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Characterization of Orlicz Sobolev Spaces

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Abstract

We give a characterization of the Orlicz Sobolev spaces $W^{1,\Phi}(\Omega)$ when $\Omega \subset \mathbb{R}^N$ is a open subset, $N \geq 1$ and $\Phi \in \triangle^2$.

Keywords: Orlicz Spaces, Orlicz Sobolev Spaces, Non-local functionals

1. Introduction

In (Gagliardo, 1957) Gagliardo has introduced the semi-norm

$$[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}} \, d\mathcal{L}^N(x) \, d\mathcal{L}^N(y) \right)^{\frac{1}{p}}$$

besides he has studied the fractional Sobolev spaces $W^{s,p}(\Omega)$ with $0 < s < 1$ and $p > 1$. It is well known that $[f]_{W^{s,p}(\Omega)}$ does not converge to $[f]_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla f|^p \, d\mathcal{L}^N(x) \right)^{\frac{1}{p}}$ when $s \to 1^-$. Moreover, if Ω is a smooth bounded domain, then in (Bourgain et al.), Bourgain, Brezis and Mironescu have proved that

$$\lim_{s \to 1^-} (1-s) [f]^p_{W^{s,p}(\Omega)} = K_{p,N} [f]^p_{W^{1,p}(\Omega)}$$

for all $f \in W^{1,p}(\Omega)$, with $p > 1$. In (Leoni & Spector, 2011), Leoni and Spector have given an alternative characteization of the Sobolev spaces using the not-local semi-norm

$$[f]_{W^{1,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{p}} \, d\mathcal{L}^N(x) \, d\mathcal{L}^N(y) \right)^{\frac{1}{p}}$$

Particularly, in (Leoni & Spector, 2011), Leoni and Spector have shown the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be open, let $f \in L^p_{loc}(\Omega)$. Then $f \in W^{1,p}{}_{loc}(\Omega)$ if and only if*

$$\lim_{\lambda \to 0} \limsup_{\varepsilon \to 0^+} \int_{\Omega_\lambda} \int_{\Omega_\lambda} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y) \, d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) < +\infty \tag{1.1}$$

moreover

$$\begin{aligned} &\lim\nolimits_{\lambda \to 0} \limsup_{\varepsilon \to 0^+} \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \tfrac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y) \, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) = \\ &= K_{N,p} \int_\Omega |\nabla f(x)|^p \, d\mathcal{L}^N(x) . \end{aligned} \tag{1.2}$$

where ρ_ε is a "good" family of mollifiers.

In this article we will extend such results in the case of Orlicz Sobolev Spaces.

The Orlicz spaces have been introduced both as generalization of the spaces L^p, both for physical motivations see (Adams, 1975; Astarita & Marrucci, 1974; Diening & Ruzika, 2007; Gosez, 1974; Lieberman, 1991; Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991). Particularly, from the nineties many results of regularity are gotten for minima of functionals with general growths defined on Orlicz Sobolev spaces, see (Breit et al., 2011; Cianchi & Fusco, 1999; Dall'Aglio et al., 1998; Diening et al, 2009; Fuchs, 2011; Fusco & Sbordone, 1990; Granucci, 2017; Klimov, 2000; Talenti, 1990; Young, 1912). The following ipoptesis will be at the base of our results.

H-1; Φ is a N-function and $\Phi \in \Delta_2 on(0, +\infty)$.

H-2; ρ_ε is a family of mollifiers such that

$$\rho_\varepsilon \geq 0, \quad \int\limits_{\mathbb{R}^N} \rho_\varepsilon \, d\mathcal{L}^N = 1, \tag{1.3}$$

$$\lim_{\varepsilon \to 0^+} \int\limits_{|x|>\delta} \rho_\varepsilon \, d\mathcal{L}^N = 0 \quad \text{for all } \delta > 0. \tag{1.4}$$

H-3; There exist $\{v_i\}_{i=1,...,N} \subset \mathbb{R}^N$ and a $\delta > 0$ such that for all $\sigma_i \in C_\delta(v_i)$ the set $\{\sigma_i\}_{i=1,...,N}$ is linearly independent, where

$$C_\delta(v) = \left\{ w \in \mathbb{R}^N - \{0\} : \frac{v}{|v|} \cdot \frac{w}{|w|} > 1 - \delta \right\} \tag{1.5}$$

and

$$\liminf_{\varepsilon \to 0^+} \int\limits_{C_\delta(v_i)} \rho_\varepsilon \, d\mathcal{L}^N > 0 \tag{1.6}$$

for all $i = 1, ..., N$.

H-4; ρ_ε is radial, that is $\rho_\varepsilon(x) = \rho_\varepsilon(|x|)$ for all $x \in \mathbb{R}^N$.

The purpose of our article is to show the followings results.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2 and H-3, let $f \in L^\Phi_{loc}(\Omega)$. Then $f \in W^1 L^\Phi_{loc}(\Omega)$ if and only if*

$$\lim_{\lambda \to 0} \limsup_{\varepsilon \to 0^+} \int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi\left(\frac{|f(x) - f(y)|}{|x-y|} \right) \rho_\varepsilon(x-y) \, d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) < +\infty. \tag{1.7}$$

Moreover, if ρ_ε satisfy H-4 and $\Phi \in \triangle_2^m$ on $(0, +\infty)$, then there exist $k_{m,N} > 0$ such thnat

$$\begin{aligned} &\lim_{\lambda \to 0^+} \lim_{\varepsilon \to 0^+} \int\limits_{\Omega_\lambda} \left(\int\limits_{\Omega_\lambda} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|} \right) \rho_\varepsilon(x-y) \, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) = \\ &= k_{m,N} \int\limits_\Omega \Phi(|\nabla f(x)|) \, d\mathcal{L}^N(x). \end{aligned} \tag{1.8}$$

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3 and H-4, let $f \in L^\Phi_{loc}(\Omega)$. Assume*

$$\limsup_{\varepsilon \to 0^+} \int\limits_\Omega \int\limits_\Omega \Phi\left(\frac{|f(x) - f(y)|}{d_\Omega(x,y)} \right) \rho_\varepsilon(d_\Omega(x,y)) \, d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) < +\infty. \tag{1.9}$$

then $f \in W^1 L^\Phi_{loc}(\Omega)$.

The Theorem 1.2 and the Theorem 1.3 are an alternative characterization of the spaces of Orlicz Sobolev using non local relations, such relations are at the base of numerous results for not local functionals with standard growths and for functionals defined on fractional Sobolev space, see (Bourgain et al., 2001; Di Castro et al., 2016; Di Castro et al., 2014; Maz'ya & Shaposhnikova, 2002; Mengesha & Spector; Milman, 2005; Ponce, 2004; Schikarra et al.; Shieh & Spector). The generalization of these theorems seems to point out the possibility to also extend such results in more general cases.

2. N-function and Orlicz Spaces

Definition 2.1. *A continuous and convex function $\Phi : [0, +\infty) \to [0, +\infty)$ is called N-function (or Young function) if it satisfies*

$$\begin{aligned} &\Phi(0) = 0 \text{ and } \Phi(t) > 0 \text{ if } t > 0; \\ &\lim_{t \to o^+} \tfrac{\Phi(t)}{t} = 0; \\ &\lim_{t \to +\infty} \tfrac{\Phi(t)}{t} = +\infty. \end{aligned} \tag{2.1}$$

For exemple the function $\Phi_{p,\beta}(t) = t^p \ln^\beta (1+t)$, for $p > 1$ and $\beta \geq 0$ or $p = 1$ and $\beta > 0$, is a N-function.

Actually, only the growth at infinity really matters in the definition of N-function.

Indeed, given a continuous and convex function $A : [0, +\infty) \to [0, +\infty)$ satisfying

$$\lim_{t\to+\infty} \frac{A(t)}{t} = +\infty$$

there exist a N-function Φ and $t_0 > 0$ such that for every $t > t_0$ there holds

$$A(t) = \Phi(t).$$

The function A is called principal part of the N-function Φ. For exemple there exists a N-function Φ such that $\Phi(t) = t^{\ln(t)}$ near infinity or there exists a N-function Φ such that $\Phi(t) = t\ln(t)$ near infinity.

The function A is called principal part of the N-function Φ. For exemple there exists a N-function Φ such that $\Phi(t) = t^{\ln(t)}$ near infinity or there exists a N-function Φ such that $\Phi(t) = t\ln(t)$ near infinity.

Definition 2.2. *If Φ_1 and Φ_2 are two N-functions we say that Φ_1 dominates Φ_2 near infinity if there exists positive constants $\varkappa$ and t_0 such that*

$$\Phi_2(t) \leq \Phi_1(\varkappa t)$$

for all $t \geq t_0$.

