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Generalization of $\mathcal{U}$-Generator and M-Subgenerator Related to Category $\sigma[M]$

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Abstract

Let $\mathcal{U}$ be a non-empty set of R-modules. R-module N is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to N, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. R-module M is a subgenerator for N if N is isomorphic to a submodule of an M-generated module. In this paper, we introduce a $\mathcal{U}_V$-generator, where V be a submodule of $\oplus_\Lambda U_\lambda$, as a generalization of $\mathcal{U}$-generator by using the concept of V-coexact sequence. We also provide a $\mathcal{U}_V$-subgenerator motivated by the concept of M-subgenerator. Furthermore, we give some properties of $\mathcal{U}_V$-generated and $\mathcal{U}_V$-subgenerated modules related to category $\sigma[M]$. We also investigate the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules. We prove that the collection of $\mathcal{U}_V$-subgenerated modules is closed under submodules and factor modules.

Keywords: $\mathcal{U}$-generator, $\mathcal{U}_V$-generator, V-coexact sequences, M-subgenerator, $\mathcal{U}_V$-subgenerator

1. Introduction

The concept of exact sequences of R-modules and R-module homomorphisms is a useful tool in the study of modules. A sequence $A \to B \to C$ is exact if $Imf = Kerg (= g^{-1}(0))$. Davvaz and Parnian-Garamaleky (1999) provide the generalization of exact sequences, i.e. quasi-exact sequences. They substitute the submodule $\{0\}$ to any submodule U of C.

Then Anvariyeh dan Davvaz (2005) investigate further results about quasi-exact sequences. They also introduce the generalization of Schanuel's Lemma. Furthermore, Davvaz and ShabaniSolt (2002) give a generalization of some notions in homological algebra. In 2002, Anvariyeh and Davvaz provide U-split sequences. They also establish several connections between U-split sequences and projective modules.

Motivated by the definition of U-exact and V-coexact sequence, Fitriani et al. (2016) provide an X-sub exact sequence, which is a generalization of exact sequence. In 2017, they introduce X-sublinearly independent module by using the concept of X-sub exact sequence.

Let $\mathcal{U}$ be a non-empty set of R-modules. An R-module N is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to N, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The trace of $\mathcal{U}$ is defined by $Tr(\mathcal{U}, M) = \sum\{Imh | h : U \to M, \text{ for some } U \in \mathcal{U}\}$. If $\mathcal{U} = \{U\}$ is a singleton, then $Tr(U, M) = \sum\{Imh | h \in Hom_R(U, M)\}$. $Tr(\mathcal{U}, M)$ is the unique largest submodule L of M generated by $\mathcal{U}$ (Wisbauer, 1991). Clearly, $Tr(\mathcal{U}, M) = M$ if and only if $\mathcal{U}$ generates M (Anderson & Fuller, 1992). For an indexed set $(M_\alpha)_{\alpha \in A}$ of modules and class of modules $\mathcal{U}$, the direct sum of the traces $Tr(\mathcal{U}, M)$ is contained in $\oplus_A M_\alpha$. The trace of M in an R-module N is the sum of all M-generated submodules of N (Clark et al., 2006).

Proposition 1 (Wisbauer, 1991) *If $(M_\alpha)_{\alpha \in A}$ is an indexed set of modules, then for each module M*

$$Tr(\mathcal{U}, \oplus_A M_\alpha) = \oplus_A Tr(\mathcal{U}, M_\alpha).$$

Furthermore, an M-subgenerated module is defined as follows.

Definition 2 (Wisbauer, 1991) Let M be an R-module. We say that an R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M -generated module.

A subcategory C of R-MOD is said to be subgenerated by M, or M is a subgenerator for C, if every object in C is subgenerated by M. Category $\sigma[M]$ is the full subcategory of $R - MOD$ whose objects are all R-modules subgenerated by M. This category is a category closely connected to M and hence reflecting properties of M.

The properties of $\sigma[M]$ given by the following proposition:

Proposition 3 (Wisbauer, 1991) *For an R-module M we have:*

1. *For N in $\sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$, i.e. $\sigma[M]$ has kernels and cokernels.*
2. *The direct sum of a family of modules in $\sigma[M]$ belong to $\sigma[M]$ and is equal to the coproduct of these modules in $\sigma[M]$.*
3. *Pullback and pushout of morphisms in $\sigma[M]$ belong to $\sigma[M]$.*

As a generalization of exact sequence of R-modules, Anvanriyeh and Davvaz (1999) defined U-exact sequences as follows: A sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ if there exists a submodule U of C such that $Im\ f = g^{-1}(U)$. In this case, the sequence is said to be U-exact (at B). If $f(V) = Ker\ g$, where V is a submodule of A, then the sequence is said to be V-coexact.

