

Generalization of \mathcal{U} -Generator and M -Subgenerator Related to Category $\sigma[M]$

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Abstract

Let \mathcal{U} be a non-empty set of R -modules. R -module N is generated by \mathcal{U} if there is an epimorphism from $\bigoplus_{\lambda \in \Lambda} U_{\lambda}$ to N , where $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. R -module M is a subgenerator for N if N is isomorphic to a submodule of an M -generated module. In this paper, we introduce a \mathcal{U}_V -generator, where V be a submodule of $\bigoplus_{\lambda \in \Lambda} U_{\lambda}$, as a generalization of \mathcal{U} -generator by using the concept of V -coexact sequence. We also provide a \mathcal{U}_V -subgenerator motivated by the concept of M -subgenerator. Furthermore, we give some properties of \mathcal{U}_V -generated and \mathcal{U}_V -subgenerated modules related to category $\sigma[M]$. We also investigate the existence of pullback and pushout of a pair of morphisms of \mathcal{U}_V -subgenerated modules. We prove that the collection of \mathcal{U}_V -subgenerated modules is closed under submodules and factor modules.

Keywords: \mathcal{U} -generator, \mathcal{U}_V -generator, V -coexact sequences, M -subgenerator, \mathcal{U}_V -subgenerator

1. Introduction

The concept of exact sequences of R -modules and R -module homomorphisms is a useful tool in the study of modules. A sequence $A \rightarrow B \rightarrow C$ is exact if $\text{Im}f = \text{Ker}g (= g^{-1}(0))$. Davvaz and Parnian-Garamaleky (1999) provide the generalization of exact sequences, i.e. quasi-exact sequences. They substitute the submodule $\{0\}$ to any submodule U of C .

Then Anvariye dan Davvaz (2005) investigate further results about quasi-exact sequences. They also introduce the generalization of Schanuel's Lemma. Furthermore, Davvaz and ShabaniSolt (2002) give a generalization of some notions in homological algebra. In 2002, Anvariye and Davvaz provide U -split sequences. They also establish several connections between U -split sequences and projective modules.

Motivated by the definition of U -exact and V -coexact sequence, Fitriani et al. (2016) provide an X -sub exact sequence, which is a generalization of exact sequence. In 2017, they introduce X -sublinearly independent module by using the concept of X -sub exact sequence.

Let \mathcal{U} be a non-empty set of R -modules. An R -module N is generated by \mathcal{U} if there is an epimorphism from $\bigoplus_{\lambda \in \Lambda} U_{\lambda}$ to N , where $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. The trace of \mathcal{U} is defined by $\text{Tr}(\mathcal{U}, M) = \sum \{\text{Im}h | h : U \rightarrow M, \text{ for some } U \in \mathcal{U}\}$. If $\mathcal{U} = \{U\}$ is a singleton, then $\text{Tr}(U, M) = \sum \{\text{Im}h | h \in \text{Hom}_R(U, M)\}$. $\text{Tr}(\mathcal{U}, M)$ is the unique largest submodule L of M generated by \mathcal{U} (Wisbauer, 1991). Clearly, $\text{Tr}(\mathcal{U}, M) = M$ if and only if \mathcal{U} generates M (Anderson & Fuller, 1992). For an indexed set $(M_{\alpha})_{\alpha \in A}$ of modules and class of modules \mathcal{U} , the direct sum of the traces $\text{Tr}(\mathcal{U}, M)$ is contained in $\bigoplus_A M_{\alpha}$. The trace of M in an R -module N is the sum of all M -generated submodules of N (Clark et al., 2006).

Proposition 1 (Wisbauer, 1991) *If $(M_{\alpha})_{\alpha \in A}$ is an indexed set of modules, then for each module M*

$$\text{Tr}(\mathcal{U}, \bigoplus_A M_{\alpha}) = \bigoplus_A \text{Tr}(\mathcal{U}, M_{\alpha}).$$

Furthermore, an M -subgenerated module is defined as follows.

Definition 2 (Wisbauer, 1991) Let M be an R -module. We say that an R -module N is subgenerated by M , or that M is a subgenerator for N , if N is isomorphic to a submodule of an M -generated module.

A subcategory C of $R\text{-MOD}$ is said to be subgenerated by M , or M is a subgenerator for C , if every object in C is subgenerated by M . Category $\sigma[M]$ is the full subcategory of $R\text{-MOD}$ whose objects are all R -modules subgenerated by M . This category is a category closely connected to M and hence reflecting properties of M .

