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Convergence Properties of Extended Newton-type Iteration Method for Generalized Equations

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Abstract

In this paper, we introduce and study the extended Newton-type method for solving generalized equation $0 \in f(x)+g(x)+\mathcal{F}(x)$, where $f : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ is Fréchet differentiable in a neighborhood Ω of a point $\bar{x}$ in $\mathcal{X}$, $g : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ is linear and differentiable at a point $\bar{x}$, and $\mathcal{F}$ is a set-valued mapping with closed graph acting in Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Semilocal and local convergence of the extended Newton-type method are analyzed.

Keywords: Generalized equations, Semilocal convergence, Lipschitz–like mappings, Extended Newton-type method, Divided difference

1. Introduction

In this study we are concerned with the problem of approximating a solution of a generalized equations. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\Omega \subseteq \mathcal{X}$. Let $f : \Omega \to \mathcal{Y}$ be a Fréchet differentiable function and its Fréchet derivative is denoted by ∇f, $g : \Omega \to \mathcal{Y}$ be a linear and differentiable function at x but may not differentiable in a neighborhood Ω of x and its first order divided difference on the points x and y is denoted by $[x, y; g]$ and $\mathcal{F} : \mathcal{X} \rightrightarrows 2^{\mathcal{Y}}$ be a set-valued mapping with closed graph. We consider here a generalized equation problem to approximate a point $x \in \Omega$ satisfying

$$0 \in f(x) + g(x) + \mathcal{F}(x). \tag{1}$$

For solving (1), Alexis & Pietrus (2008) introduced the following Newton-like method:

$$\begin{aligned} 0 \quad \in \quad & f(x_k) + g(x_k) + (\nabla f(x_k) + [2x_{k+1} - x_k, x_k; g])(x_{k+1} - x_k) \\ & + \mathcal{F}(x_{k+1}), \text{ for } k = 0, 1, \ldots \end{aligned} \tag{2}$$

and obtained local convergence of this method. In particular, the authors obtained superlinear and quadratic convergence of the method (2) when ∇f is Lipschitz continuous. To solve (1), Rashid, Wang & Li (2012) established local convergence results for the method (2) under the weaker conditions than Alexis & Pietrus (2008). Specifically, Rashid, Wang & Li (2012) extended the results by fixing a gap in the proof of Theorem 1 in Alexis & Pietrus (2008).

Moreover, for solving (1), Hilout, Alexis, & Piétrus (2006) considered the following sequence

$$\begin{cases} x_0 \text{ and } x_1 \text{ are given starting points} & \\ y_k = \alpha x_k + (1-\alpha)x_{k-1}; & \alpha \text{ is fixed in } (0,1) \\ 0 \in f(x_k) + [y_k, x_k; f](x_{k+1} - x_k) + \mathcal{F}(x_{k+1}) & \end{cases}$$

and they proved the convergence of this method is superlinear when f is only continuous and differentiable at x^*. Furthermore, it should be mentioned that Argyros (2004) has studied local as well as semilocal convergence analysis for two-point Newton-like methods in a Banach space setting under very general Lipschitz type conditions for solving (1) in the case when $\mathcal{F} = \{0\}$. When $g = 0$, this study has been extended by Rashid (2017a, 2017b, 2018).

Let $x \in \mathcal{X}$ and the subset of $\mathcal{X}$, denoted by $\mathcal{N}(x)$, is defined by

$$\mathcal{N}(x) = \{d \in \mathcal{X} \colon 0 \in f(x) + g(x) + (\nabla f(x) + [x + d, x; g])d + \mathcal{F}(x + d)\}.$$

Argyros & Hilout (2008) associated the following Newton-like method (See Algorithm 1) for solving the generalized equation (1):

Algorithm 1 (The Newton-like Method)

Step 1. Select $x_0 \in \mathcal{X}$, and put $k := 0$.
Step 2. If $0 \in \mathcal{N}(x_k)$, then stop; otherwise, go to Step 3.
Step 3. If $0 \notin \mathcal{N}(x_k)$, choose d_k such that $d_k \in \mathcal{N}(x_k)$.
Step 4. Set $x_{k+1} := x_k + d_k$.
Step 5. Replace k by $k + 1$ and go to Step 2.

Argyros & Hilout (2008) obtained the quadratic convergence of the sequence generated by Algorithm 1 when ∇f is Lipschitz continuous.