Let $\mathcal{U}$ be a family of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The aim of this paper is to generalize the concept of $\mathcal{U}$-generator to a $\mathcal{U}_V$-generator, where V is a submodule of $\oplus_\Lambda U_\lambda$. Furthermore, we provide a $\mathcal{U}_V$-subgenerator as a generalization of M-subgenerator. We also investigate the properties of $\mathcal{U}_V$-generated modules and $\mathcal{U}_V$-subgenerated modules related to the properties of the category $\sigma[M]$.

2. Results

2.1 $\mathcal{U}_V$-Generated Modules

Let $\mathcal{U}$ be a family of R-modules. It is possible that an R-module M is not a $\mathcal{U}$-generated module, i.e. there no epimorphism from $\oplus_\Lambda U_\lambda$ to M, but we can define an epimorphism from a submodule $V \oplus_\Lambda U_\lambda$ to M. Therefore we can generalize the concept of a $\mathcal{U}$-generated module to a $\mathcal{U}_V$-generated module by using the definition of V-coexact sequence.

Definition 4 Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. We say that an R-module N is generated by $\mathcal{U}_V$ if there exists an epimorphism $V \to N \to 0$.

A set $\{U_\lambda\}_\Lambda$ is called $\mathcal{U}_V$-generator for N. Furthermore, the set $\{U_\lambda\}_\Lambda$ is called minimal $\mathcal{U}_V$-generator for N if

$$\Lambda = min\{\Lambda_V | N \text{ is } \mathcal{U}_V - \text{generated}, V \subseteq \oplus_{\Lambda_V} U_\lambda\}.$$

If we take $V = \oplus_\Lambda U_\lambda$, then a $\mathcal{U}_V$-generated module is a $\mathcal{U}$-generated module. Clearly, every $\mathcal{U}$-generated module is $\mathcal{U}_V$-generated. But, a $\mathcal{U}_V$-generated module need not be a $\mathcal{U}$-generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then $\mathbb{Z}$-module $\mathbb{Z}$ is a $\mathcal{U}_{\mathbb{Z}}$-generated module. But, we can not define an epimorphism from $\mathbb{Q}$ to $\mathbb{Z}$ and hence $\mathbb{Z}$-module $\mathbb{Z}$ is not a $\mathcal{U}$-generated module.

Now, we give some examples of $\mathcal{U}_V$-generated modules. *Example 1*

1. Let $\mathcal{U}$ be the set of all free R-modules and P be projective R-module. Since P is projective, P is a direct summand of a free module F. Hence P is $\mathcal{U}_F$-generated module.
2. Let $\mathcal{U} = \{\mathbb{Z}_p | p\text{prime}\}$, a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_6$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. In general, $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.
3. Let $\mathcal{U} = \{\mathbb{Q}\}$. $\mathbb{Z}$-module $\mathbb{Z}_n$, $n \geq 2$, is $\mathcal{U}_V$-generated, where $V = \mathbb{Z}$.
4. Let R be a commutative ring with unit and $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a family of R-modules, where $U_\lambda = Hom_R(R, M_\lambda)$, for every $\lambda \in \Lambda$.
   Based on Adkins & Weintraub (1992), we can define
   $$\phi : Hom_R(R, M) \to M,$$
   where $\phi(f) := f(1)$. Then M_λ is $\mathcal{U}_{U_\lambda}$-generated.
5. Let $\mathcal{U} = \{\mathbb{Z}_n | n \in \mathbb{Z}\}$ be a family of $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be $\mathbb{Z}$-modules. Then M is $\mathcal{U}_N$-generated and N is $\mathcal{U}_M$-generated.

If there exists a finite index set $E \subseteq \Lambda$ such that M is $\mathcal{U}_V$-generated and V is a submodule of $\oplus_E U_e$, then we define a finitely $\mathcal{U}_V$-generated module as follows:

Definition 5 Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. If there exists a finite index set $E \subseteq \Lambda$ such that $V \subseteq \oplus_E U_e$ and M is $\mathcal{U}_V$-generated, then R-module N is said to be finitely $\mathcal{U}_V$-generated.

Example 2 Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a finitely $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.

