $\mathcal{L}$.

The converse of Proposition 7 doesn’t hold in general. For example, $B$ is an $(e_{0.3}, e_{0.3} \lor q_{0.6})$-fuzzy filter. But $B$ isn’t an $(e_{0.3}, e_{0.3})$-fuzzy filter, since $b_{0.65} e_{0.3} B$, but $b_{0.65} e_{0.3} B$.

**Theorem 8** Let $A$ be a fuzzy subset of $\mathcal{L}$. Then $A$ is an $(e_h, e_h)$-fuzzy filter if and only if $A$ is a fuzzy filter of $\mathcal{L}$.

**Proof** Let $A$ be an $(e_h, e_h)$-fuzzy filter of $\mathcal{L}$. Since $x_{A(x)} \in A$ for any $x \in L$, we have $I_{A(x)} e_h A$, that is $A(I) \geq A(x)$. 

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Let $x_r \in h A$, and then $A(x) \geq t > h$. We have $max(A(x), h) \geq min[A(x), \delta] = A(x) \geq t > h$. Then $A(I) > h$. Hence $\forall r \in (h, \delta]$, $I_r \in \mathcal{L}$. Therefore $A(I) > h$. Therefore $A(I) = max[A(I), h] \geq min[A(x), \delta]$. There are two cases need to be discussed:

Case I: When $r \in (h, \delta]$, then $2\delta + \delta > r$. We have $A(I) \geq min[A(x), \delta] \geq min[r, \delta] = r > h$ or $A(I) \geq min[A(x), \delta] \geq min[\delta, 2\delta - r] = \delta > h$, then $I_r \in \mathcal{L}$. Hence $\exists y \in A(I)$, $y \in \mathcal{L}$. Therefore, we have $I_r \in \forall r \in (h, \delta]$. That is, $I \in \mathcal{L}$.

Let $x, x \rightarrow y \in \mathcal{L}$, $x \in h A$, and $(x \rightarrow y), y \in h A$. There are four cases need to be discussed:

(a) If $x_r \in h A$ and $(x \rightarrow y), y \in h A$, then $y_r \in \forall r \in (h, \delta]$. Hence $y \in \mathcal{L}$.

(b) If $x_r \in h A$ and $(x \rightarrow y), q_r \in \forall r \in (h, \delta]$, then $A(x) \geq r > h$ and $A(x \rightarrow y) + r > 2\delta$. It follows that $A(x) \geq r > h$ and $A(x \rightarrow y) > 2\delta - r$. Since $A$ is an $(e_h, e_h \forall r \in (h, \delta]$, we have $max[A(x), h] \geq min[A(x), \delta] > min[r, 2\delta - r, \delta]$. If $r \in (h, \delta]$, then $2\delta - r \geq \delta$. Therefore $max[A(y), h] \geq min[A(x), \delta] > min[r, 2\delta - r, \delta] > r$, that is $A(y) \geq r > h$, so $y_r \in h A$. Hence $y \in \forall r \in (h, \delta]$. Therefore $A(I) = max[A(x), h] \geq min[A(x), \delta] > min[r, 2\delta - r, \delta] = 2\delta - r > 2\delta - 1 = h$, so $A(y) > h$. Therefore $A(y) = max[A(x), h] \geq min[A(x), \delta] > min[r, 2\delta - r, \delta] = 2\delta - r$, so $A(y) > r$. It follows that $y \in \forall r \in (h, \delta]$. Therefore, we have $y \in \mathcal{L}$.

(3) If $x, y \in \forall r \in (h, \delta]$, $y \in h A$, then, similar with proof of (2), we can obtain $y \in \mathcal{L}$.

(4) If $x, y \in \forall r \in (h, \delta]$, then $A(x) \geq r > 2\delta$ and $A(x \rightarrow y) > 2\delta$. It follows that $A(x) \geq r > h$ and $A(x \rightarrow y) > 2\delta - r$. Since $A$ is an $(e_h, e_h \forall r \in (h, \delta]$, we have $max[A(y), h] \geq min[A(x), \delta] > min[2\delta - r, \delta]$. The remaining discussion is analogous to the (2). We have $y \in \mathcal{L}$.
Sum up above, \( \Sigma \) is a filter of \( \mathcal{L} \).

Conversely, Let \( \Sigma \) is a filter of \( \mathcal{L} \), then \( l \in \Sigma \), that is \( I, \varepsilon_h \land q_\beta A \).