Under some suitable conditions around a solution x^* of the generalized equation (1), Argyros & Hilout (2008) showed in their Theorem 4.1 that there exists a neighborhood U of x^* such that, for any point in U, there exists a sequence generated by Algorithm 1 which is quadratically convergent to the solution x^*. This reflects that the convergence result, established in Argyros & Hilout (2008), guarantees the existence of a convergent sequence. Therefore, for any initial point near to a solution, the sequences generated by Algorithm 1 are not uniquely defined and not every generated sequence is convergent. Hence, in view of numerical computation, this kind of methods are not convenient in practical application. This difficulties inspired us to introduce a method "so-called" extended Newton-type (EN-type) method. Thus, we propose the following EN-type method:

Algorithm 2 (The EN-type Method))

Step 1. Select $\eta \in [1, \infty)$, $x_0 \in \mathcal{X}$, and put $k := 0$.
Step 2. If $0 \in \mathcal{N}(x_k)$, then stop; otherwise, go to Step 3.
Step 3. If $0 \notin \mathcal{N}(x_k)$, choose d_k such that $d_k \in \mathcal{N}(x_k)$ and

$$\|d_k\| \leq \eta \operatorname{dist}(0, \mathcal{N}(x_k)).$$

Step 4. Set $x_{k+1} := x_k + d_k$.
Step 5. Replace k by $k + 1$ and go to Step 2.

The difference between Algorithms 1 and 2 is that Algorithm 2 generates at least one sequence and every generated sequence is convergent but this does not happen for Algorithm 1. Since the sequences generated by Algorithm 1 are not uniquely defined, in comparison with Algorithms 1 and 2, we can infer that Algorithm 2 is more precise than Algorithm 1 in numerical computation.

If the set $\mathcal{N}(x)$ is replaced by the set

$$D(x) := \{d \in \mathcal{X} \colon 0 \in f(x) + g(x) + (\nabla f(x) + [2d + x, x; g])d + \mathcal{F}(x + d)\},$$

then the Algorithm 2 reduces to the same algorithm corresponding one given by Rashid (2014).

There have been studied many fruitful works on semilocal convergence analysis for the Gauss-Newton method in the case when $\mathcal{F} = \{0\}$ and $g = 0$ (see Dedieu & Kim (2002); Dedieu & Shub (2000); Xu & Li (2008), for more details) or when $\mathcal{F} = C$ and $g = 0$ (see Li & Ng (2007), for details).

In the case when $g = 0$, Rashid, Yu, Li & Wu (2013) introduced Gauss-Newton-type method to solve the generalized equation (1) and established its semilocal convergence. Moreover, in the same case, Rashid introduced different kinds of methods for solving (1) and obtained their semilocal and local convergence; see for examples (Rashid (2016); Rashid & Sardar (2015); Rashid (2015)). However, in our best knowledge, there is no other study on semilocal convergence analysis discovered for the Algorithm 1.

The purpose of this study is to analyze the semilocal convergence of the extended Newton-type method defined by Algorithm 2. The main tool is the Lipschitz-like property of set-valued mappings, which was introduced by Aubin (1984). in the context of nonsmooth analysis and studied by many mathematicians (see for example, Alexis & Piétrus (2008); Argyros & Hilout (2008); Dontchev (1996a); Hilout, Alexis,& Piétrus (2006); Piétrus (2000b)) and the references therein. The main results are the convergence criteria, established in Section 3, which, based on the attraction region around the initial point, provide some sufficient conditions ensuring the convergence to a solution of any sequence generated by Algorithm 2. As a result, local convergence results for the extended Newton-type method are obtained.

This paper is organized as follows: In section 2, we recall a few necessary preliminary results and also recall a fixed-point theorem which has been proved by Dontchev & Hager (1994). This fixed-point theorem is the main tool to prove the existence of the sequence generated by Algorithm 2. In section 3, we consider the extended Newton-type method as well as the concept of Lipschitz-like property to show the existence and the convergence of the sequence generated by Algorithm 2. In the last section, a summary of the major results of this study are given.