Definition 2.3. *If Φ_1 and Φ_2 are two N-functions we say that Φ_1 and Φ_2 are equivalent near infinity ($\Phi_1 \sim \Phi_2$) if and only if there exists positive constants $\varkappa_1$, $\varkappa_2$ and t_0 such that*

$$\Phi_1(\varkappa_1 t) \leq \Phi_2(t) \leq \Phi_1(\varkappa_2 t)$$

for all $t \geq t_0$.

Remark 2.4. *If $0 < \lim\limits_{t\to+\infty} \frac{\Phi_1(t)}{\Phi_2(t)} < +\infty$ then Φ_1 and Φ_2 are equivalent near infinity.*

Let us introduce two important classes of N-functions.

Definition 2.5. *A N-function Φ is of class $\triangle_2$ ($\Phi \in \triangle_2$) if exist $k > 1$ and $t_0 > 0$ such that*

$$\Phi(2t) \leq k\Phi(t) \qquad \forall t \in (t_0, +\infty). \tag{2.2}$$

Definition 2.6. *A N-function Φ is of class $\triangle_2^m$ $\left(\Phi \in \triangle_2^m\right)$, with $m > 1$, if exists $t_0 > 0$ such that for every $\lambda > 1$*

$$\Phi(\lambda t) \leq \lambda^m \Phi(t) \qquad \forall t \in (t_0, +\infty). \tag{2.3}$$

Definition 2.7. *A N-function Φ is of class $\triangle_2$ globally in $(0, +\infty)$ if (2.2) holds for every $t > 0$.*

Definition 2.8. *A N-function Φ is of class $\triangle_2^m$ globally in $(0, +\infty)$, with $m > 1$, if (2.3) holds for every $t > 0$.*

Remark 2.9. *If Φ is a N-function and $\Phi \in \triangle_2$ then there exists a N-function Φ_1 such that $\Phi \sim \Phi_1$ and $\Phi_1 \in \triangle_2$ globally in $(0, +\infty)$, see (Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991).*

Definition 2.10. *A N-function Φ is of class ∇_2 ($\Phi \in \nabla_2$) if exist $l > 1$ and $t_0 > 0$ such that*

$$\Phi(t) \leq \frac{\Phi(lt)}{2l} \qquad \forall t \in (t_0, +\infty). \tag{2.4}$$

Definition 2.11. *A N-function Φ is of class ∇_2^r $\left(\Phi \in \nabla_2^r\right)$, with $r > 1$, if exists $t_0 > 0$ such that for every $\lambda > 1$*

$$\lambda^r \Phi(t) \leq \Phi(\lambda t) \qquad \forall t \in (t_0, +\infty). \tag{2.5}$$

Definition 2.12. *A N-function Φ is of class ∇_2 globally in $(0, +\infty)$ if (2.4) holds for every $t > 0$.*

Definition 2.13. *A N-function Φ is of class ∇_2^r globally in $(0, +\infty)$, with $r > 1$, if (2.5) holds for every $t > 0$.*

Remark 2.14. *If Φ is a N-function and $\Phi \in \nabla_2$ then there exists a N-function Φ_1 such that $\Phi \sim \Phi_1$ and $\Phi_1 \in \nabla_2$ globally in $(0, +\infty)$, see (Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991).*

The N-functions $\Phi \in \triangle_2^m$ are characterized by the following result.

Lemma 2.15. *Let Φ be a N-function and let $\dot{\Phi}_+$ be its right derivative. For $m > 1$ the following properties are equivalent:*

(i) $\Phi(\lambda t) \leq \lambda^m \Phi(t)$*, for every* $t > 00$*, for every* $\lambda > 1$*;*

(ii) $t\dot{\Phi}_+(t) \leq m\Phi(t)$*, for every* $t > 0$*;*

(iii) *the function* $\frac{\Phi(t)}{t^m}$ *is nonincreasing on* $(0, +\infty)$.

Proof. See (Dall'Aglio et al., 1998; Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991). □

The N-functions $\Phi \in \nabla_2^r$ are characterized by the following result.

Lemma 2.16. *Let Φ be a N-function and let $\dot{\Phi}_-$ be its left derivative. For $r > 1$ the following properties are equivalent:*

(i)' $\Phi(\lambda t) \geq \lambda^r \Phi(t)$*, for every* $t > 0$*, for every* $\lambda > 1$*;*

(ii)' $t\dot{\Phi}_-(t) \geq r\Phi(t)$*, for every* $t > 0$*;*

(iii)' *the function* $\frac{\Phi(t)}{t^r}$ *is nondecreasing on* $(0, +\infty)$.

Proof. See (Dall'Aglio et al., 1998; Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991). □

Remark 2.17. *We observe that*

$$\Delta_2 = \bigcup_{m>1} \triangle_2^m$$

and

$$\nabla_2 = \bigcup_{r>1} \nabla_2^r.$$

Remark 2.18. *If Φ is a N-function then $t\dot{\Phi}_-(t) \geq r\Phi(t)$ for every $t > 0$. By Lemma 2.16 it follows that $\Phi \in \nabla_2^1$ on $(0, +\infty)$.*

Now we give some alternative characterizations of the N-funtions of class $\triangle_2^m$ globally in $(0, +\infty)$ and of class ∇_2^r globally in $(0, +\infty)$.

Proposition 2.19. *Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$ if and only if $\Phi^{-1}(aw) \leq a^{\frac{1}{m}}\Phi^{-1}(w)$ for every $w \in (0, +\infty)$ and $a \in (0, 1)$.*

Proof. If Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$, then we have $\Phi(\lambda t) \leq \lambda^m \Phi(t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^m} \leq \Phi\left(\frac{s}{\lambda}\right)$ and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^m}\right) \leq \frac{s}{\lambda}$ for every $s \in (0, +\infty)$ and $\lambda > 1$. Let us put $s = \Phi^{-1}(w)$ then we have $\Phi^{-1}\left(\frac{w}{\lambda^m}\right) \leq \frac{\Phi^{-1}(w)}{\lambda}$ for every $w \in (0, +\infty)$ and $\lambda > 1$. Let us put $\frac{1}{\lambda^m} = a$ then we have $\Phi^{-1}(aw) \leq a^{\frac{1}{m}}\Phi^{-1}(w)$ for every $w \in (0, +\infty)$ and $a \in (0, 1)$. The converse follows in a similar manner. □

Proposition 2.20. *Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$ if and only if $\lambda\Phi^{-1}(s) \leq \Phi^{-1}(\lambda^m s)$ for every $s \in (0, +\infty)$ and $\lambda > 1$.*

Proof. If Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$, then $\Phi(\lambda t) \leq \lambda^m \Phi(t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. It follows that $\lambda t \leq \Phi^{-1}(\lambda^m \Phi(t))$ then if $t = \Phi^{-1}(s)$ we get $\lambda\Phi^{-1}(s) \leq \Phi^{-1}(\lambda^m s)$ for every $s \in (0, +\infty)$ and $\lambda > 1$. The converse follows in a similar manner. □

Now we get the following characterization of the N-functions of class $\triangle_2^m$ globally in $(0, +\infty)$, this characterization is not present in the bibliography note to the author and we will use it in the paper.

Proposition 2.21. *Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$ if and only if $a^m \Phi(s) \leq \Phi(as)$ for every $s \in (0, +\infty)$ and $a \in (0, 1)$.*

Proof. If Φ is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$, then $\Phi(\lambda t) \leq \lambda^m \Phi(t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. It follows that $\frac{1}{\lambda^m}\Phi(\lambda t) \leq \Phi(t)$ then if $t = \frac{s}{\lambda}$ we get $\frac{1}{\lambda^m}\Phi(s) \leq \Phi\left(\frac{s}{\lambda}\right)$ for every $s \in (0, +\infty)$ and $\lambda > 1$. If we put $a = \frac{1}{\lambda} \in (0, 1)$ we get $a^m \Phi(s) \leq \Phi(as)$ for every $s \in (0, +\infty)$ and $a \in (0, 1)$. The converse follows in a similar manner. □

Now we get the following characterization of the N-functions of class ∇_2^r globally in $(0, +\infty)$, this characterization is not present in the bibliography note to the author.

Proposition 2.22. *Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$ if and only if*

$$\Phi^{-1}\left(\frac{w}{\lambda^r}\right) \geq \frac{\Phi^{-1}(w)}{\lambda} \tag{2.6}$$

for every $w \in (0, +\infty)$ and $\lambda > 1$.