Let \( x_r, \varepsilon_h A \) and \( (x \rightarrow y)_r, \varepsilon_h A \), that is \( A(x) \geq r > h \) and \( A(x \rightarrow y) \geq t > h \). We have \( A(x) \geq r \geq \min(t, r) > h \) and

\[
A(x \rightarrow y) \geq t \geq \min(t, r) > h,
\]

then \( x_{\min(t, r)} \in \varepsilon_h A \) and \((x \rightarrow y)_{\min(t, r)} \in \varepsilon_h A \). We can obtain \( x, x \rightarrow y \in \Sigma \). Since \( \Sigma \) is a filter, we have \( y \in \Sigma \), that is \( y_{\min(t, r)} \in \varepsilon_h \land q_\beta A \). Therefore \( A \) is an \((\varepsilon_h, \varepsilon_h \land q_\beta)\)-fuzzy filter of \( \mathcal{L} \).

**Corollary 11** Let \( A \) be fuzzy set of \( \mathcal{L} \).

1. \( A \) is an \((\varepsilon, \varepsilon \land q_\beta)\)-fuzzy filter if and only \( U(A; r)(\neq \emptyset) \) is a filter of \( \mathcal{L} \) for any \( r \in (0, 0.5] \), where \( U(A; r) = \{ x \in L | x_r \in A \} \).

2. \( A \) is an \((\varepsilon, \varepsilon \land q_\beta)\)-fuzzy filter if and only \( V(A; r)(\neq \emptyset) \) is a filter of \( \mathcal{L} \) for any \( r \in (0.5, 1] \), where \( V(A; r) = \{ x \in L | x_r \in q_\beta A \} \).

3. \( A \) is an \((\varepsilon, \varepsilon \land q_\beta)\)-fuzzy filter if and only \( \Sigma(A; r)(\neq \emptyset) \) is a filter of \( \mathcal{L} \) for any \( r \in [0, 1] \), where \( \Sigma(A; r) = \{ x \in L | x_r \in q_\beta A \} \).

**4. \((\overline{\alpha}, \underline{\beta})\)-fuzzy filters**

**Definition 2** A fuzzy subset \( A \) on \( \mathcal{L} \) is said to be an \((\overline{\alpha}, \underline{\beta})\)-fuzzy filter, if it satisfies, for any \( x, y \in L, t, r \in (h, 1] \) and \( h < \delta \):

(F7) \( I_1 \overline{\alpha} A \) implies \( x \underline{\beta} A \).

(F8) if \( y_{\min(t, r)} \in \overline{\alpha} A \), then \( x_{\overline{\alpha} \underline{\beta}} A \) or \((x \rightarrow y)_{\overline{\alpha} \underline{\beta}} A \), where \( \overline{\alpha}, \overline{\beta} \in [\overline{\varepsilon}_h, \overline{\varepsilon}_h, \underline{\varepsilon}_h \land \underline{q}_\beta, \overline{\varepsilon}_h \land \underline{q}_\beta] \) but \( \overline{\alpha} \neq \overline{\varepsilon}_h \land \underline{q}_\beta \).

In Definition 2, the case \( \overline{\alpha} = \overline{\varepsilon}_h \land \underline{q}_\beta \) can be omitted, the same reason with Definition 1.

**Example 3** In Example 1, we define a fuzzy set \( A \) as follows:

\[
A(O) = 0.4, A(I) = A(b) = A(c) = 0.9, A(a) = A(d) = 0.5.
\]

It is routine to verify \( A \) is an \((\overline{\varepsilon}_h, \overline{\varepsilon}_h \land \underline{q}_\beta)\)-fuzzy filter of \( \mathcal{L} \).

**Theorem 12** Let \( A \) be a fuzzy subset of \( \mathcal{L} \), then \( A \) is an \((\overline{\varepsilon}_h, \overline{\varepsilon}_h \land q_\beta)\)-fuzzy filter if and only if for any \( x, y \in L, t, r \in (h, 1] \) and \( h < \delta \),

1. \( \max(A(I), \delta) \geq A(x) \),

2. \( \max(A(y), \delta) \geq \min(A(x), A(x \rightarrow y)) \).