2. Preliminaries

In this section we give some notations and collect some results that will be helpful to prove our main results. Throughout the whole study, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two real or complex Banach spaces. Let $x \in \mathcal{X}$. Let $\mathbb{B}(x, r) = \{u \in X : \|u - x\| \leq r\}$ be denote the closed ball centered at x with radius $r > 0$. Let $\mathcal{F} : X \rightrightarrows 2^{\mathcal{Y}}$ be a set-valued mapping with closed graph. The domain of $\mathcal{F}$, denoted by dom$\mathcal{F}$, is defined by

$$\mathrm{dom}\mathcal{F} := \{x \in \mathcal{X} : \mathcal{F}(x) \neq \emptyset\}.$$

The inverse of $\mathcal{F}$, denoted by $\mathcal{F}^{-1}$, is defined by

$$\mathcal{F}^{-1}(y) := \{x \in \mathcal{X} : y \in \mathcal{F}(x)\} \quad \text{for each } y \in \mathcal{Y}.$$

and the graph of $\mathcal{F}$, denoted by gph$\mathcal{F}$, is defined by

$$\mathrm{gph}\mathcal{F} := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y \in \mathcal{F}(x)\}.$$

Let $A, B \subseteq \mathcal{X}$. The distance from a point $x \in \mathcal{X}$ to a set A is defined by

$$\mathrm{dist}(x, A) := \inf_{a \in A} \|x - a\|.$$

Moreover, the excess from the set A to the set B is defined by

$$e(B, A) := \sup_{b \in B}\{\mathrm{dist}(b, A)\}.$$

The space of linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and the norms are denoted by $\| \cdot \|$.

Now, we recall some definitions, results and then state the Banach fixed point theorem. We begin with the definition of the first order divided difference operators. The notion of divided differences of nonlinear operators is given by Argyros (2007), which is given below:

Definition 2.1. *Let $g \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then g is said to have the first order divided difference on the points x and y in $\mathcal{X}$ $(x \neq y)$ if the following properties hold:*

(a) $[x, y; g](y - x) = g(y) - g(x)$ *for* $x \neq y$*;*

(b) *if g is Fréchet differentiable at $x \in \mathcal{X}$, then* $[x, x; g] = \nabla g(x)$.

Recall from Rashid, Yu, Li & Wu (2013), the notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings. These notions were introduced by Aubin (see, Aubin (1984); Aubin & Frankowska (1990), for more details) and have been studied extensively.

Definition 2.2. *Let $\Gamma : \mathcal{Y} \rightrightarrows 2^{\mathcal{X}}$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in \mathrm{gph}\Gamma$. Let $r_{\bar{x}}$, $r_{\bar{y}}$ and μ are positive constants. Then Γ is said to be*

(a) *Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant μ if the following inequality holds:*

$$e(\Gamma(y_1) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \Gamma(y_2)) \leq \mu\|y_1 - y_2\| \quad \textit{for every } y_1, y_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}}).$$

(b) *pseudo-Lipschitz around $(\bar{y}, \bar{x})$ if there exist constants $a > 0$, $b > 0$ and $\mu' > 0$ such that Γ is Lipschitz-like on $\mathbb{B}(\bar{y}, b)$ relative to $\mathbb{B}(\bar{x}, a)$ with constant μ'.*

Remark 2.1. *The set-valued mapping Γ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant $\mu > 0$, which is equivalent to the following statement: if for every $y_1, y_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ and for every $x_1 \in \Gamma(y_1) \cap \mathbb{B}(\bar{x}, r_{\bar{x}})$, there exists $x_2 \in \Gamma(y_2)$ such that*

$$\|x_1 - x_2\| \leq \mu\|y_1 - y_2\|.$$

The following lemma is due to Lemma 2.1 of Rashid, Yu, Li & Wu (2013). This lemma is useful and its proof is a little bit similar to that for Theorem 1.49(i) of Mordukhovich (2006).

Lemma 2.1. *Let* $\Gamma : \mathcal{Y} \rightrightarrows 2^{\mathcal{X}}$ *be a set-valued mapping and let* $(\bar{y}, \bar{x}) \in \operatorname{gph} \Gamma$. *Assume that* Γ *is Lipschitz-like on* $\mathbb{B}(\bar{y}, r_{\bar{y}})$ *relative to* $\mathbb{B}(\bar{x}, r_{\bar{x}})$ *with constant* μ. *Then*

$$\operatorname{dist}(x, \Gamma(y)) \leq \mu \operatorname{dist}(y, \Gamma^{-1}(x))$$

holds for every $x \in \mathbb{B}(\bar{x}, r_{\bar{x}})$ *and* $y \in \mathbb{B}(\bar{y}, \frac{r_{\bar{y}}}{3})$ *satisfying* $\operatorname{dist}(y, \Gamma^{-1}(x)) \leq \dfrac{r_{\bar{y}}}{3}$.