Then, we will give some basic properties of $\mathcal{U}_V$-generated modules. Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. We define:

$$\mathcal{U}(N) = \{V \subseteq \oplus_\Lambda U_\lambda, U_\lambda \in \mathcal{U} | N \text{ is } \mathcal{U}_V\text{-generated}\}.$$

In this set, we collect all submodules V of $\oplus_\Lambda U_\lambda$ such that N is a $\mathcal{U}_V$-generated module. In the following proposition, we prove that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

Proposition 6 *Let $\mathcal{U}$ be a non-empty set of R-modules, V_λ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \Lambda$ for every $\lambda \in \Lambda$. If N_λ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, then $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated.*

Proof. Since N_λ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, the sequences $V_\lambda \to N_\lambda \to 0$ is exact for every $\lambda \in \Lambda$. Therefore, the sequence

$$\oplus_\Lambda V_\lambda \to \oplus_\Lambda N_\lambda \to 0$$

is exact. Hence, $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated. So, we can say that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

As a corollary of Proposition 6, we obtain:

Corollary 7 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N_i is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then $\oplus_{i=1}^n X_i$ is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $i = 1, 2, ..., n$ and $\lambda \in \Lambda$.*

In the following proposition, we will show that if $V \in \mathcal{U}(N)$, for an R-module N, then V is in $\mathcal{U}(N')$, for every homomorphic image N' of N.

Proposition 8 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_V$-generated, then N' is $\mathcal{U}_V$-generated, for every homomorphic image N' of N.*

Proof. If R-module N is $\mathcal{U}_V$-generated, then the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{f} N \to 0$$

is V-coexact. Let N' be homomorphic image of N, then there is an epimorphism $p : N \to N'$. Hence, $g = p \circ f$ is a homomorphism from V to N'. Since f and p are epimorphisms, then g is an epimorphism. So, N' is $\mathcal{U}_V$-generated.

In the next proposition, we will prove that $\mathcal{U}_V(N)$ is closed under direct sum, i.e. if V_λ is in $\mathcal{U}(N)$ for every $\lambda \in \Lambda$, then $\oplus_{\lambda\in\Lambda} V_\lambda$ is in $\mathcal{U}(N)$.

Proposition 9 *Let $\mathcal{U}$ be a non-empty set of R-modules and V_α be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$ for every $\lambda \in \Lambda$. If R-module M is $\mathcal{U}_{V_\alpha}$-generated, for every $\alpha \in A$, then M is $\mathcal{U}_{\oplus_{\alpha\in A} V_\alpha}$-generated.*

Proof. Since R-module M is $\mathcal{U}_{V_\alpha}$-generated for every $\alpha \in A$, there is an epimorphism f_α such that the sequence: $V_\alpha \xrightarrow{f_\alpha} M \to 0$ is exact for every $\alpha \in A$. We can define $f : \oplus_{\alpha\in A} V_\alpha \to M$, where $f((v_\alpha)_A) = f_{\alpha_i}(v_{\alpha_i})$, $\alpha_i \in A$. From this, we have f is an epimorphism from $\oplus_{\alpha\in A}$ to M. Hence, M is $\mathcal{U}_{\oplus_{\alpha\in A} V_\alpha}$-generated.

As a corollary of Proposition 9, we obtain:

Proposition 10 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module M is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then M is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$ for every $i = 1, 2, ..., n$.*

If $V_2 \in \mathcal{U}(N)$ and $V_1 \in \mathcal{U}(V_2)$ i.e. N is $\mathcal{U}_{V_1}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, with modules V_1 and V_2 are submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, then we will show that $V_1 \in \mathcal{U}(N)$, i.e. N is $\mathcal{U}_{V_1}$-generated module.

Proposition 11 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_{V_2}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, then N is $\mathcal{U}_{V_1}$-generated, where V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

Proof. Since N is $\mathcal{U}_{V_2}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, there exists epimorphisms $\alpha : V_2 \to N$ and $\beta : V_1 \to V_2$. So, we can define $g = \alpha \circ \beta : V_1 \to N$. Since α and β are epimorphisms, g is an epimorphism. Finally, N is $\mathcal{U}_{V_1}$-generated.

As a corollary we obtain:

Corollary 12 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_V$-generated and V is $\mathcal{U}$-generated, then N is $\mathcal{U}$-generated, where V be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

Proof. Since R-module N is $\mathcal{U}_V$-generated and V is $\mathcal{U}$-generated, by Proposition 11, we have N is $\mathcal{U}_{\oplus_\Lambda U_\lambda}$-generated. In other words, N is $\mathcal{U}$-generated.