Proof. If Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$, then we have $\lambda^r \Phi(t) \leq \Phi(\lambda t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^r} \geq \Phi\left(\frac{s}{\lambda}\right)$ and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^r}\right) \geq \frac{s}{\lambda}$ for every $s \in (0, +\infty)$ and $\lambda > 1$. Let us put $s = \Phi^{-1}(w)$ then we have $\Phi^{-1}\left(\frac{w}{\lambda^r}\right) \geq \frac{\Phi^{-1}(w)}{\lambda}$ for every $w \in (0, +\infty)$ and $\lambda > 1$. The converse follows in a similar manner. □

Remark 2.23. *If we choose $\lambda = \kappa^{\frac{1}{r}} > 1$ then we can write the inequality (2.6) this way*

$$\Phi^{-1}\left(\frac{w}{\kappa}\right) \geq \frac{\Phi^{-1}(w)}{\kappa^{\frac{1}{r}}} \tag{2.7}$$

for every $w \in (0, +\infty)$ and $\kappa > 1$.

Another characterization of the functions N-function of class ∇_2^r globally in $(0, +\infty)$, it is the following.

Proposition 2.24. *Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$ if and only if $\Phi(at) \leq a^r \Phi(t)$ for every $t \in (0, +\infty)$ and $a \in (0, 1)$.*

Proof. If Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$, then we have $\lambda^r \Phi(t) \leq \Phi(\lambda t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. If we put $t = \frac{s}{\lambda}$ we get $\Phi\left(\frac{s}{\lambda}\right) \leq \frac{1}{\lambda^r}\Phi(s)$, then if $a = \frac{1}{\lambda}$ it follows that $\Phi(as) \leq a^r \Phi(s)$ for every $s \in (0, +\infty)$ and $a \in (0, 1)$. The converse follows in a similar manner. □

Proposition 2.25. *Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$ if and only if $\Phi^{-1}(aw) \geq a^{\frac{1}{r}} \Phi^{-1}(w)$ for every $w \in (0, +\infty)$ and $a \in (0, 1)$.*

Proof. If Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$, then we have $\lambda^r \Phi(t) \leq \Phi(\lambda t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. Let us put $t = \frac{s}{\lambda}$ then we have $\frac{\Phi(s)}{\lambda^r} \geq \Phi\left(\frac{s}{\lambda}\right)$ and $\Phi^{-1}\left(\frac{\Phi(s)}{\lambda^r}\right) \geq \frac{s}{\lambda}$ for every $s \in (0, +\infty)$ and $\lambda > 1$. Let us put $s = \Phi^{-1}(w)$ then we have $\Phi^{-1}\left(\frac{w}{\lambda^r}\right) \geq \frac{\Phi^{-1}(w)}{\lambda}$ for every $w \in (0, +\infty)$ and $\lambda > 1$. Let us put $\frac{1}{\lambda^r} = a$ then we have $\Phi^{-1}(aw) \geq a^{\frac{1}{r}} \Phi^{-1}(w)$ for every $w \in (0, +\infty)$ and $a \in (0, 1)$. The converse follows in a similar manner. □

Proposition 2.26. *Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$ if and only if $\Phi^{-1}(\lambda^r s) \leq \lambda \Phi^{-1}(s)$ for every $s \in (0, +\infty)$ and $\lambda > 1$.*

Proof. f Φ is a N-function of class ∇_2^r globally in $(0, +\infty)$, then we have $\lambda^r \Phi(t) \leq \Phi(\lambda t)$ for every $t \in (0, +\infty)$ and $\lambda > 1$. It follows that $\Phi^{-1}(\lambda^r \Phi(t)) \leq \lambda t$ then if $t = \Phi^{-1}(s)$ we get $\Phi^{-1}(\lambda^r s) \leq \lambda \Phi^{-1}(s)$ for every $s \in (0, +\infty)$ and $\lambda > 1$. The converse follows in a similar manner. □

Remark 2.27. *f Φ is a N-function and $\Phi \in \triangle_2$ then there exists a N-function Φ_1 such that $\Phi \sim \Phi_1$ and Φ_1, is a N-function of class $\triangle_2^m$ globally in $(0, +\infty)$.*

Lemma 2.28. *Let $g(t)$, $h(t)$ be a non-negative and increasing functions on $[0, +\infty)$ then*

$$g(t)\, h(s) \leq g(t)\, h(t) + g(s)\, h(s)$$

for every s, $t \in [0, +\infty)$.

Proof. If $s \leq t$ then $g(t)\, h(s) \leq g(t)\, h(t) \leq g(t)\, h(t) + g(s)\, h(s)$. If $t \leq s$ then $g(t)\, h(s) \leq g(s)\, h(s) \leq g(t)\, h(t) + g(s)\, h(s)$. □

Remark 2.29. *Since $\dot{\Phi}(t) \leq \dot{\Phi}_{+}(t)$ for every $t > 0$, where $\dot{\Phi}(t)$ is the weak derivative of Φ and $\dot{\Phi}_{+}(t)$ is the right derivative of Φ, then by Lemma 2 and Lemma 4 we have*

$$a\dot{\Phi}(b) \leq a\dot{\Phi}_{+}(b) \leq a\dot{\Phi}_{+}(a) + b\dot{\Phi}_{+}(b) \leq m(\Phi(a) + \Phi(b)).$$

Now we can introduce Orlicz spaces and Orlicz Sobolev Spaces, L^{Φ} and W^1L^{Φ}. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded and open set, the Orlicz class $K^{\Phi}(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ (equivalence classes modulo equality $\mathcal{L}^N$ a.e. in Ω) satisfying $\int_{\Omega} \Phi(|u|)\, d\mathcal{L}^N < +\infty$. The Orlicz space $L^{\Phi}(\Omega)$ is defined to be the linear hull of $K^{\Phi}(\Omega)$, thus it consists of all measurable functions u such that $\lambda u \in K^{\Phi}(\Omega)$ for some $\lambda > 0$. Moreover, the equality $K^{\Phi}(\Omega) \equiv L^{\Phi}(\Omega)$ holds if and only if $\Phi \in \triangle_2$.

Definition 2.30. *If $\Omega \subset \mathbb{R}^N$ is a bounded open set and $\Phi \in \triangle_2$ then we define*

$$W^1L^{\Phi}(\Omega) = \left\{u \in L^{\Phi}(\Omega) : \partial_i u \in L^{\Phi}(\Omega) \text{ for } i = 1, ..., N\right\}$$

where $\partial_i u$ are the weak derivatives of u for $i = 1, ..., N$.

Theorem 2.31. *Let $\Phi \in \triangle_2$, then $L^{\Phi}(\Omega)$ and $W^1L^{\Phi}(\Omega)$ are Banach spaces with the following norms*

$$\|u\|_{\Phi,\Omega} = \inf\left(k > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{k}\right) d\mathcal{L}^N \leq 1\right)$$

and

$$\|u\|_{1,\Phi,\Omega} = \|u\|_{\Phi,\Omega} + \sum_{i=1}^{N} \|\partial_i u\|_{\Phi,\Omega}.$$

We observe that if $\Phi(t) = t^p$, with $p > 1$, then $\|u\|_{\Phi,\Omega} = \|u\|_{p,\Omega}$, where $\|u\|_{p,\Omega} = \int_{\Omega} |u|^p\, d\mathcal{L}^N$. In general, however, so simple relationships do not be had among the Luxemburg norm $\|u\|_{\Phi,\Omega}$ and the integral $\int_{\Omega} \Phi(|u|)\, d\mathcal{L}^N$, this creates some difficulties to use the Luxemburg norm and the Hölder inequality then we are forced to introduce some suitable tricks to proceed. For greater details on Orlicz spaces, Orlicz-Sobolev spaces and Luxemburg norm we refer (Adams, 1975; Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991)

3. Lemmas

Fix $\psi \in C_c^{\infty}\left(\mathbb{R}^N\right)$ such that $\int_{\mathbb{R}^N} \psi\, dx = 1$ and $\mathrm{supp}(\psi) \Subset B_1(0)$. For $\delta > 0$ define

$$\psi_{\delta}(x) = \frac{1}{\delta^N}\psi\left(\frac{x}{\delta}\right)$$

Given a open set $\Omega \subset \mathbb{R}^N$ and a function $f \in L^1_{loc}(\Omega)$, for every $x \in \Omega_{\delta} = \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$ define the mollification of the function f by

$$f_{\delta}(x) = (f * \psi_{\delta})(x) = \int_{\mathbb{R}^N} f(y)\psi_{\delta}(x-y)\, dy$$

Remark 3.1. *Since Φ is a convex function then*

$$\Phi(u) \geq \Phi(a) + \dot{\Phi}(a)(u-a)$$

for every $u, a > 0$. Since $\psi_{\delta}(x-y) \geq 0$ we get

$$\psi_{\delta}(x-y)\Phi(u) \geq \psi_{\delta}(x-y)\Phi(a) + \psi_{\delta}(x-y)\dot{\Phi}(a)(u-a)$$

If we choose $u = |f(y)|$ and $a = \int_{\mathbb{R}^N} |f(z)|\psi_{\delta}(x-z)\, dz$ and if we integrate on $\mathbb{R}^N$, since $\int_{\mathbb{R}^N} \psi_{\delta}(z-y)\, dz = 1$, it follows

$$\int_{\mathbb{R}^N} \Phi(|f(y)|\psi_{\delta}(x-y))\, dy \geq \Phi\left(\int_{\mathbb{R}^N} |f(y)|\psi_{\delta}(x-y)\, dy\right)$$

for every $x \in \Omega$ with $dist(x, \partial\Omega) > \delta$.