The properties of $\sigma[M]$ given by the following proposition:

Proposition 3 (Wisbauer, 1991) *For an R -module M we have:*

1. *For N in $\sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$, i.e. $\sigma[M]$ has kernels and cokernels.*
2. *The direct sum of a family of modules in $\sigma[M]$ belong to $\sigma[M]$ and is equal to the coproduct of these modules in $\sigma[M]$.*
3. *Pullback and pushout of morphisms in $\sigma[M]$ belong to $\sigma[M]$.*

As a generalization of exact sequence of R -modules, Anvanriyeh and Davvaz (1999) defined U -exact sequences as follows: A sequence of R -modules $A \xrightarrow{f} B \xrightarrow{g} C$ if there exists a submodule U of C such that $Im f = g^{-1}(U)$. In this case, the sequence is said to be U -exact (at B). If $f(V) = Ker g$, where V is a submodule of A , then the sequence is said to be V -coexact.

Let \mathcal{U} be a family of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$, where $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. The aim of this paper is to generalize the concept of \mathcal{U} -generator to a \mathcal{U}_V -generator, where V is a submodule of $\oplus_{\Lambda} U_{\lambda}$. Furthermore, we provide a \mathcal{U}_V -subgenerator as a generalization of M -subgenerator. We also investigate the properties of \mathcal{U}_V -generated modules and \mathcal{U}_V -subgenerated modules related to the properties of the category $\sigma[M]$.

2. Results

2.1 \mathcal{U}_V -Generated Modules

Let \mathcal{U} be a family of R -modules. It is possible that an R -module M is not a \mathcal{U} -generated module, i.e. there no epimorphism from $\oplus_{\Lambda} U_{\lambda}$ to M , but we can define an epimorphism from a submodule $V \subseteq \oplus_{\Lambda} U_{\lambda}$ to M . Therefore we can generalize the concept of a \mathcal{U} -generated module to a \mathcal{U}_V -generated module by using the definition of V -coexact sequence.

Definition 4 Let \mathcal{U} be a non-empty set of R -modules, V be a submodule of $\oplus_{\Lambda} U_{\lambda}$, where $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. We say that an R -module N is generated by \mathcal{U}_V if there exists an epimorphism $V \rightarrow N \rightarrow 0$.

A set $\{U_{\lambda}\}_{\Lambda}$ is called \mathcal{U}_V -generator for N . Furthermore, the set $\{U_{\lambda}\}_{\Lambda}$ is called minimal \mathcal{U}_V -generator for N if

$$\Lambda = \min\{\Lambda_V | N \text{ is } \mathcal{U}_V\text{-generated, } V \subseteq \oplus_{\Lambda_V} U_{\lambda}\}.$$

If we take $V = \oplus_{\Lambda} U_{\lambda}$, then a \mathcal{U}_V -generated module is a \mathcal{U} -generated module. Clearly, every \mathcal{U} -generated module is \mathcal{U}_V -generated. But, a \mathcal{U}_V -generated module need not be a \mathcal{U} -generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then \mathbb{Z} -module \mathbb{Z} is a $\mathcal{U}_{\mathbb{Z}}$ -generated module. But, we can not define an epimorphism from \mathbb{Q} to \mathbb{Z} and hence \mathbb{Z} -module \mathbb{Z} is not a \mathcal{U} -generated module.

Now, we give some examples of \mathcal{U}_V -generated modules. *Example 1*

1. Let \mathcal{U} be the set of all free R -modules and P be projective R -module. Since P is projective, P is a direct summand of a free module F . Hence P is \mathcal{U}_F -generated module.
2. Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$, a family of \mathbb{Z} -modules. \mathbb{Z} -module \mathbb{Z}_6 is a \mathcal{U}_V -generated, where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. In general, \mathbb{Z} -module \mathbb{Z}_{pq} is a \mathcal{U}_V -generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.
3. Let $\mathcal{U} = \{\mathbb{Q}\}$. \mathbb{Z} -module \mathbb{Z}_n , $n \geq 2$, is \mathcal{U}_V -generated, where $V = \mathbb{Z}$.
4. Let R be a commutative ring with unit and $\mathcal{U} = \{U_{\lambda}\}_{\Lambda}$ be a family of R -modules, where $U_{\lambda} = Hom_R(R, M_{\lambda})$, for every $\lambda \in \Lambda$.
Based on Adkins & Weintraub (1992), we can define

$$\phi : Hom_R(R, M) \rightarrow M,$$

where $\phi(f) := f(1)$. Then M_{λ} is $\mathcal{U}_{U_{\lambda}}$ -generated.