**Proof** Assume that (F7) hold and there exists \( x \in L \) such that \( \max(A(I), \delta) < A(x) = t \). Then \( t \in (\delta, 1] \) and \( I_1 \overline{\alpha} A \). It follows that \( x_{\overline{\alpha} \underline{\beta}} A \) from (F7). Hence \( A(x) < t \) or \( A(x) + t \leq 2\delta \), we have \( t \leq \delta \) for \( A(x) = t \), contradiction. Therefore, \( \max(A(I), \delta) \geq A(x) \), (1) is valid.

Assume that there exist \( x, y \in L \) such that \( \max(A(y), \delta) < \min(A(x), A(x \rightarrow y)) = t \), then \( A(y) < t \) and \( t \in (\delta, 1] \). It follows that \( y_{\min(t, r)} \in \overline{\alpha} A \). But \( x_r, \varepsilon_h A \) and \((x \rightarrow y)_r, \varepsilon_h A \). By (F8), we have \( x_{\overline{\alpha} \underline{\beta}} A \) or \((x \rightarrow y)_{\overline{\alpha} \underline{\beta}} A \). It follows that \( A(x) \geq t \) and \( A(x) + t \leq 2\delta \), we have that \( t \geq \delta \), contradiction. Therefore, (2) holds.

Conversely, assume that there exist \( x, y \in L, t, r \in (h, 1] \) such that \( I_1 \overline{\alpha} A \), but \( x_{\overline{\alpha} \underline{\beta}} A \), then \( A(I) < t, A(x) \geq t > h \) and \( A(x) + t \geq 2\delta \). Therefore, \( A(x) \geq \delta \). Thus \( \max(A(x), \delta) \leq \max(t, \delta) \leq \max(A(x), t) = A(x) \), contradiction. That is, \( I_1 \overline{\alpha} A \) implies \( x_{\overline{\alpha} \underline{\beta}} A \).

Let \( y_{\min(t, r)} \in \overline{\alpha} A \), then \( A(y) < t \). There are two cases to be discussed.

(a) If \( A(y) \geq \min(A(x), A(x \rightarrow y)) \), then \( \min(t, r) > \min(A(x), A(x \rightarrow y)) \). It follows that \( A(x) < t \) or \( A(x \rightarrow y) < t \), that is, \( x_{\overline{\alpha} \underline{\beta}} A \) or \((x \rightarrow y)_{\overline{\alpha} \underline{\beta}} A \). Of course, \( x_{\overline{\alpha} \underline{\beta}} A \) or \((x \rightarrow y)_{\overline{\alpha} \underline{\beta}} A \).

(b) If \( A(y) < \min(A(x), A(x \rightarrow y)) \), then \( \delta \geq \max(A(y), \delta) \geq \min(A(x), A(x \rightarrow y)) \). Assume that \( x_{\overline{\alpha} \underline{\beta}} A \) and \((x \rightarrow y)_{\overline{\alpha} \underline{\beta}} A \), then \( A(x) \geq r \) and \( A(x) + r > 2\delta, A(x \rightarrow y) \geq r \) and \( A(x \rightarrow y) + r > 2\delta \). It follows that \( A(x) > \delta \) and \( A(x \rightarrow y) > \delta \). Hence \( \min(A(x), A(x \rightarrow y)) > \delta \), which contradicts with \( \min(A(x), A(x \rightarrow y)) \leq \delta \). Therefore, \( x_{\overline{\alpha} \underline{\beta}} A \).

**Theorem 13** Let \( h < \delta \) and \( A \) be a fuzzy subset of \( \mathcal{L} \). Then \( U(A; \alpha), \alpha \in (\delta, 1] \) is a filter of \( \mathcal{L} \) if and only if \( A \) satisfies

1. \( (\forall x \in L)(\max(A(I), \delta)) \geq A(x) \),

2. \( (\forall x, y \in L)(\max(A(x), \delta)) \geq \min(A(x), A(x \rightarrow y)) \), where \( U(A; \alpha) = \{ x \in L | A(x) \geq \alpha \} \).

**Proof** Assume that \( U(A; \alpha), \alpha \in (\delta, 1] \) is a filter of \( \mathcal{L} \). If there exists \( a \in L \) such that \( \max(A(I), \delta) < A(a) \), then \( A(a) \in (\delta, 1] \).
and \( a \in U(A; A(a)) \), but \( A(I) < A(a) \), that is \( I \not\in U(A; A(a)) \), which contradicts with \( U(A; A(a)) \) is a filter. Hence (1) is valid.