Lemma 3.2. *Let ρ_ε satisfy H-2 and H-3, let $\{\mu_\varepsilon\} \subset \mathcal{M}\left(S^{N-1}\right)$ the measure defined by*

$$\mu_\varepsilon(F) = \int_F \int_0^{+\infty} \rho_\varepsilon(t\sigma)\, t^{N-1}\, dt\, d\mathcal{H}^{N-1}(\sigma) \tag{3.1}$$

for every Borel subset $F \subset S^{N-1}$. Then there exist a subsequence ε_j, with $\varepsilon_j \to 0^+$, $\mu \in \mathcal{M}\left(S^{N-1}\right)$ such that $\mu_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$ for $\varepsilon_j \to 0^+$. Moreover, for every N-function $\Phi \in \Delta_2$, there exists $\alpha_\Phi > 0$ such that for every $v \in \mathbb{R}^N$ we have

$$\int_{S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu(\sigma) \geq \alpha_\Phi \Phi(|v|). \tag{3.2}$$

Proof. Using polar coordinates and H-2 we have

$$\mu_\varepsilon\left(S^{N-1}\right) = \int_{S^{N-1}} \int_0^{+\infty} \rho_\varepsilon(t\sigma)\, t^{N-1}\, dt\, d\mathcal{H}^{N-1}(\sigma) = \int_{\mathbb{R}^N} \rho_\varepsilon\, d\mathcal{L}^N = 1$$

then $\|\mu_\varepsilon\|_{\mathcal{M}(S^{N-1})} = 1$ and so there exists $\mu \in \mathcal{M}\left(S^{N-1}\right)$ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$ and $\|\mu\|_{\mathcal{M}(S^{N-1})} = 1$. Let $\{v\}_{i=1,\ldots,N}$ be the linearly independent set of vectors given in H-3, We claim there exists $\varepsilon_0 > 0$ with the property that for all $v \in \mathbb{R}^N$ there exists an i such that

$$|v \cdot \sigma| \geq \varepsilon_0 |v|$$

for all $\sigma \in C_\delta(v_i) \cap S^{N-1}$. By rescaling we restrict ourselves to the case $v \in S^{N-1}$, and we proceed by contradiction. If not, then there exist a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ tending to zero, $w_k \in S^{N-1}$ and $\sigma_{i,k} \in C_\delta(v_i)$, $i = 1, ..., N$, so that up to a subsequence, which we will not relabel, $w_k \to w_0 \in S^{N-1}$ and $\sigma_{i,k} \to \sigma_{i,0} \in C_\delta(v_i)$, with $\left|w_0 \cdot \sigma_{i,0}\right| = 0$ for all $i = 1, ..., N$. However, since $\{\sigma_{i,0}\}_{i=1,\ldots,N}$ form a linearly independet set, (see Remark 1.4 of (Leoni & Spector, 2011)), we have a contradiction. Definne

$$c = \min_{i=1,\ldots,N} \liminf_{j\to+\infty} \int_{C_\delta(v_i)} \rho_{\varepsilon_j}\, d\mathcal{L}^N$$

then $c > 0$. Given $v \in \mathbb{R}^N$, let i such that $|v \cdot \sigma| \geq \varepsilon_0 |v|$ for all $\sigma \in C_\delta(v_i) \cap S^{N-1}$, then

$$\begin{aligned}
\int_{S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu_{\varepsilon_j}(\sigma) &\geq \int_{C_\delta(v_i)\cap S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu_{\varepsilon_j}(\sigma) \\
&\geq \int_{C_\delta(v_i)\cap S^{N-1}} \Phi(\varepsilon_0 |v|)\, d\mu_{\varepsilon_j}(\sigma) \\
&= \Phi(\varepsilon_0 |v|)\, \mu_{\varepsilon_j}\left(C_\delta(v_i) \cap S^{N-1}\right) \\
&= \Phi(\varepsilon_0 |v|) \int_{C_\delta(v_i)\cap\cap S^{N-1}} \int_0^{+\infty} \rho_\varepsilon(t\sigma)\, t^{N-1}\, dt\, d\mathcal{H}^{N-1}(\sigma)
\end{aligned}$$

By Tonelli's theorem we get

$$\int_{S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu_{\varepsilon_j} \geq \Phi(\varepsilon_0 |v|) \int_{C_\delta(v_i)} \rho_{\varepsilon_j}\, d\mathcal{L}^N \geq c\Phi(\varepsilon_0 |v|)$$

Letting $j \to +\infty$, since $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$, it follows

$$\int_{S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu(\sigma) \geq c\Phi(\varepsilon_0 |v|)$$

since Φ is a N function, then by Remark refR1 $\Phi \in \nabla_2^1$ and using Lemma 2.16 (i) and Proposition 2.21 we have

$$\int_{S^{N-1}} \Phi(|v \cdot \sigma|)\, d\mu(\sigma) \geq c(\varepsilon_0)\, \Phi(|v|)$$

where $c(\varepsilon_0) = c \min\left\{\varepsilon_0, \varepsilon_0^m\right\}$ □

Definition 3.3. *For every fixed $\eta > 0$ we define*

$$\rho_\varepsilon^\eta = \rho_\varepsilon \chi_{B_\eta(0)}. \tag{3.3}$$

We hav the following properties of ρ_ε^η

$$\rho_\varepsilon^\eta < \rho_\varepsilon, \tag{3.4}$$

$$\rho_\varepsilon^\eta \geq 0, \quad \int_{\mathbb{R}^N} \rho_\varepsilon^\eta \, d\mathcal{L}^N \leq 1, \tag{3.5}$$

$$\lim_{\varepsilon \to 0^+} \int_{|x|>\delta} \rho_\varepsilon^\eta \, d\mathcal{L}^N = 0 \quad \text{for all } \delta > 0, \tag{3.6}$$

$$\lim_{\varepsilon \to 0^+} \int_E |x| \rho_\varepsilon^\eta (x) \, d\mathcal{L}^N (x) = 0 \tag{3.7}$$

for every $E \subset \mathbb{R}^N$ bounded and measurable. Now we can define the measure

$$\mu_{\varepsilon,\eta}(F) = \int_F \int_0^{+\infty} \rho_\varepsilon^\eta (t\sigma) \, t^{N-1} \, dt \, d\mathcal{H}^{N-1}(\sigma) \tag{3.8}$$

for every Borel subset $F \subset S^{N-1}$. Applying the Randon Nikodym theorem, for $\mathcal{H}^{N-1}$ a.e. $\sigma \in S^{N-1}$

$$\frac{d\mu_{\varepsilon,\eta}}{d\mathcal{H}^{N-1}}(\sigma) = \int_0^{+\infty} \rho_\varepsilon^\eta (t\sigma) \, t^{N-1} \, dt = \int_0^{\eta} \rho_\varepsilon (t\sigma) \, t^{N-1} \, dt. \tag{3.9}$$

Lemma 3.4. *Let ρ_ε satisfy H-2, Let $\{\mu_\varepsilon\} \subset \mathcal{M}\left(S^{N-1}\right)$ the measure defined in (3.1). If $\mu_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu \in \mathcal{M}\left(S^{N-1}\right)$ for $\varepsilon_j \to 0^+$ then, for every $\eta > 0$, $\mu_{\varepsilon,\eta} \overset{*}{\rightharpoonup} \mu \in \mathcal{M}\left(S^{N-1}\right)$, where $\mu_{\varepsilon,\eta}$ are the measures defined in (3.8).*