5. Let $\mathcal{U} = \{\mathbb{Z}_n | n \in \mathbb{Z}\}$ be a family of \mathbb{Z} -modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be \mathbb{Z} -modules. Then M is \mathcal{U}_N -generated and N is \mathcal{U}_M -generated.

If there exists a finite index set $E \subseteq \Lambda$ such that M is \mathcal{U}_V -generated and V is a submodule of $\oplus_E U_e$, then we define a finitely \mathcal{U}_V -generated module as follows:

Definition 5 Let \mathcal{U} be a non-empty set of R -modules and N be an R -module. If there exists a finite index set $E \subseteq \Lambda$ such that $V \subseteq \oplus_E U_e$ and M is \mathcal{U}_V -generated, then R -module N is said to be finitely \mathcal{U}_V -generated.

Example 2 Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of \mathbb{Z} -modules. \mathbb{Z} -module \mathbb{Z}_{pq} is a finitely \mathcal{U}_V -generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.

Then, we will give some basic properties of \mathcal{U}_V -generated modules. Let \mathcal{U} be a non-empty set of R -modules and N be an R -module. We define:

$$\mathcal{U}(N) = \{V \subseteq \oplus_\Lambda U_\lambda, U_\lambda \in \mathcal{U} | N \text{ is } \mathcal{U}_V\text{-generated}\}.$$

In this set, we collect all submodules V of $\oplus_\Lambda U_\lambda$ such that N is a \mathcal{U}_V -generated module. In the following proposition, we prove that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

Proposition 6 Let \mathcal{U} be a non-empty set of R -modules, V_λ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \Lambda$ for every $\lambda \in \Lambda$. If N_λ is \mathcal{U}_{V_λ} -generated, for every $\lambda \in \Lambda$, then $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$ -generated.

Proof. Since N_λ is \mathcal{U}_{V_λ} -generated, for every $\lambda \in \Lambda$, the sequences $V_\lambda \rightarrow N_\lambda \rightarrow 0$ is exact for every $\lambda \in \Lambda$. Therefore, the sequence

$$\oplus_\Lambda V_\lambda \rightarrow \oplus_\Lambda N_\lambda \rightarrow 0$$

is exact. Hence, $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$ -generated. So, we can say that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

As a corollary of Proposition 6, we obtain:

Corollary 7 Let \mathcal{U} be a non-empty set of R -modules. If R -module N_i is \mathcal{U}_{V_i} -generated for every $i = 1, 2, \dots, n$, then $\oplus_{i=1}^n X_i$ is $\mathcal{U}_{\oplus_{i=1}^n V_i}$ -generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $i = 1, 2, \dots, n$ and $\lambda \in \Lambda$.

In the following proposition, we will show that if $V \in \mathcal{U}(N)$, for an R -module N , then V is in $\mathcal{U}(N')$, for every homomorphic image N' of N .

Proposition 8 Let \mathcal{U} be a non-empty set of R -modules. If R -module N is \mathcal{U}_V -generated, then N' is \mathcal{U}_V -generated, for every homomorphic image N' of N .

Proof. If R -module N is \mathcal{U}_V -generated, then the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{f} N \rightarrow 0$$

is V -coexact. Let N' be homomorphic image of N , then there is an epimorphism $p : N \rightarrow N'$. Hence, $g = p \circ f$ is a homomorphism from V to N' . Since f and p are epimorphisms, then g is an epimorphism. So, N' is \mathcal{U}_V -generated.

In the next proposition, we will prove that $\mathcal{U}_V(N)$ is closed under direct sum, i.e. if V_λ is in $\mathcal{U}(N)$ for every $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} V_\lambda$ is in $\mathcal{U}(N)$.

Proposition 9 Let \mathcal{U} be a non-empty set of R -modules and V_α be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$ for every $\lambda \in \Lambda$. If R -module M is \mathcal{U}_{V_α} -generated, for every $\alpha \in A$, then M is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$ -generated.

