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On a System of Nonlinear Optical Wave Equations in Two-Spatial Dimension

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Abstract

We show that for a system of nonlinear weakly dispersive wave equations in two-spatial dimension, which is a model in nonlinear optics, the local L^2 norm of its solutions decays to zero as time approaches infinity.

Keywords: Decay, optical wave, two-dimensional

1. Introduction

The equation

$$u_t - u_x + u_{xxx} + \mathrm{i}|u|^2\mathrm{u} = 0 \tag{1.1}$$

which describes a model of nonlinear modulated dispersive wave was first discovered to have a very important application in optical communication by Wai, Menyuk, Lee, & Chen (1986), Wai, Menyuk, Chen, & Lee (1987), and Wai, Chen, & Lee (1990).

In this paper, we will discuss the asymptotic behavior of the wave of a weakly dispersive medium, which is described by the two-dimensional generalization of the equation (1.1). The wave is still propagating in the x-direction but with a weakly perturbation in the y-direction.

$$u_t - u_x + u_{xxx} + \mathrm{i}|u|^2\mathrm{u} = -\, w_y, \tag{1.2.a}$$

$$w_x = u_y \tag{1.2.b}$$

We will show that the smooth solutions decay in time in the local L^2 norm. The motivation to study this system of equations is similar to the study of the well-known Kadomtsev-Petviashivili equation (Kadomtsev & Petviashvil 1970)

$$u_t + 6\mathrm{u}u_x + u_{xxx} = -\, w_y$$

$$w_x = u_y$$

to the well-known Korteweg-de Vries equation (Korteweg & de Vries 1895)

$$u_t + 6\mathrm{u}u_x + u_{xxx} = 0.$$

This paper is the first paper to study the proposed system of equations (1.2 a) & (1.2 b).

2. Two Conservation Laws

Conservation laws are very important in the physics. The famous Noether's Theorem (Noether 1918)

states that every invariance of motion of a physical system has a corresponding conservation law. Furthermore, every conservation law is a constraint to the governing system of equations of the physical system. Here we present two conservation laws of (1.2 a) and (1.2b).

Multiplying both sides of equations (1.2.a) and (1.2.b) by u^*, where u^* is the complex conjugate of u, and then taking the real part of the equation, we get

$$\partial((|u|)^2)/\partial\mathrm{t} = \partial\mathrm{C}/\partial\mathrm{x} + \partial\mathrm{D}/\partial\mathrm{y} \tag{2.1}$$

where C = $|u|^2 - (|u|)^2)_{xx} + 3|u_x|^2 + |u|^2$ and D = $-\mathrm{w}u^* - w^*\mathrm{u}$

Thus

$$\int_{R^2}\int |u|^2 (x,y,t)\,dxdy \;=\; constant. \tag{2.2}$$

where w^* is the complex conjugate of w.

Another conservation law can be derived as

$$\partial(u^*u_x - uu_x^*)/\partial t = \partial E/\partial x + \partial F/\partial y$$

where E = $-u_{xx}^*u_x + u_{xx}u_x^* - i|u|^4 - w_y^*u + w_yu^* + u^*u_x - uu_x^* - u^*u_{xxx} + uu_{xxx}^* + u_x^*u_{xx} - u_xu_{xx}^*$ and F = $u^*_yu - u_yu^* - u^*u_y + uu^*_y$

3. Time Decay of the Local Energy

Theorem. Assume that the solutions u and w are C^3 functions such that |u|, $|u_x|$, $|u_{xx}|$ and |w| all approach zero as |x| and |y| are approaching infinity. Then, given r > 0, $\iint_{M(r)} |u|^2 dxdy \to 0,$ as $t \to \infty$.

where M(r) = { (x, y) | $x^2 + y^2 \le r^2$ }

Proof:

We shall use the method in Lin (1982) to prove this result. Multiplying both sides of (2.1) by a C^3 function A(x, y) such that |A|, $|A_x|$, $|A_y|$, $|A_{xx}|$ and $|A_{xxx}|$ are all bounded, we get

$$\partial(A|u|^2)/\partial t + A_x|u|^2 - A_{xxx}|u|^2 + 3A_x|u_x|^2 - A_ywu^* - A_yw^*u + A_x|w|^2$$
$$= \partial F/\partial x + \partial G/\partial y$$

where F = $A|u|^2 - A(|u|^2)_x + A_x(|u|^2)_x - A_{xx}(|u|^2) + 3A|u_x|^2 + A_x|w|^2$ and G = $- Awu^* - Aw^*u$.