Proof. We begin by proving that $\mu_{\varepsilon_j,\eta} - \mu_{\varepsilon_j} \to 0$ in $\mathcal{M}\left(S^{N-1}\right)$. For $f \in C\left(S^{N-1}\right)$, with $\|f\|_\infty = 1$ we have

$$\begin{aligned}
\left| \int_{S^{N-1}} f \, d\mu_{\varepsilon_j,\eta} - \int_{S^{N-1}} f \, d\mu_{\varepsilon_j} \right| &= \left| \int_{S^{N-1}} \int_\eta^{+\infty} f(\sigma) \rho_\varepsilon^\eta (t\sigma) \, t^{N-1} \, dt \, d\mathcal{H}^{N-1}(\sigma) \right| \\
&\leq \int_{S^{N-1}} \int_\eta^{+\infty} \rho_{\varepsilon_j} (t\sigma) \, t^{N-1} \, dt \, d\mathcal{H}^{N-1}(\sigma) \\
&= \int_{|x|>\eta} \rho_{\varepsilon_j}(x) \, d\mathcal{L}^N
\end{aligned}$$

then

$$\left\| \mu_{\varepsilon_j,\eta} - \mu_{\varepsilon_j} \right\|_{\mathcal{M}(S^{N-1})} \leq \int_{|x|>\eta} \rho_{\varepsilon_j}(x) \, d\mathcal{L}^N \to 0$$

for $j \to +\infty$, thus $\mu_{\varepsilon_j,\eta} - \mu_{\varepsilon_j} \to 0$ in $\mathcal{M}\left(S^{N-1}\right)$. Since $\mu_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$, it follows that $\mu_{\varepsilon_j,\eta} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$. □

Remark 3.5. *By the definition (3.3) of ρ_ε^η we hace the following properties*

$$\begin{aligned}
&\rho_\varepsilon^\eta \geq 0 \\
&\int_{\mathbb{R}^N} \rho_\varepsilon^\eta (x) \, d\mathcal{L}^N \leq 1 \\
&\lim_{\varepsilon \to 0^+} \int_{|x|>\delta} \rho_\varepsilon^\eta (x) \, d\mathcal{L}^N = 0 \quad \textit{for all } \delta > 0 \\
&\lim_{\varepsilon \to 0^+} \int_E |x| \rho_\varepsilon^\eta (x) \, d\mathcal{L}^N = 0 \quad \textit{for every } E \subset \mathbb{R}^N \textit{ bounded and measurable.}
\end{aligned}$$

Definition 3.6. *We define*

$$E^r = \left\{ x \in \mathbb{R}^N : dist(x, E) < r \right\} \tag{3.10}$$

and

$$E_r = \left\{ x \in \mathbb{R}^N : |x| < \frac{1}{r}; \, dist(x, \partial E) > r \right\} \tag{3.11}$$

Lemma 3.7. *Let $A \subset \mathbb{R}^N$ be open and bounded and let $f \in C^2\left(\overline{A^\eta}\right)$ for some $\eta > 0$, then*

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \le C_f |x-y|^2 \tag{3.12}$$

for all $x \in A$ and $y \in A^\eta$ where $C_f > 0$ depends upon $\|f\|_{C^2\left(\overline{A^\eta}\right)}$.

Proof. See (Leoni & Spector, 2011). □

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3, let $A \subset \Omega$ be open and bounded with $dist(A, \partial\Omega) > 0$, let $f \in C^2\left(\overline{A^\eta}\right)$ where $0 < \eta < dist(A, \partial\Omega)$, let $\varepsilon_j \to 0^+$ and assume that $\mu_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}\left(S^{N-1}\right)$; then for every $x \in A$ we have*

$$\lim_{j\to+\infty} \int_{A^\eta} \Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) = \int_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|)\, d\mu(\sigma) \tag{3.13}$$

where $\rho^\eta_{\varepsilon_j}$ is the family introduced in (3.3).

Proof. Set $M_f = \|\nabla f\|_{L^\infty(\bar{A}^\eta)}$, let $0 \le s, t \le M_f$ then $|\Phi(s) - \Phi(t)| \le \dot{\Phi}(\xi)|s-t|$, choose $s = \left|\frac{f(x)-f(y)}{|x-y|}\right|$ and $t = \left|\nabla f(x) \frac{x-y}{|x-y|}\right|$ then

$$\begin{aligned}
\left|\Phi\left(\left|\frac{f(x)-f(y)}{x-y}\right|\right) - \Phi\left(\left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right)\right| &\le \dot{\Phi}\left(M_f\right)\left|\left|\frac{f(x)-f(y)}{|x-y|}\right| - \left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right| \\
&\le \dot{\Phi}\left(M_f\right)\left|\frac{f(x)-f(y)-\nabla f(x)(x-y)}{|x-y|}\right|
\end{aligned}$$

Using Lemma 3.7 it follows

$$\left|\Phi\left(\left|\frac{f(x)-f(y)}{x-y}\right|\right) - \Phi\left(\left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right)\right| \le \dot{\Phi}\left(M_f\right) C_f |x-y|$$

therefore

$$\begin{aligned}
&\int_{A^\eta} \left|\Phi\left(\left|\tfrac{f(x)-f(y)}{x-y}\right|\right) - \Phi\left(\left|\nabla f(x) \tfrac{x-y}{|x-y|}\right|\right)\right| \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) \le \\
&\le \dot{\Phi}\left(M_f\right) C_f \int_{A^\eta} |x-y| \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)
\end{aligned}$$

By Remark 3.5 we get

$$\begin{aligned}
&\limsup_{j\to+\infty} \int_{A^\eta} \Phi\left(\left|\tfrac{f(x)-f(y)}{x-y}\right|\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) = \\
&= \limsup_{j\to+\infty} \int_{A^\eta} \Phi\left(\left|\nabla f(x) \tfrac{x-y}{|x-y|}\right|\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)
\end{aligned}$$

Since $\rho^\eta_{\varepsilon_j}(x-y) = 0$ if $|x-y| > \eta$ then

$$\begin{aligned}
\int_{A^\eta} \Phi\left(\left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) &= \int_{B_\eta(x)} \Phi\left(\left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) \\
&= \int_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|) \int_\eta^{+\infty} \rho^\eta_\varepsilon(t\sigma)\, t^{N-1}\, dt\, d\mathcal{H}^{N-1}(\sigma) \\
&= \int_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|)\, d\mu_{\varepsilon_j,\eta}(\sigma)
\end{aligned}$$

Since $\Phi(|\nabla f(x) \cdot \sigma|)$ is continuous by Lemma 3.4 it follows

$$\lim_{j\to+\infty} \int_{A^\eta} \Phi\left(\left|\nabla f(x) \frac{x-y}{|x-y|}\right|\right) \rho^\eta_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) = \int_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|)\, d\mu(\sigma).$$

□

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3, let $A \subset \Omega$ be open and bounded with $dist(A, \partial\Omega) > 0$, let $f \in W^1L^\Phi(\Omega)$; then for all $0 < \eta < \frac{1}{3} dist(A, \partial\Omega)$ we have*

$$\begin{aligned}&\int_{A^\eta}\left(\int_{A^\eta}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \leq \\ &\leq \int_{A^{2\eta}}\left(\int_{B_\eta(0)}\Phi\left(\left|\nabla f(x)\cdot\tfrac{h}{|h|}\right|\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x).\end{aligned} \tag{3.14}$$

Proof. Making the change of variables $y = x + h$, since $\rho_{\varepsilon_j}^\eta(y-x) = 0$ if $|y-x| > \eta$ then

$$\begin{aligned}&\int_{A^\eta}\left(\int_{A^\eta}\Phi\left(\tfrac{|f(y)-f(x)|}{|y-x|}\right)\rho_\varepsilon^\eta(y-x)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)\\ &= \int_{A^\eta}\left(\int_{B_\eta(x)}\Phi\left(\tfrac{|f(x+h)-f(x)|}{|h|}\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)\end{aligned}$$

For $0 < \delta < \eta < \frac{dist(A,\partial\Omega)}{3}$ the function f_δ is well defined in A^η, then we have

$$\begin{aligned}&\int_{A^\eta}\left(\int_{A^\eta}\Phi\left(\tfrac{|f_\delta(y)-f_\delta(x)|}{|y-x|}\right)\rho_\varepsilon^\eta(y-x)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) = \\ &= \int_{A^\eta}\left(\int_{B_\eta(0)}\Phi\left(\tfrac{|f_\delta(x+h)-f_\delta(x)|}{|h|}\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)\\ &= \int_{A^\eta}\left(\int_{B_\eta(0)}\Phi\left(\int_0^1\left|\nabla f_\delta(x+th)\cdot\tfrac{h}{|h|}\right|dt\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)\\ &= I\end{aligned}$$

Using Jensen's inequality we get

$$I \leq \int_{A^\eta}\left(\int_{B_\eta(0)}\int_0^1\Phi\left(\left|\nabla f_\delta(x+th)\cdot\frac{h}{|h|}\right|\right)dt\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)$$