Corollary 12 *Let $\mathcal{U}$ be a non-empty set of R-modules and $V \subset \oplus_\Lambda U_\lambda$, with modules $U_\lambda \in \mathcal{U}$. If R-module M is $\mathcal{U}_V$-subgenerated and V is a $\mathcal{U}$-generated module, then the sequence*

$$\oplus_\Lambda U_\lambda \to M \to 0$$

is V-coexact.

Proof. Since R-module M is $\mathcal{U}_V$-subgenerated, there is an epimorphism $\alpha : V \to M$. By asumption, V is a $\mathcal{U}$-generated module. So, there is an epimorphism $\pi : \oplus_\Lambda U_\lambda \to V$. Hence, $g = \alpha \circ \pi$ is an epimorphism from $\oplus_\Lambda U_\lambda$ to M such that $g|_V = \alpha$. We have the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{g} M \to 0$$

is V-coexact.

Corollary 13 *Let $\mathcal{U}$ be a non-empty set of semisimple R-modules. If R-module M is $\mathcal{U}_V$-generated, then M is $\mathcal{U}$-generated, where V is a submodule of $\oplus_\Lambda U_\lambda$.*

Proof. We assume that R-module M is a $\mathcal{U}_V$-generated. Since every submodule of semisimple module $\oplus_\Lambda U_\lambda$ is a direct summand, M is $\mathcal{U}$-generated by using Proposition 11.

2.2 $\mathcal{U}_V$-Subgenerated Modules

We already know that an M-subgenerated module is a generalization of a $\mathcal{U}$-generated module. In the similar way, we can obtain a $\mathcal{U}_V$-subgenerated module as a generalization of $\mathcal{U}_V$-generated module.

Definition 14 *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$. We say that an R-module N is subgenerated by $\mathcal{U}_V$ if N isomorphic to a submodule of a $\mathcal{U}_V$-generated module.*

M-subgenerated module is a special case of $\mathcal{U}_V$-subgenerated modules by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. By Definition 14, every $\mathcal{U}_V$-generated module is a $\mathcal{U}_V$-subgenerated module. But the converse need not be true. For example, let $\mathcal{U}$ the set of all $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathcal{U}_\mathbb{Q}$-subgenerated. But, $\mathbb{Z}$-module $\mathbb{Z}$ is not $\mathcal{U}_\mathbb{Q}$-generated.

Proposition 15 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated and N is a direct summand of a $\mathcal{U}_V$-generated module, then N is $\mathcal{U}_V$-generated module.*

Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. In $\sigma[M]$, Wisbauer (1991) collect all R-modules subgenerated by M. In the similar way, we will collect all R-modules subgenerated by $\mathcal{U}_V$, we denote it by $\sigma_V(\mathcal{U})$:

$$\sigma_V(\mathcal{U}) = \{N | N \text{ is } \mathcal{U}_V\text{-subgenerated}\}.$$

The full subcategory $\sigma[M]$ of $R - MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. Next, we will show that $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules.

Proposition 16 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then N' is a $\mathcal{U}_V$-subgenerated module, for every submodule N' of N.*

Proof. Since N is a $\mathcal{U}_V$-subgenerated, then N isomorphic to a submodule of a $\mathcal{U}_V$-generated module. So, there is an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and N is isomorphic to a submodule of K. Let N' be a submodule of N. We have N' is somorphic to a submodule of K and N' is a $\mathcal{U}_V$-subgenerated module.

Proposition 17 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then N/L is $\mathcal{U}_V$-subgenerated module, for every factor module N/L of N.*

Proof. Since N is a $\mathcal{U}_V$-subgenerated, there is a $\mathcal{U}_V$-generated module K and an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and N is isomorphic to a submodule of K. Let L be a submodule of N. We have L is isomorphic to a submodule of K and hence N/L is is isomorphic to a submodule of K/L', where $L \cong L'$. Since K/L' is a $\mathcal{U}_V$-generated module, we get N/L is a $\mathcal{U}_V$-subgenerated module.

As a corolarry of Proposition 16 and 17, we obtain:

Corollary 18 *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$ and*

$$0 \to K \to L \to M \to 0$$

be an exact sequence of R-modules. If L is a $\mathcal{U}_V$-subgenerated module, then K and M are $\mathcal{U}_V$-subgenerated modules.