Proof. Since R -module M is \mathcal{U}_{V_α} -generated for every $\alpha \in A$, there is an epimorphism f_α such that the sequence: $V_\alpha \xrightarrow{f_\alpha} M \rightarrow 0$ is exact for every $\alpha \in A$. We can define $f : \oplus_{\alpha \in A} V_\alpha \rightarrow M$, where $f((v_\alpha)_A) = f_{\alpha_i}(v_{\alpha_i})$, $\alpha_i \in A$. From this, we have f is an epimorphism from $\oplus_{\alpha \in A} V_\alpha$ to M . Hence, M is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$ -generated.

As a corollary of Proposition 9, we obtain:

Proposition 10 Let \mathcal{U} be a non-empty set of R -modules. If R -module M is \mathcal{U}_{V_i} -generated for every $i = 1, 2, \dots, n$, then M is $\mathcal{U}_{\oplus_{i=1}^n V_i}$ -generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$ for every $i = 1, 2, \dots, n$.

If $V_2 \in \mathcal{U}(N)$ and $V_1 \in \mathcal{U}(V_2)$ i.e. N is \mathcal{U}_{V_1} -generated and V_2 is \mathcal{U}_{V_1} -generated, with modules V_1 and V_2 are submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, then we will show that $V_1 \in \mathcal{U}(N)$, i.e. N is \mathcal{U}_{V_1} -generated module.

Proposition 11 Let \mathcal{U} be a non-empty set of R -modules. If R -module N is \mathcal{U}_{V_2} -generated and V_2 is \mathcal{U}_{V_1} -generated, then N is \mathcal{U}_{V_1} -generated, where V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.

Proof. Since N is \mathcal{U}_{V_2} -generated and V_2 is \mathcal{U}_{V_1} -generated, there exists epimorphisms $\alpha : V_2 \rightarrow N$ and $\beta : V_1 \rightarrow V_2$. So, we can define $g = \alpha \circ \beta : V_1 \rightarrow N$. Since α and β are epimorphisms, g is an epimorphism. Finally, N is \mathcal{U}_{V_1} -generated.

As a corollary we obtain:

Corollary 12 Let \mathcal{U} be a non-empty set of R -modules. If R -module N is \mathcal{U}_V -generated and V is \mathcal{U} -generated, then N is \mathcal{U} -generated, where V be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.

Proof. Since R -module N is \mathcal{U}_V -generated and V is \mathcal{U} -generated, by Proposition 11, we have N is $\mathcal{U}_{\oplus_{\Lambda} U_{\lambda}}$ -generated. In other words, N is \mathcal{U} -generated.

Corollary 12 *Let \mathcal{U} be a non-empty set of R -modules and $V \subset \oplus_{\Lambda} U_{\lambda}$, with modules $U_{\lambda} \in \mathcal{U}$. If R -module M is \mathcal{U}_V -subgenerated and V is a \mathcal{U} -generated module, then the sequence*

$$\oplus_{\Lambda} U_{\lambda} \rightarrow M \rightarrow 0$$

is V -coexact.

Proof. Since R -module M is \mathcal{U}_V -subgenerated, there is an epimorphism $\alpha : V \rightarrow M$. By assumption, V is a \mathcal{U} -generated module. So, there is an epimorphism $\pi : \oplus_{\Lambda} U_{\lambda} \rightarrow V$. Hence, $g = \alpha \circ \pi$ is an epimorphism from $\oplus_{\Lambda} U_{\lambda}$ to M such that $g|_V = \alpha$. We have the sequence

$$\oplus_{\Lambda} U_{\lambda} \xrightarrow{g} M \rightarrow 0$$

is V -coexact.

Corollary 13 *Let \mathcal{U} be a non-empty set of semisimple R -modules. If R -module M is \mathcal{U}_V -generated, then M is \mathcal{U} -generated, where V is a submodule of $\oplus_{\Lambda} U_{\lambda}$.*

Proof. We assume that R -module M is a \mathcal{U}_V -generated. Since every submodule of semisimple module $\oplus_{\Lambda} U_{\lambda}$ is a direct summand, M is \mathcal{U} -generated by using Proposition 11.

2.2 \mathcal{U}_V -Subgenerated Modules

We already know that an M -subgenerated module is a generalization of a \mathcal{U} -generated module. In the similar way, we can obtain a \mathcal{U}_V -subgenerated module as a generalization of \mathcal{U} -generated module.