Assume that there exist \( a, b \in L \) such that \( \max[A(b), \delta] < \min[A(a), A(a \rightarrow b)] = \beta \), then \( \beta \in (\delta, 1] \) and \( A(a) \geq \beta \), \( A(a \rightarrow b) \geq \beta \), that is \( a, a \rightarrow b \in U(A; \beta) \). Since \( U(A; \beta) \) is a filter of \( L \), we have \( b \in U(A; \beta) \), that is \( A(b) \geq \beta \). Therefore \( \max[A(b), \delta] \geq \max[\beta, \delta] = \beta \). But \( \max[A(b), \delta] < \beta \), contradiction. Hence (2) holds.

Conversely, assume that \( A \) satisfies (1)(2). Let \( \alpha \in (\delta, 1] \), for any \( x \in U(A; \alpha) \), we have \( \max[A(I), \alpha] \geq A(x) \geq \alpha > \delta \). Then \( A(I) > \delta \), it follows that \( \max[A(I), \alpha] = A(I) \geq A(x) \geq \alpha \), therefore \( I \in U(A; \alpha) \).

Let \( x, y \in L \) and \( x, x \rightarrow y \in U(A; a) \), then \( A(x) \geq \delta \), \( A(x \rightarrow y) \geq \delta \). Since \( \max[A(y), \delta] \) \( \geq \min[A(x), A(x \rightarrow y)] \geq \alpha > \delta \) for any \( x, y \in L \), we have \( A(y) > \delta \). Hence \( \max[A(y), \delta] = A(y) \geq \min[A(x), A(x \rightarrow y)] \geq \alpha \), then \( y \in U(A; \alpha) \). Therefore \( U(A; \alpha)(\alpha \in (\delta, 1]) \) is a filter of \( L \).

By the Theorem 12 and Theorem 13, we have the following theorem:

**Theorem 14** Let \( A \) be a fuzzy subset of \( L \). If \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter, then \( U(A; \alpha) \) is a filter of \( L \), where \( U(A; \alpha) = \{ x \in L | A(x) \geq \alpha \} \) and \( \alpha \in (h, \delta] \).

**Proposition 15** Let \( A \) be a fuzzy subset of \( L \). If \( A \) is an \((\overline{e_h} \vee \overline{q_h}), \overline{e_h} \vee \overline{q_h})\)-fuzzy filter, then \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter.

The converse of Proposition 15 doesn’t hold in general. For example, in Example 4, \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter, but \( A \) isn’t an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter. Therefore \( A \) isn’t an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter.

**Proposition 16** Let \( A \) be a fuzzy subset of \( L \). If \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter, then \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter.

The converse of Proposition 16 doesn’t hold in general. For example, in Example 1, we define a fuzzy subset \( B \) as follows: \( B(I) = B(b) = B(\in) = 0.7 \), \( B(O) = B(a) = (d) = 0.8 \). It is routine to verify \( B \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter. But \( B \) isn’t an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter. Since \( B(I) = 0.7 + 0.45 < 0 \times 0.6 \), that is, \( b \in O_0 \) \( \overline{e_h} \) \( \overline{q_h} \). But \( B(d) = 0.8 \) and \( B(d) + 0.45 > 2 \times 0.6 \), that is, \( d \in O_0 \) \( \overline{e_h} \) \( \overline{q_h} \). Therefore, \( d \in O_0 \) \( \overline{e_h} \) \( \overline{q_h} \)

**Theorem 17** Let \( A \) be a fuzzy subset of \( L \).

1. \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter if and only if \( A(\neq \emptyset) \) is a filter of \( L \) for any \( r \in (\delta, 1] \).
2. \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter if and only if \( A(\neq \emptyset) \) is a filter of \( L \) for any \( r \in (h, \delta] \).

**Proof** Let \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \) and \( I \notin A_r \), then \( I \notin A_r \), that is \( A(I) < r \). Since \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \), we have \( \max[A(I), \alpha] \geq A(x) \) for any \( x \in A_r \), it follows that \( r = \max[r, \delta] \geq \max[A(I), \delta] \geq A(x) \). Thus \( A(x) < r \). But \( x \notin A_r \), that is \( A(x) \geq r \), contradiction. Therefore \( I \in A_r \).

Assume that \( x, x \rightarrow y \in A_r \), that is \( A(x) \geq r \) and \( A(x \rightarrow y) \geq r \). Since \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \), we have \( \max[A(y), \delta] \geq \min[A(x), A(x \rightarrow y)] \geq r \). It follows that \( A(y) \geq r \) for \( r \in (\delta, 1] \). Hence \( y \in A_r \), that is \( y \in A_r \). Therefore, \( A \) is a filter of \( L \).