If R-module N_1 and N_2 are $\mathcal{U}_V$-subgenerated, then we have two exact sequences: $V \to M_1 \to 0$ and $V \to M_1 \to 0$. Furthermore, N_1 and N_2 are isomorphic to submodules of M_1 and M_2, respectively. Hence $Tr(V, M_1) = M_1$ and $Tr(V, M_2) = M_2$. By Proposition 1, we have $Tr(V, M_1 \oplus M_2) = Tr(V, M_1) \oplus Tr(V, M_2) = M_1 \oplus M_2$. But, $N_1 \oplus N_2$ need not be a $\mathcal{U}_V$-subgenerated module. By Proposition 6, we have $N_1 \oplus N_2$ is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

In the following proposition, we will show the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules.

Proposition 19 *Let $\mathcal{U}$ be a non-empty set of R-modules. If N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, then pullback of $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where V_1, V_2 are submodules of $\oplus_\Lambda U_\lambda$.*

Proof. Since N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $f_1 : N_1 \to M$, $f_2 : N_2 \to M$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. Based on Wisbauer (1991), pullback of (f_1, f_2) is a submodule of $N_1 \oplus N_2$. Since every submodule of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pullback of (f_1, f_2) is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

Proposition 20 *Let $\mathcal{U}$ be a non-empty set of R-modules. If N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, then pushout of $g_1 : X \to N_1$ and $g_2 : X \to N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where V_1, V_2 are submodules of $\oplus_\Lambda U_\lambda$.*

Proof. Since N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $g_1 : X \to N_1$, $g_2 : X \to N_2$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. Based on Wisbauer (1991), pushout of (g_1, g_2) is a factor module of $N_1 \oplus N_2$. Since every factor module of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pushout of (g_1, g_2) is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

A submodule N of R-module M is called fully invariant if $f(N)$ is contained in N for every R-endomorphism f of M. M is called a duo module provided every submodule of M is fully invariant (Özcan et al., 2006).

The following theorem shows that the properties of R-modules in $\sigma_V\mathcal{U}$ are reflecting the properties of V.

Theorem 21 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$.*

1. *If R-module U is V-injective (V-projective), then U is N-injective (N-projective), for every $N \in \sigma_V(\mathcal{U})$.*
2. *If V is semisimple, then every module in $\sigma_V(\mathcal{U})$ is semisimple.*
3. *If V is Noetherian (Artinian), then N is Noetherian (Artinian), for every $N \in \sigma_V(\mathcal{U})$.*
4. *If V is a duo module, quasi-injective and quasi-projective, then N is a duo module, V-projective and V-injective, for every $N \in \sigma_V(\mathcal{U})$.*

Proof.

1. Let $N \in \sigma_V\mathcal{U}$. Then N is isomorphic to a submodule of $\mathcal{U}_V$-generated module, say M. We have the following exact sequence:
$$0 \to Ker\, f \to V \xrightarrow{f} M \to 0.$$
Based on Wisbauer (1991), if U is V-injective, then U is M-injective. Therefore by Wisbauer (1991) 16.3, U is N-injective.
2 and 3 can be shown in a similar way to 1.

4 Based on Özcan et. al. (2006), if V is a duo module and quasi-injective, then every submodule of V is a duo module. Futhermore, if V is a duo module and quasi-projective, then every homomorphic image of V is a duo module. From 1, we have N is V-projective and V-injective, for every N in $\sigma_V(\mathcal{U})$.

3. Conclusions

A $\mathcal{U}_V$-generator is a generalization of $\mathcal{U}$-generator. If an R-module N is $\mathcal{U}_V$-generated, then every homomorphic image of N is also $\mathcal{U}_V$-generated. Furthermore, direct sums of $\mathcal{U}_V$-generated R-modules are $\mathcal{U}_{V'}$-generated, for some submodules V' of $\oplus_\Lambda U_\lambda$. In the set $\mathcal{U}(N)$, we collect all submodules V of $\oplus_\Lambda U_\lambda$ such that N is a $\mathcal{U}_V$-generated module and we have $\mathcal{U}(N)$ is closed under direct sums.

In the set $\sigma_V(\mathcal{U})$, we collect all R-modules subgenerated by $\mathcal{U}_V$. The full subcategory $\sigma[M]$ of $R-MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. The set $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules. Furthermore, the properties of R-modules in $\sigma_V(\mathcal{U})$ are reflecting the properties of V.

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