Definition 14 *Let \mathcal{U} be a non-empty set of R -modules, V be a submodule of $\oplus_{\Lambda} U_{\lambda}$. We say that an R -module N is subgenerated by \mathcal{U}_V if N is isomorphic to a submodule of a \mathcal{U}_V -generated module.*

M -subgenerated module is a special case of \mathcal{U}_V -subgenerated modules by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. By Definition 14, every \mathcal{U}_V -generated module is a \mathcal{U}_V -subgenerated module. But the converse need not be true. For example, let \mathcal{U} the set of all \mathbb{Z} -modules. \mathbb{Z} -module \mathbb{Z} is $\mathcal{U}_{\mathbb{Q}}$ -subgenerated. But, \mathbb{Z} -module \mathbb{Z} is not $\mathcal{U}_{\mathbb{Q}}$ -generated.

Proposition 15 *Let \mathcal{U} be a non-empty set of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$. If R -module N is \mathcal{U}_V -subgenerated and N is a direct summand of a \mathcal{U}_V -generated module, then N is \mathcal{U}_V -generated module.*

Let \mathcal{U} be a non-empty set of R -modules and N be an R -module. In $\sigma[M]$, Wisbauer (1991) collect all R -modules subgenerated by M . In the similar way, we will collect all R -modules subgenerated by \mathcal{U}_V , we denote it by $\sigma_V(\mathcal{U})$:

$$\sigma_V(\mathcal{U}) = \{N | N \text{ is } \mathcal{U}_V\text{-subgenerated}\}.$$

The full subcategory $\sigma[M]$ of $R\text{-MOD}$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. Next, we will show that $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules.

Proposition 16 *Let \mathcal{U} be a non-empty set of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$. If R -module N is \mathcal{U}_V -subgenerated, then N' is a \mathcal{U}_V -subgenerated module, for every submodule N' of N .*

Proof. Since N is a \mathcal{U}_V -subgenerated, then N is isomorphic to a submodule of a \mathcal{U}_V -generated module. So, there is an epimorphism:

$$V \xrightarrow{f} K \rightarrow 0$$

and N is isomorphic to a submodule of K . Let N' be a submodule of N . We have N' is isomorphic to a submodule of K and N' is a \mathcal{U}_V -subgenerated module.

Proposition 17 *Let \mathcal{U} be a non-empty set of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$. If R -module N is \mathcal{U}_V -subgenerated, then N/L is \mathcal{U}_V -subgenerated module, for every factor module N/L of N .*

Proof. Since N is a \mathcal{U}_V -subgenerated, there is a \mathcal{U}_V -generated module K and an epimorphism:

$$V \xrightarrow{f} K \rightarrow 0$$

and N is isomorphic to a submodule of K . Let L be a submodule of N . We have L is isomorphic to a submodule of K and hence N/L is isomorphic to a submodule of K/L' , where $L \cong L'$. Since K/L' is a \mathcal{U}_V -generated module, we get N/L is a \mathcal{U}_V -subgenerated module.

As a corollary of Proposition 16 and 17, we obtain:

Corollary 18 Let \mathcal{U} be a non-empty set of R -modules, V be a submodule of $\oplus_{\Lambda} U_{\lambda}$ and

$$0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$$

be an exact sequence of R -modules. If L is a \mathcal{U}_V -subgenerated module, then K and M are \mathcal{U}_V -subgenerated modules.

If R -module N_1 and N_2 are \mathcal{U}_V -subgenerated, then we have two exact sequences: $V \rightarrow M_1 \rightarrow 0$ and $V \rightarrow M_2 \rightarrow 0$. Furthermore, N_1 and N_2 are isomorphic to submodules of M_1 and M_2 , respectively. Hence $Tr(V, M_1) = M_1$ and $Tr(V, M_2) = M_2$. By Proposition 1, we have $Tr(V, M_1 \oplus M_2) = Tr(V, M_1) \oplus Tr(V, M_2) = M_1 \oplus M_2$. But, $N_1 \oplus N_2$ need not be a \mathcal{U}_V -subgenerated module. By Proposition 6, we have $N_1 \oplus N_2$ is a $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module.

In the following proposition, we will show the existence of pullback and pushout of a pair of morphisms of \mathcal{U}_V -subgenerated modules.