We close this section with the following lemma. This lemma is a fixed point statement which has been proved by Dontchev & Hager (1994) and employing the standard iterative concept for contracting mapping. This lemma will be used to prove the existence of the sequence generated by Algorithm 2.

Lemma 2.2. *Let* $\Phi : \mathcal{X} \rightrightarrows 2^{\mathcal{X}}$ *be a set-valued mapping. Let* $x^* \in \mathcal{X}$, $r > 0$ *and* $0 < \lambda < 1$ *be such that*

$$\operatorname{dist}(x^*, \Phi(x^*)) < r(1 - \lambda) \tag{3}$$

and

$$e(\Phi(x_1) \cap \mathbb{B}(x^*, r), \Phi(x_2)) \leq \lambda \|x_1 - x_2\| \quad \textit{for all } x_1, x_2 \in \mathbb{B}(x^*, r). \tag{4}$$

Then Φ *has a fixed point in* $\mathbb{B}(x^*, r)$, *that is, there exists* $x \in \mathbb{B}(x^*, r)$ *such that* $x \in \Phi(x)$. *Moreover, if* Φ *is single-valued, then the fixed point of* Φ *in* $\mathbb{B}(x^*, r)$ *is unique i.e.* $x = \Phi(x)$.

The previous lemma is a generalization of a fixed point theorem which has been given by Ioffe & Tikhomirov (1979), where in assertion (b) the excess e is replaced by Hausdorff distance.

3. Convergence analysis of EN-type Method

Let Ω be a subset of $\mathcal{X}$. Suppose that $f : \Omega \to \mathcal{Y}$ is a Fréchet differentiable function on a neighborhood Ω of $\bar{x}$ with its derivative denoted by ∇f, $g \colon \Omega \to \mathcal{Y}$ is linear and differentiable at $\bar{x}$ and let $\mathcal{F} : \mathcal{X} \rightrightarrows 2^{\mathcal{Y}}$ be a set-valued mapping with closed graph. This section is devoted to prove the existence and convergence of the sequences generated by extended Newton-type method, defined by the Algorithm 2, on a neighborhood Ω of a point $\bar{x}$.

Fix $x \in X$. Then for every $x \in X$, we have that

$$\begin{aligned} g(x) + [x + d, x; g]d &= g(x) - [x + d, x; g](x - (x + d)) \\ &= g(x) - (g(x) - g(x + d)) = g(x + d). \end{aligned} \tag{5}$$

Therefore, we define the mapping $\mathcal{G}_x$ by

$$\mathcal{G}_x(\cdot) := f(x) + g(\cdot) + \nabla f(x)(\cdot - x) + \mathcal{F}(\cdot).$$

It follows, from the construction of $\mathcal{N}(x)$, that

$$\mathcal{N}(x) = \{d \in \mathcal{X} : 0 \in \mathcal{G}_x(x + d)\}.$$

Moreover, for any $z \in \mathcal{X}$ and $y \in \mathcal{Y}$, we have the following equivalence:

$$z \in \mathcal{G}_x^{-1}(y) \text{ if and only if } y \in f(x) + g(z) + \nabla f(x)(z - x) + \mathcal{F}(z). \tag{6}$$

In particular, let $(\bar{x}, \bar{y}) \in \operatorname{gph}\mathcal{G}_{\bar{x}}$. Then, the closed graphness of $\mathcal{G}_{\bar{x}}$ imply that

$$\bar{x} \in \mathcal{G}_{\bar{x}}^{-1}(\bar{y}). \tag{7}$$

The following result establishes the equivalence between $(f + g + \mathcal{F})^{-1}$ and $\mathcal{G}_{\bar{x}}^{-1}$. This result is the modification of Rashid & Sardar (2015).