Since $|h| < \eta$ by Tonelli's theorem it follows

$$I \leq \int_{A^{2\eta}}\left(\int_{B_\eta(0)}\Phi\left(\left|\nabla f_\delta(y)\cdot\frac{h}{|h|}\right|\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(y)$$

and

$$\begin{aligned}&\int_{A^\eta}\left(\int_{A^\eta}\Phi\left(\tfrac{|f_\delta(y)-f_\delta(x)|}{|y-x|}\right)\rho_\varepsilon^\eta(y-x)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \leq \\ &\leq \int_{A^{2\eta}}\left(\int_{B_\eta(0)}\Phi\left(\left|\nabla f_\delta(y)\cdot\tfrac{h}{|h|}\right|\right)\rho_\varepsilon^\eta(h)\,d\mathcal{L}^N(h)\right)d\mathcal{L}^N(y)\end{aligned}$$

letting $\delta \to 0$ by Fatou's lemma and Lebesgue dominated convergence theorem we obtain (3.14). □

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3, let $A \subset \Omega$ be open and bounded with $dist(A, \partial\Omega) > 0$, let $f \in L^\Phi(\Omega)$; then for all $0 < \delta < \eta < \frac{1}{3} dist(A, \partial\Omega)$ we have*

$$\begin{aligned}&\int_{A}\left(\int_{A}\Phi\left(\tfrac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \leq \\ &\int_{A^\eta}\left(\int_{A^\eta}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\,d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)\end{aligned} \tag{3.15}$$

where f_δ is the mollification of f.

Proof. Let us consider

$$I_0 = \int\limits_A \left(\int\limits_A \Phi\left(\frac{|f_\delta(x) - f_\delta(y)|}{|x-y|} \right) \rho_\varepsilon^\eta(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x)$$

then, we get

$$I_0 = \int\limits_A \left(\int\limits_A \Phi\left(\frac{\left| \int\limits_{B_\delta(0)} \psi_\delta(z) f(x-z) - f(y-z)\, dz \right|}{|x-y|} \right) \rho_\varepsilon^\eta(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x)$$

and

$$I_0 \leq \int\limits_A \left(\int\limits_A \Phi\left(\int\limits_{B_\delta(0)} \psi_\delta(z) \frac{|f(x-z) - f(y-z)|}{|x-y|}\, dz \right) \rho_\varepsilon^\eta(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x).$$

Since Φ is a convex function, using Remark 3.1, it follows

$$I_0 \leq \int\limits_A \left(\int\limits_A \left(\int\limits_{B_\delta(0)} \psi_\delta(z) \Phi\left(\frac{|f(x-z) - f(y-z)|}{|x-y|} \right) dz \right) \rho_\varepsilon^\eta(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x)$$

Using Tonelli's theorem we get

$$I_0 \leq \int\limits_{B_\delta(0)} \int\limits_A \left(\int\limits_A \Phi\left(\frac{|f(x-z) - f(y-z)|}{|x-y|} \right) \rho_\varepsilon^\eta(x-y)\, d\mathcal{L}^N(y) \right) \psi_\delta(z)\, d\mathcal{L}^N(x)\, dz$$

Then making the change of variables $w = x + z$, $u = y + z$, for $z \in B_\delta(0)$, since the integrand is non-negative, we have

$$I_0 \leq \int\limits_{B_\delta(0)} \int\limits_A \left(\int\limits_A \Phi\left(\frac{|f(w) - f(u)|}{|w-u|} \right) \rho_\varepsilon^\eta(w-u)\, d\mathcal{L}^N(u) \right) d\mathcal{L}^N(w)\, \psi_\delta(z)\, dz$$

Since $\int\limits_{\mathbb{R}^N} \psi_\delta(z-y)\, dz = 1$, it follows

$$I_0 \leq \int\limits_A \left(\int\limits_A \Phi\left(\frac{|f(w) - f(u)|}{|w-u|} \right) \rho_\varepsilon^\eta(w-u)\, d\mathcal{L}^N(u) \right) d\mathcal{L}^N(w)$$

□

4. Proof of Theorem 1.2

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3, let $\Phi \in \Delta_2$, let $f \in L^\Phi_{loc}(\Omega)$. Assume*

$$\lim_{\lambda \to 0} \limsup_{\varepsilon \to 0^+} \int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi\left(\frac{|f(x) - f(y)|}{|x-y|} \right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\, d\mathcal{L}^N(x) < +\infty. \tag{4.1}$$

Then $f \in W^1 L^\Phi_{loc}(\Omega)$ and $\nabla f \in L^\Phi(\Omega)$. Moreover there exist $\varepsilon_j \to 0^+$ and a probability measure μ in $\mathcal{M}\left(S^{N-1}\right)$ such that for every $0 < \eta < \frac{\lambda}{3}$ then

$$\begin{aligned} &\lim\nolimits_{\lambda \to 0} \liminf_{\varepsilon \to 0^+} \int\limits_{\Omega_\lambda^\eta} \left(\int\limits_{\Omega_\lambda^{2\eta}} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|} \right) \rho_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \geq \\ &\geq \int\limits_\Omega \left(\int\limits_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|)\, d\mu(\sigma) \right) d\mathcal{L}^N(x). \end{aligned} \tag{4.2}$$

Proof. We define

$$C = \lim_{\lambda\to 0}\limsup_{\varepsilon\to 0^+}\int_{\Omega_\lambda}\int_{\Omega_\lambda}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) < +\infty$$

by the monotonicity of the integrals over Ω_λ we have that for any $\eta < \frac{\lambda}{3}$,

$$\limsup_{\varepsilon\to 0^+}\int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^{2\eta}}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) \le C$$

where $\Omega_\lambda^\eta = (\Omega_\lambda)^\eta$, since $\rho_\varepsilon^\eta \le \rho_\varepsilon$, we get

$$\limsup_{\varepsilon\to 0^+}\int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^{2\eta}}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) \le C$$

Fix $0 < \eta < \frac{\lambda}{3}$, for any $0 < \delta < \eta$ apply Lemma 3.10 we obtain

$$I_\delta \le \int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^{2\eta}}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x)$$

where

$$I_\delta = \int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^{\eta}}\Phi\left(\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x)$$

Let μ_ε be the measures defined in (3.1). By Lemma 3.2 there exist a subsequence $\{\varepsilon_j\}$, with $\varepsilon_j \to 0$, and a probability measure $\mu \in \mathcal{M}(S^{N-1})$ such that $\mu_{\varepsilon_j} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(S^{N-1})$. Since $f_\delta \in C^2\left(\overline{\Omega_\lambda^{2\eta}}\right)$ with $\Omega_\lambda^{2\eta}$ open and bounded, by Lemma 3.8 , for every $x \in \Omega_\lambda$,

$$\lim_{j\to+\infty}\int_{\Omega_\lambda^\eta}\Phi\left(\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\right)\rho_\varepsilon^\eta(x-y)\ d\mathcal{L}^N(y) = \int_{S^{N-1}}\Phi(|\nabla f_\delta(x)\cdot\sigma|)\ d\mu(\sigma)$$

Applying Fatou's lemma we have

$$\begin{aligned}&\int_{\Omega_\lambda^\eta}\left(\int_{S^{N-1}}\Phi(|\nabla f_\delta(x)\cdot\sigma|)\ d\mu(\sigma)\right)d\mathcal{L}^N(x) \le\\ &\le \liminf_{j\to+\infty}\int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^\eta}\Phi\left(\tfrac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\right)\rho_{\varepsilon_j}^\eta(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x)\\ &\le \int_{\Omega_\lambda^\eta}\int_{\Omega_\lambda^{2\eta}}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}^\eta(x-y)\ d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) \le C\end{aligned}$$

so that

$$\int_{\Omega_\lambda}\left(\int_{S^{N-1}}\Phi(|\nabla f_\delta(x)\cdot\sigma|)\ d\mu(\sigma)\right)d\mathcal{L}^N(x) \le C$$

Then Lemma 3.2 implies

$$\int_{\Omega_\lambda}\Phi(|\nabla f_\delta(x)|)\ d\mathcal{L}^N(x) \le \frac{C}{\alpha}$$

for some $\alpha > 0$, independent of λ. Using the P-D theorem we get $\nabla f_\delta \rightharpoonup V$ in $L^1_{loc}(\Omega)$. Since as $\delta \to 0$, $f_\delta \to f$ in $L^1_{loc}(\Omega)$ then $f \in W^1L^\Phi_{loc}(\Omega)$ and $\nabla f = V \in L^\Phi\left(\Omega_\lambda, \mathbb{R}^N\right)$. Finally, letting $\lambda \to 0$ we have $\nabla f \in L^\Phi\left(\Omega, \mathbb{R}^N\right)$. □