Proposition 19 Let \mathcal{U} be a non-empty set of R -modules. If N_1 is \mathcal{U}_{V_1} -subgenerated and N_2 is \mathcal{U}_{V_2} -subgenerated, then pullback of $f_1 : N_1 \rightarrow N$ and $f_2 : N_2 \rightarrow N$ is $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module, where V_1, V_2 are submodules of $\oplus_{\Lambda} U_{\lambda}$.

Proof. Since N_1 is \mathcal{U}_{V_1} -subgenerated and N_2 is \mathcal{U}_{V_2} -subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated. Let $f_1 : N_1 \rightarrow M$, $f_2 : N_2 \rightarrow M$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated modules. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module. Based on Wisbauer (1991), pullback of (f_1, f_2) is a submodule of $N_1 \oplus N_2$. Since every submodule of $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated, the pullback of (f_1, f_2) is a $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module.

Proposition 20 Let \mathcal{U} be a non-empty set of R -modules. If N_1 is \mathcal{U}_{V_1} -subgenerated and N_2 is \mathcal{U}_{V_2} -subgenerated, then pushout of $g_1 : X \rightarrow N_1$ and $g_2 : X \rightarrow N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module, where V_1, V_2 are submodules of $\oplus_{\Lambda} U_{\lambda}$.

Proof. Since N_1 is \mathcal{U}_{V_1} -subgenerated and N_2 is \mathcal{U}_{V_2} -subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated. Let $g_1 : X \rightarrow N_1$, $g_2 : X \rightarrow N_2$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated modules. Based on Wisbauer (1991), pushout of (g_1, g_2) is a factor module of $N_1 \oplus N_2$. Since every factor module of $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated, the pushout of (g_1, g_2) is a $\mathcal{U}_{V_1 \oplus V_2}$ -subgenerated module.

A submodule N of R -module M is called fully invariant if $f(N)$ is contained in N for every R -endomorphism f of M . M is called a duo module provided every submodule of M is fully invariant (Özcan et al., 2006).

The following theorem shows that the properties of R -modules in $\sigma_V \mathcal{U}$ are reflecting the properties of V .

Theorem 21 Let \mathcal{U} be a non-empty set of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$, $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$.

1. If R -module U is V -injective (V -projective), then U is N -injective (N -projective), for every $N \in \sigma_V(\mathcal{U})$.
2. If V is semisimple, then every module in $\sigma_V(\mathcal{U})$ is semisimple.
3. If V is Noetherian (Artinian), then N is Noetherian (Artinian), for every $N \in \sigma_V(\mathcal{U})$.
4. If V is a duo module, quasi-injective and quasi-projective, then N is a duo module, V -projective and V -injective, for every $N \in \sigma_V(\mathcal{U})$.

Proof.

1. Let $N \in \sigma_V \mathcal{U}$. Then N is isomorphic to a submodule of \mathcal{U}_V -generated module, say M . We have the following exact sequence:

$$0 \rightarrow Ker f \rightarrow V \xrightarrow{f} M \rightarrow 0.$$

Based on Wisbauer (1991), if U is V -injective, then U is M -injective. Therefore by Wisbauer (1991) 16.3, U is N -injective.

2 and 3 can be shown in a similar way to 1.

- 4 Based on Özcan et. al. (2006), if V is a duo module and quasi-injective, then every submodule of V is a duo module. Furthermore, if V is a duo module and quasi-projective, then every homomorphic image of V is a duo module. From 1, we have N is V -projective and V -injective, for every N in $\sigma_V(\mathcal{U})$.

3. Conclusions

A \mathcal{U}_V -generator is a generalization of \mathcal{U} -generator. If an R -module N is \mathcal{U}_V -generated, then every homomorphic image of N is also \mathcal{U}_V -generated. Furthermore, direct sums of \mathcal{U}_V -generated R -modules are $\mathcal{U}_{V'}$ -generated, for some submodules V' of $\bigoplus_{\lambda} U_{\lambda}$. In the set $\mathcal{U}(N)$, we collect all submodules V of $\bigoplus_{\lambda} U_{\lambda}$ such that N is a \mathcal{U}_V -generated module and we have $\mathcal{U}(N)$ is closed under direct sums.

In the set $\sigma_V(\mathcal{U})$, we collect all R -modules subgenerated by \mathcal{U}_V . The full subcategory $\sigma[M]$ of $R - MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. The set $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules. Furthermore, the properties of R -modules in $\sigma_V(\mathcal{U})$ are reflecting the properties of V .

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