Lemma 3.1. *Let* $(\bar{x}, \bar{y}) \in \operatorname{gph}(f + g + \mathcal{F})$. *Suppose that* f *is Fréchet differentiable in an open neighborhood* Ω *of* $\bar{x}$ *and* ∇f *is continuous at* $\bar{x}$. *Assume that* g *is Fréchet differentiable at* $\bar{x}$ *and admits first order divided difference. Then the following are equivalent:*

(i) *The mapping* $(f + g + \mathcal{F})^{-1}$ *is pseudo-Lipschitz at* $(\bar{y}, \bar{x})$;

(ii) *The mapping $\mathcal{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz at $(\bar{y}, \bar{x})$.*

Proof. Define a function $h : \mathcal{X} \to \mathcal{Y}$ by

$$h(x) := -f(x) + f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}).$$

The proof is similar to that of Rashid & Sardar (2015), because the proof does not depend on the property of g. □

For our convenience, let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\mathbb{B}(\bar{x}, r_{\bar{x}}) \subseteq \Omega \cap \operatorname{dom} \mathcal{F}$. Assume that the function g is Fréchet differentiable at $\bar{x}$ and admits a first order divided difference, that is, there exist $\nu > 0$ such that for all $x, y, u, v \in \mathbb{B}(\bar{x}, r_{\bar{x}})$ $(x \neq y, u \neq v)$,

$$\|[x, y; g] - [u, v; g]\| \leq \nu(\|x - u\| + \|y - v\|),$$

and the mapping $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M, that is,

$$e(\mathcal{G}_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}(y_2)) \leq M\|y_1 - y_2\| \quad \text{for any } y_1,\ y_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}}). \tag{8}$$

Moreover, the closed graph property of $\mathcal{G}_{\bar{x}}$ implies that $f + g + \mathcal{F}$ is continuous at $\bar{x}$ for $\bar{y}$ i.e. the following condition is hold:

$$\lim_{x \to \bar{x}} \operatorname{dist}(\bar{y}, f(x) + g(x) + \mathcal{F}(x)) = 0. \tag{9}$$

Let $\varepsilon > 0$ and write

$$\bar{r} := \min\Big\{r_{\bar{y}} - 2\varepsilon r_{\bar{x}}, \frac{r_{\bar{x}}(1 - M\varepsilon)}{4M}\Big\}. \tag{10}$$

Then

$$\bar{r} > 0 \text{ if and only if } \varepsilon < \min\Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}},\ \frac{1}{M}\Big\}. \tag{11}$$

The following lemma plays a crucial role for convergence analysis of the extended Newton-type method. The proof is a refinement of Lemma 3.1 in Rashid, Yu, Li & Wu (2013).

Lemma 3.2. *Suppose that $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M. Let ε be defined by* (11) *and $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Assume that ∇f is continuous on $\mathbb{B}((\bar{x}, \frac{r_{\bar{x}}}{2})$. Let $\bar{r}$ be defined by* (10) *such that* (11) *is true. Then $\mathcal{G}_{x}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \bar{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-M\varepsilon}$, that is,*

$$e(\mathcal{G}_{x}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}), \mathcal{G}_{x}^{-1}(y_2)) \leq \frac{M}{1 - M\varepsilon}\|y_1 - y_2\| \quad \textit{for any } \ y_1,\ y_2 \in \mathbb{B}(\bar{y}, \bar{r}).$$

Proof. Let

$$y_1,\ y_2 \in \mathbb{B}(\bar{y}, \bar{r}) \quad \text{and} \quad x' \in \mathcal{G}_{x}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}). \tag{12}$$

It is enough to show that there exist $x'' \in \mathcal{G}_{x}^{-1}(y_2)$ such that

$$\|x' - x''\| \leq \frac{M}{1 - M\varepsilon}\|y_1 - y_2\|.$$

To this finish , we will verify that there exists a sequence $\{x_n\} \subset \mathbb{B}(\bar{x}, r_{\bar{x}})$ such that

$$y_2 \in f(x) + g(x_n) + \nabla f(x)(x_{n-1} - x) + \nabla f(\bar{x})(x_n - x_{n-1}) + \mathcal{F}(x_n), \tag{13}$$

and

$$\|x_n - x_{n-1}\| \leq M\|y_1 - y_2\|(M\varepsilon)^{n-2} \tag{14}$$

hold for each $n = 2, 3, 4, \ldots$. We proceed by mathematical induction on n. Letting

$$u_i := y_i - f(x) - \nabla f(x)(x_1 - x) + f(\bar{x}) + \nabla f(\bar{x})(x_1 - \bar{x}) \quad \text{for each } i = 1, 2.$$