Conversely, suppose that there exists \( x \in L \) such that \( \max[A(I), \delta] < A(x) \) \( x \rightarrow y \). Then \( r \in (\delta, 1] \), \( A(I) < r \). That is \( I \notin A_r \). Since \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \), \( A(I) < r \) and \( A(x) \geq r \). By hypothesis \( r \in (h, \delta] \), we have \( \delta < r < \delta + \delta \). Thus \( A(I) < \delta + \delta \), contradiction. Hence \( I \notin A_r \).

Assume that \( x, x \rightarrow y \in A_r \), that \( A(x) + r > 2 \delta \) and \( A(x \rightarrow y) + r > 2 \delta \). Since \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \), we have \( \max[A(y), \delta] \geq \min[A(x), A(x \rightarrow y)] > 2 \delta - r \). Since \( 2 \delta - r > \delta \) for \( r \in (h, \delta] \), it follows that \( A(y) > 2 \delta - \delta \), that is \( A(y) + r > 2 \delta \), thus \( y \in A_r \). Therefore, \( A \) is a filter of \( L \).

Conversely, suppose that there exists \( x, y \in L \) such that \( \max[A(I), \delta] < A(x) \) \( x \rightarrow y \). Then \( r \in (\delta, 1] \), \( A(I) < r \). By hypothesis \( r \in (\delta, 1] \), \( A(I) < r \). Thus \( A(x) > 2 \delta - r \delta \), that is \( A(x) + r > 2 \delta \). Since \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \), we have \( \max[A(y), \delta] \geq \min[A(x), A(x \rightarrow y)] > 2 \delta - r \). Since \( 2 \delta - r > \delta \) for \( r \in (h, \delta] \), it follows that \( A(y) > 2 \delta - \delta \), that is \( A(y) + r > 2 \delta \), thus \( y \in A_r \). Therefore, \( A \) is a filter of \( L \).

In Theorem 17, taking \( h = 0, \delta = 0.5 \), we have following Corollaries:

**Corollary 18** Let \( A \) be a fuzzy subset of \( L \).

1. \( A \) is an \((\overline{e_h}, \overline{e_h} \vee \overline{q_h})\)-fuzzy filter of \( L \) if and only if \( U(A; r)(\neq \emptyset) \) is a filter of \( L \) for any \( r \in (0.5, 1] \).

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(2) $A$ is an $(\epsilon_h, \epsilon_b \lor q_h)$-fuzzy filter of $\mathcal{L}$ if and only if $Q(A; r)(\neq \emptyset)$ is a filter of $\mathcal{L}$ for any $r \in (0, 0.5]$.

5. Conclusion

In order to research the many-valued logical system whose propositional value is given in a lattice, Xu initiated the concept of lattice implication algebras. Hence for development of this many-valued logical system, it is needed to make clear the structure of lattice implication algebras. In the notion of lattice implication algebra, the partial order can be applied to describe the incomparability and the implication operation can be used to represent the transfer of incomparability. In this paper, we extend the belongs to ($\in$) relation and quasi-coincidence with ($q$) relation between fuzzy points and a fuzzy subsets, the concept of $(\alpha, \beta)$-fuzzy filters and $(\overline{\alpha}, \overline{\beta})$-fuzzy filters of lattice implication algebras is introduced and some related properties are investigated. Some equivalent characterizations of these generalized fuzzy filters are derived. Finally, we discussed relations among these generalized fuzzy filters. This idea of this paper can be applied to the fuzzy implicative filter, fuzzy ultra-filter, and so on. We generalized some results in (Y.B. Jun, 2007, J.M. Zhan, 2009).

Based on these results, we will study primeness and maximality in the $(\epsilon_h, \epsilon_b \lor q_h)$-fuzzy setting. From the view of universal algebra, we will try to investigated the unifying definition of $(\epsilon_h, \epsilon_b \lor q_h)$-fuzzy filter on all logical algebras and study their common properties.

References


B. Davvaz. (2006). $(\epsilon, \epsilon \lor q)$-fuzzy subnear-rings and ideals. Soft Computing, 10, 206-211.


Table 1. Operators ‘’ and → in $L$

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Figure 1. Hasse Diagram of $L = \{O, a, b, c, d, I\}$