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be open, let Φ and ρ_ε satisfy H-1, H-2, H-3, let $f \in W^1 L^\Phi_{loc}(\Omega)$ and $\nabla f \in L^\Phi\left(\Omega, \mathbb{R}^N\right)$; then it follows*

$$\begin{aligned} &\int_{\Omega^\eta_\lambda} \left(\int_{\Omega^\eta_\lambda} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \le \\ &\le \int_\Omega \Phi(|\nabla f(x)|)\, d\mathcal{L}^N(x) + \\ &+ \tfrac{2^m}{\eta^m} \int_{\Omega^\eta_\lambda} \Phi(|f(x)|)\, d\mathcal{L}^N(x) \int_{|h|>\eta} \rho_\varepsilon(h)\, d\mathcal{L}^N(h) \end{aligned} \tag{4.3}$$

Proof. Fix $0 < \lambda < 1$ and $0 < \eta < \frac{\lambda}{2}$; consider

$$\begin{aligned} &\int_{\Omega^\eta_\lambda} \left(\int_{\Omega^\eta_\lambda} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) = \\ &\int_{\Omega^\eta_\lambda} \left(\int_{|x-y|<\eta} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) + \\ &\int_{\Omega^\eta_\lambda} \left(\int_{|x-y|>\eta} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \end{aligned}$$

Considering

$$\begin{aligned} B &= \int_{\Omega^\eta_\lambda} \left(\int_{|x-y|>\eta} \Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \\ &\le \frac{2^{m-1}}{\eta^m} \int_{\Omega^\eta_\lambda} \left(\int_{|x-y|>\eta} (\Phi(|f(x)|) + \Phi(|f(y)|))\, \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \end{aligned}$$

By Fubini and Tonelli Theorems we get

$$\begin{aligned} B \le\ & \frac{2^{m-1}}{\eta^m} \int_{\Omega^\eta_\lambda} \Phi(|f(x)|) \left(\int_{|x-y|>\eta} \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) + \\ & + \frac{2^{m-1}}{\eta^m} \int_{\Omega^\eta_\lambda} \Phi(|f(y)|) \left(\int_{|x-y|>\eta} \rho_\varepsilon(x-y)\, d\mathcal{L}^N(x) \right) d\mathcal{L}^N(y) \end{aligned}$$

and

$$B \le \frac{2^m}{\eta^m} \int_{\Omega^\eta_\lambda} \Phi(|f(x)|)\, d\mathcal{L}^N(x) \int_{|h|>\eta} \rho_\varepsilon(h)\, d\mathcal{L}^N(h)$$

Moreover applaying Lemma 3.9 we get

$$\begin{aligned} &\int_{\Omega^\eta_\lambda} \left(\int_{|x-y|<\eta} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \\ &\le \int_{\Omega^\eta_\lambda} \left(\int_{|x-y|<\eta} \Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right) \rho^\eta_\varepsilon(x-y)\, d\mathcal{L}^N(y) \right) d\mathcal{L}^N(x) \\ &\le \int_{\Omega^\eta_\lambda} \left(\int_{B_\eta(0)} \Phi\left(\left|\nabla f(x) \cdot \tfrac{h}{|h|}\right|\right) \rho^\eta_\varepsilon(h)\, d\mathcal{L}^N(h) \right) d\mathcal{L}^N(x) \\ &\le \int_\Omega \Phi(|\nabla f(x)|)\, d\mathcal{L}^N(x) \end{aligned}$$

then we get (4.3). □

Now we can show the Theorem 1.2

Proof. (Proof of Theorem 1.2) Let $f \in L^{\Phi}_{loc}(\Omega)$, assume

$$\lim_{\lambda\to 0}\limsup_{\varepsilon\to 0^+}\int_{\Omega_\lambda}\int_{\Omega_\lambda}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\, d\mathcal{L}^N(x) < +\infty$$

then applaying Theorem 4.1 we have $f \in W^1L^{\Phi}_{loc}(\Omega)$ and $\nabla f \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$. Moreover we get

$$\begin{aligned}&\liminf_{j\to+\infty}\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^{2\eta}_\lambda}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \geq\\ &\geq \int_\Omega\left(\int_{S^{N-1}}\Phi(|\nabla f(x)\cdot\sigma|)\, d\mu(\sigma)\right)d\mathcal{L}^N(x).\end{aligned} \tag{4.4}$$

Conversely let $f \in W^1L^{\Phi}_{loc}(\Omega)$ and $\nabla f \in L^{\Phi}\left(\Omega, \mathbb{R}^N\right)$; then using Theorem 4.2 we have

$$\begin{aligned}&\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^\eta_\lambda}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \leq\\ &\leq \int_\Omega\Phi(|\nabla f(x)|)\, d\mathcal{L}^N(x)+\\ &+\tfrac{2^m}{\eta^m}\int_{\Omega^\eta_\lambda}\Phi(|f(x)|)\, d\mathcal{L}^N(x)\int_{|h|>\eta}\rho_\varepsilon(h)\, d\mathcal{L}^N(h)\end{aligned}$$

Since, by Remark 3.5, $\lim_{\varepsilon\to 0^+}\int_{|h|>\eta}\rho_\varepsilon(h)\, d\mathcal{L}^N(h) = 0$ we get

$$\limsup_{\varepsilon\to 0^+}\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^\eta_\lambda}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) \leq \int_\Omega\Phi(|\nabla f(x)|)\, d\mathcal{L}^N(x)$$

and

$$\lim_{\lambda\to 0^+}\limsup_{\varepsilon\to 0^+}\int_{\Omega^\eta_\lambda}\left(\int_{\Omega^\eta_\lambda}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x) < +\infty$$

Moreover, since, by Lemma 3.9,

$$\begin{aligned}&\int_{\Omega^\eta_\lambda}\left(\int_{|x-y|<\eta}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)\\ &\leq \int_{\Omega^\eta_\lambda}\left(\int_{|x-y|<\eta}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho^\eta_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)\\ &\leq \int_{\Omega^\eta_\lambda}\left(\int_{B_\eta(0)}\Phi\left(\left|\nabla f(x)\cdot\tfrac{h}{|h|}\right|\right)\rho^\eta_\varepsilon(h)\, d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)\end{aligned}$$

we have

$$\begin{aligned}&\liminf_{j\to+\infty}\int_{\Omega^\eta_\lambda}\left(\int_{|x-y|<\eta}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)\\ &\leq \liminf_{j\to+\infty}\int_{\Omega^{2\eta}_\lambda}\left(\int_{B_\eta(0)}\Phi\left(\left|\nabla f(x)\cdot\tfrac{h}{|h|}\right|\right)\rho^\eta_{\varepsilon_j}(h)\, d\mathcal{L}^N(h)\right)d\mathcal{L}^N(x)\\ &= \liminf_{j\to+\infty}\int_{\Omega^{2\eta}_\lambda}\left(\int_{S^{N-1}}\Phi(|\nabla f(x)\cdot\sigma|)\, d\mu^\eta_{\varepsilon_j}(\sigma)\right)d\mathcal{L}^N(x)\\ &= \int_{\Omega^{2\eta}_\lambda}\left(\int_{S^{N-1}}\Phi(|\nabla f(x)\cdot\sigma|)\, d\mu(\sigma)\right)d\mathcal{L}^N(x)\end{aligned} \tag{4.5}$$

Conbining (4.4) and (4.5) we get

$$
\begin{aligned}
&\int_{\Omega_\lambda}\left(\int_{S^{N-1}} \Phi\left(|\nabla f(x)\cdot\sigma|\right)\, d\mu(\sigma)\right) d\mathcal{L}^N(x)\\
\leq\ & \liminf_{j\to+\infty}\int_{\Omega_\lambda^\eta}\left(\int_{|x-y|<\eta}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)\right) d\mathcal{L}^N(x)\\
\leq\ & \limsup_{j\to+\infty}\int_{\Omega_\lambda^\eta}\left(\int_{|x-y|<\eta}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\, d\mathcal{L}^N(y)\right) d\mathcal{L}^N(x)\\
\leq\ & \int_{\Omega}\left(\int_{S^{N-1}} \Phi\left(|\nabla f(x)\cdot\sigma|\right)\, d\mu(\sigma)\right) d\mathcal{L}^N(x)
\end{aligned}
$$

sending $\lambda\to 0$ it follows

$$
\begin{aligned}
&\lim_{\lambda\to0^+}\lim_{\varepsilon\to0^+}\int_{\Omega_\lambda}\left(\int_{\Omega_\lambda}\Phi\left(\tfrac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\, d\mathcal{L}^N(y)\right)d\mathcal{L}^N(x)=\\
&=\int_\Omega\left(\int_{S^{N-1}}\Phi\left(|\nabla f(x)\cdot\sigma|\right)\, d\mu(\sigma)\right)d\mathcal{L}^N(x)
\end{aligned}
\tag{4.6}
$$