Integrating both sides with respect to the entire x-y space, we get

$$\partial\left(\iint_{R^2} A|u|^2\,dxdy\right)/\partial t + \iint_{R^2}\left[\left(A_x - A_{xxx}\right)|u|^2 + A_x|w|^2 + 3A_x|u_x|^2 - A_y\left(wu^* + w^*u\right)\right]dxdy = 0$$

Now we integrate both sides with respect to t from 0 to T, T > 0, to get

$$\int_0^T\iint_{R^2}\left[\left(A_x - A_{xxx}\right)|u|^2 + A_x|W|^2 + 3A_x|u_x|^2 - A_y\left(wu^* + w^*u\right)\right]dxdydt$$
$$= \iint_{R^2} A|u|^2(x,y,0)\,dxdy - \iint_{R^2} A|u|^2(x,y,T)\,dxdy$$

Using (2.2), we get

$$\int_0^T\iint_{R^2}\left[\left(A_x - A_{xxx} - |A_y|\right)|u|^2 + \left(A_x - |A_y|\right)|w|^2 + 3A_x|u_x|^2\right]dxdydt \le c_1$$

where c_1 is a constant depending on the initial data and the bound for A. Note that c_1 doesn't depend on T.

We now choose A(x,y) = $\arctan(\frac{x}{3} + \frac{y}{6})$. Then

$$(A_x - A_{xxx} - |A_y|) > 0,\ A_x - |A_y| > 0, \text{ and } A_x > 0.$$

Let r > 0. We get

$$\int_0^\infty\iint_{M(3r)}\left[|u|^2 + |w|^2 + |u_x|^2\right]dxdydt \le c_2 \tag{3.1}$$

where c_2 depends on the initial data, A, and r.

Let B be a C^3function which depends only on x and y, B = 0 if $x^2 + y^2 \geq 9r^2$, B= 1 if $x^2 + y^2 \leq 4r^2$, and $0 \leq B \leq 1$.

Then

$$\left| \iint_{M(3r)} B\left(\partial\left(|u|^2\right)/\partial t\right) dxdy \right| \leq c_3 \iint_{M(3r)} \left[|u|^2 + |w|^2 + |u_x|^2\right] dxdy$$

where c_3 depends only on the upper bounds for $|B_x|$, $|B_{xxx}|$, and $|B_y|$.

Let 0 < s < t, then

$$(t-s)\iint_{M(r)} |u|^2 dxdy \leq (t-s)\iint_{M(r)} B|u|^2 dxdy \leq \int_s^t \iint_{M(3r)} B|u|^2 dxdydz + \int_s^t (z-s)\left|\iint_{M(3r)} B\partial\left(|u|^2\right)/\partial t dxdy\right| dz$$

Let s = t – 1, then

$$\int_{M(r)} |u|^2 dxdy \leq c_4 \int_{t-1}^{t} \iint_{M(3r)} \left[|u|^2 + |w|^2 + |u_x|^2\right] dxdydz$$

for some constant c_4 which is independent of t.

Thus by (3.1), $\iint_{M(r)} |u|^2 dxdy$ goes to zero as t goes to infinity. M(r)

3. Conclusion

The result of this paper is the first paper in the weakly dispersion in another direction of the system of equations that models the zero-group-dispersion wavelength of a single-mode optical fiber. There remains several problems for the future study of this system of equations such as the well-posedness, special solutions, symmetries, etc. Moreover, we have found two conservation laws. It would be very interesting to find more conservation laws, if any additional conservation law exists.

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