From (12) we have that

$$\|x - x'\| \leq \|x - \bar{x}\| + \|\bar{x} - x'\| \leq r_{\bar{x}}.$$

Since ∇f is continuous around $\bar{x}$ with the constant ε, it gives that

$$\begin{aligned}\|f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})\| &= \|\int_0^1 [\nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x})](x - \bar{x})dt\| \\ &\leq \int_0^1 \|\nabla f(\bar{x} + t(x - \bar{x})) - \nabla f(\bar{x})\|\|x - \bar{x}\|dt \\ &\leq \varepsilon\|x - \bar{x}\| \int_0^1 dt = \varepsilon\|x - \bar{x}\|,\end{aligned}$$

It follows, from (12) and the relation $\bar{r} \leq r_{\bar{y}} - 2\varepsilon r_{\bar{x}}$ by (10), that

$$\begin{aligned}\|u_i - \bar{y}\| &\leq \|u_i - \bar{y}\| + \|f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})\| + \|(\nabla f(x) - \nabla f(\bar{x}))(x - x')\| \\ &\leq \bar{r} + \varepsilon(\|x - \bar{x}\| + \|x - x'\|) \\ &\leq \bar{r} + \varepsilon(\frac{r_{\bar{x}}}{2} + r_{\bar{x}}) \leq r_{\bar{y}}.\end{aligned}$$

The above inequality implies that $u_i \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ for each $i = 1, 2$. Denote $x_1 := x'$. Then $x_1 \in \mathcal{G}_x^{-1}(y_1)$ by (12) and it follows from (6) that

$$y_1 \in f(x) + g(x_1) + \nabla f(x)(x_1 - x) + \mathcal{F}(x_1).$$

The alternative form of the above inclusion is as follows:

$$y_1 + f(\bar{x}) + \nabla f(\bar{x})(x_1 - \bar{x})) \in f(x) + g(x_1) + \nabla f(x)(x_1 - x) + \mathcal{F}(x_1) + f(\bar{x}) + \nabla f(\bar{x})(x_1 - \bar{x}).$$

By the definition of u_1, this yields that

$$u_1 \in f(\bar{x}) + g(x_1) + \nabla f(\bar{x})(x_1 - \bar{x}) + \mathcal{F}(x_1).$$

Hence $x_1 \in \mathcal{G}_{\bar{x}}^{-1}(u_1)$ by (6). This gives, for (12), that

$$x_1 \in \mathcal{G}_{\bar{x}}^{-1}(u_1) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}).$$

Since $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$, then for every $u_1, u_2 \in \mathbb{B}(\bar{y}, r_{\bar{y}})$, we have through (8) that there exists $x_2 \in \mathcal{G}_{\bar{x}}^{-1}(u_2)$ such that

$$\|x_2 - x_1\| \leq M\|u_1 - u_2\| = M\|y_1 - y_2\|.$$

In addition, by the construction of u_2 and $x_1 = x'$, we obtain that

$$x_2 \in \mathcal{G}_{\bar{x}}^{-1}(u_2) = \mathcal{G}_{\bar{x}}^{-1}(y_2 - f(x) - \nabla f(x)(x_1 - x) + f(\bar{x}) + \nabla f(\bar{x})(x_1 - \bar{x})).$$

This, together with (6), gives that

$$y_2 \in f(x) + g(x_2) + \nabla f(x)(x_1 - x) + \nabla f(\bar{x})(x_2 - x_1) + \mathcal{F}(x_2).$$

This implies that (13) and (14) are true with the constructed points x_1 and x_2.

Suppose that the points $x_1, x_2, \ldots, x_k$ are constructed so that (13) and (14) are true for $n = 2, 3, \ldots, k$. We have to construct the point x_{k+1} such that (13) and (14) are also true for $n = k + 1$. For showing this, let, for each $i = 0, 1$,

$$u_i^k := y_2 - f(x) - \nabla f(x)(x_{k+i-1} - x) + f(\bar{x}) + \nabla f(\bar{x})(x_{k+i-1} - \bar{x}).$$

Then, for the above inductional assumption, we get

$$\begin{aligned}\|u_0^k - u_1^k\| &= \|(\nabla f(\bar{x}) - \nabla f(x))(x_k - x_{k-1})\| \\ &\leq \varepsilon\|x_k - x_{k-1}\| \leq \|y_1 - y_2\|(M\varepsilon)^{k-1}.\end{aligned} \tag{15}$$