When ρ_ε satisfy H-4 we get $\mu=\mathcal{H}^{N-1}$ then

$$
\int_{S^{N-1}}\Phi\left(|\nabla f(x)\cdot\sigma|\right)\, d\mu(\sigma)=\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\left|\frac{\nabla f(x)}{|\nabla f(x)|}\cdot\sigma\right|\right)\, d\mathcal{H}^{N-1}(\sigma)\tag{4.7}
$$

Since $\frac{\nabla f(x)}{|\nabla f(x)|}\in S^{N-1}$ using the rotational invariance of $\mathcal{H}^{N-1}$ then

$$
\int_{S^{N-1}}\Phi\left(|\nabla f(x)\cdot\sigma|\right)\, d\mu(\sigma)=\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\,|e_1\cdot\sigma|\right)\, d\mathcal{H}^{N-1}(\sigma)
$$

Since $|e_1\cdot\sigma|\leq 1$, by Remark 2.18, we get

$$
\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\,|e_1\cdot\sigma|\right)\, d\mathcal{H}^{N-1}(\sigma)\leq\Phi\left(|\nabla f(x)|\right)\int_{S^{N-1}}|e_1\cdot\sigma|\, d\mathcal{H}^{N-1}(\sigma)\tag{4.8}
$$

Moredover by Jensen inequality we have

$$
\frac{1}{\mathcal{H}^{N-1}\left(S^{N-1}\right)}\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\,|e_1\cdot\sigma|\right)\, d\mathcal{H}^{N-1}(\sigma)\geq\Phi\left(\frac{|\nabla f(x)|}{\mathcal{H}^{N-1}\left(S^{N-1}\right)}\int_{S^{N-1}}|e_1\cdot\sigma|\, d\mathcal{H}^{N-1}(\sigma)\right)
$$

If we suppose $\Phi\in$ on $(0,+\infty)$, since $\frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\int_{S^{N-1}}|e_1\cdot\sigma|\, d\mathcal{H}^{N-1}(\sigma)\leq\frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\int_{S^{N-1}}|e_1||\sigma|\, d\mathcal{H}^{N-1}(\sigma)\leq 1$, using Proposition 2.21 it follows

$$
\frac{1}{\mathcal{H}^{N-1}\left(S^{N-1}\right)}\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\,|e_1\cdot\sigma|\right)\, d\mathcal{H}^{N-1}(\sigma)\geq\Phi\left(|\nabla f(x)|\right)\left(\Lambda_N\right)^m
$$

where $\Lambda_N=\frac{1}{\mathcal{H}^{N-1}(S^{N-1})}\int_{S^{N-1}}|e_1\cdot\sigma|\, d\mathcal{H}^{N-1}(\sigma)$, then

$$
\int_{S^{N-1}}\Phi\left(|\nabla f(x)|\,|e_1\cdot\sigma|\right)\, d\mathcal{H}^{N-1}(\sigma)\geq w_{m,N}\cdot\Phi\left(|\nabla f(x)|\right)\int_{S^{N-1}}|e_1\cdot\sigma|\, d\mathcal{H}^{N-1}(\sigma)\tag{4.9}
$$

where $w_{m,N} = \left(\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \int\limits_{S^{N-1}} |e_1 \cdot \sigma| \; d\mathcal{H}^{N-1}(\sigma) \right)^{m-1} > 0$. Using (4.8) and (4.9) it follows

$$w_m \le \frac{\int\limits_{\Omega} \left(\int\limits_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|) \; d\mu(\sigma) \right) d\mathcal{L}^N(x)}{\int\limits_{S^{N-1}} |e_1 \cdot \sigma| \; d\mathcal{H}^{N-1}(\sigma) \int\limits_{\Omega} \Phi(|\nabla f(x)|) \; d\mathcal{L}^N(x)} \le 1$$

then there exits $\Gamma_{m,N} \in [w_{m,N}, 1]$ such that

$$\int\limits_{\Omega} \left(\int\limits_{S^{N-1}} \Phi(|\nabla f(x) \cdot \sigma|) \; d\mu(\sigma) \right) d\mathcal{L}^N(x) = \Gamma_{m,N} \int\limits_{S^{N-1}} |e_1 \cdot \sigma| \; d\mathcal{H}^{N-1}(\sigma) \int\limits_{\Omega} \Phi(|\nabla f(x)|) \; d\mathcal{L}^N(x)$$

(1.8) follows if we define $k_{m,N} = \Gamma_{m,N} \int\limits_{S^{N-1}} |e_1 \cdot \sigma| \; d\mathcal{H}^{N-1}(\sigma)$. □

The proof of the Theorem 1.3 follows the ideas introduced in [17], we introduce it only for completeness.

Proof. (Proof of Theorem 1.3) Let $f \in L^{\Phi}(\Omega)$ satisfy (1.9) then for every $\eta \in \left(0, \frac{\lambda}{3}\right)$ and every $\Omega_\lambda \subset \Omega$ we get

$$\limsup_{\epsilon \to 0} \int\limits_{\Omega_\lambda^\eta} \int\limits_{\Omega_\lambda^{2\eta}} \Phi\left(\frac{|f(x) - f(y)|}{d_\Omega(x, y)}\right) \rho_\epsilon^\eta(d_\Omega(x, y)) \, d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) < +\infty$$

Since $\rho_\epsilon^\eta(d_\Omega(x, y)) = 0$ if $d_\Omega(x, y) > \eta$ and $|x - y| \le d_\Omega(x, y) \le \eta < \frac{\lambda}{3}$ then for $x \in \Omega_\lambda^\eta$ and $y \in \Omega_\lambda^{2\eta}$ the segment containing x and y is conteined in Ω, it follows that $d_\Omega(x, y) = |x - y|$ and

$$\limsup_{\epsilon \to 0} \int\limits_{\Omega_\lambda^\eta} \int\limits_{\Omega_\lambda^{2\eta}} \Phi\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \rho_\epsilon^\eta(|x - y|) \, d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) < +\infty$$

then by Theorem 1.2 $f \in W^{1,\Phi}(\Omega)$. □

Now we observe as the hypothesis H-1 is technical and as we are able weakened.

Remark 4.3. *If $\Phi \in \triangle_2^m$ on $(t_0, +\infty)$ with $t_0 > 0$ then there exists $\Phi_1 \in \triangle_2^m$ on $(0, +\infty)$ such that $\Phi \sim \Phi_1$. Moreover there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}^+$ such that*

$$\Phi(t) \le c_1 \Phi_1(t) + c_2 \tag{4.10}$$

for every $t > 0$ and

$$\Phi_1(t) \le c_3 \Phi(t) + c_4 \tag{4.11}$$

for every $t > 0$. Let us consider

$$\int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \rho_\varepsilon(x - y) \; d\mathcal{L}^N(y) \, d\mathcal{L}^N(x)$$

and

$$\int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi_1\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \rho_\varepsilon(x - y) \; d\mathcal{L}^N(y) \, d\mathcal{L}^N(x)$$

using (4.10) and (4.11) we get

$$\begin{aligned} \int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi\left(\tfrac{|f(x) - f(y)|}{|x - y|}\right) \rho_\varepsilon(x - y) \; d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) \quad &\le c_1 \int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \Phi_1\left(\tfrac{|f(x) - f(y)|}{|x - y|}\right) \rho_\varepsilon(x - y) \; d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) \\ &+ c_2 \int\limits_{\Omega_\lambda} \int\limits_{\Omega_\lambda} \rho_\varepsilon(x - y) \; d\mathcal{L}^N(y) \, d\mathcal{L}^N(x) \end{aligned} \tag{4.12}$$

and

$$\int_{\Omega_\lambda}\int_{\Omega_\lambda}\Phi_1\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) \le c_3\int_{\Omega_\lambda}\int_{\Omega_\lambda}\Phi\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) + c_4\int_{\Omega_\lambda}\int_{\Omega_\lambda}\rho_\varepsilon(x-y)\,d\mathcal{L}^N(y)\,d\mathcal{L}^N(x) \tag{4.13}$$

Since $W^1L^{\Phi}(\Omega) = W^1L^{\Phi_1}(\Omega)$*, see (Adams, 1975; Krasnosel'skij & Rutickii, 1961; Rao & Ren, 1991), then, by (4.12) and (4.13), Theorem 1.2 holds olso if* Φ *is a N-function and* $\Phi \in \triangle_2$ *on* $(t_0, +\infty)$ *with* $t_0 > 0$.

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