We have from (12) that $\|x_1 - \bar{x}\| \leq \frac{r_{\bar{x}}}{2}$ and $\|y_1 - y_2\| \leq 2\bar{r}$. This, together with (14), implies that

$$\begin{aligned}\|x_k - \bar{x}\| &\leq \sum_{i=2}^{k} \|x_i - x_{i-1}\| + \|x_1 - \bar{x}\| \\ &\leq 2M\bar{r}\sum_{i=2}^{k}(M\varepsilon)^{i-2} + \frac{r_{\bar{x}}}{2} \\ &\leq \frac{2M\bar{r}}{1 - M\varepsilon} + \frac{r_{\bar{x}}}{2}.\end{aligned}$$

Note by (10) that $4M\bar{r} \le r_{\bar{x}}(1 - M\varepsilon)$. Therefore, we have from the above inequality that

$$\|x_k - \bar{x}\| \le r_{\bar{x}}. \tag{16}$$

Moreover, we obtain that

$$\|x_k - x\| \le \|x_k - \bar{x}\| + \|\bar{x} - x\| \le \frac{3}{2} r_{\bar{x}}. \tag{17}$$

Furthermore, using (12) and (17), one has that, for each $i = 0, 1$,

$$\begin{aligned}
&\|u_i^k - \bar{y}\| \\
\le\ & \|y_2 - \bar{y}\| + \|f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})\| + \|(\nabla f(x) - \nabla f(\bar{x}))(x - x_{k+i-1})\| \\
\le\ & \bar{r} + \varepsilon(\|x - \bar{x}\| + \|x - x_{k+i-1}\|) \le \bar{r} + \varepsilon(\frac{r_{\bar{x}}}{2} + \frac{3r_{\bar{x}}}{2}) \\
=\ & \bar{r} + 2\varepsilon r_{\bar{x}}.
\end{aligned}$$

By the relation $\bar{r} \le r_{\bar{y}} - 2\varepsilon r_{\bar{x}}$ in (10), it follows that $\|u_i^k - \bar{y}\| \le r_{\bar{y}}$. This shows that $u_i^k \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ for each $i = 0, 1$. By our assumption (13) is true for $n = k$. Thus, we have that

$$y_2 \in f(x) + g(x_k) + \nabla f(x)(x_{k-1} - x) + \nabla f(\bar{x})(x_k - x_{k-1}) + \mathcal{F}(x_k).$$

The above inequality can be written as follows:

$$\begin{aligned}
y_2 + f(\bar{x}) + \nabla f(\bar{x})(x_{k-1} - \bar{x}) \quad \in \quad & f(x) + \nabla f(x)(x_{k-1} - x) + f(\bar{x}) + g(x_k) \\
& + \nabla f(\bar{x})(x_k - x_{k-1}) + \mathcal{F}(x_k) + \nabla f(\bar{x})(x_{k-1} - \bar{x}).
\end{aligned}$$

Then by the construction of u_0^k, we have that $u_0^k \in f(\bar{x}) + g(x_k) + \nabla f(\bar{x})(x_k - \bar{x}) + \mathcal{F}(x_k)$. This together with (6) implies that $x_k \in \mathcal{G}_{\bar{x}}^{-1}(u_0^k)$. It follows from (16) that

$$x_k \in \mathcal{G}_{\bar{x}}^{-1}(u_0^k) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}).$$

By Lipschit-like property of $\mathcal{G}_{\bar{x}}^{-1}$, there exists an element $x_{k+1} \in \mathcal{G}_{\bar{x}}^{-1}(u_1^k)$ such that

$$\|x_{k+1} - x_k\| \le M\|u_0^k - u_1^k\|.$$

Then by (15), it follows that

$$\|x_{k+1} - x_k\| \le M\|y_1 - y_2\|(M\varepsilon)^{k-1}. \tag{18}$$

By the construction of u_1^k, we get that

$$x_{k+1} \in \mathcal{G}_{\bar{x}}^{-1}(u_1^k) = \mathcal{G}_{\bar{x}}^{-1}(y_2 - f(x) - \nabla f(x)(x_k - x) + f(\bar{x}) + \nabla f(\bar{x})(x_k - \bar{x})).$$

This, together with (6), implies that

$$y_2 \in f(x) + g(x_{k+1}) + \nabla f(x)(x_k - x) + \nabla f(\bar{x})(x_{k+1} - x_k) + \mathcal{F}(x_{k+1}).$$

The inequality (18) together with the above inclusion completes the induction step and confirming the existence of a sequence $\{x_k\}$ which satisfies (13) and (14).

Since $M\varepsilon < 1$, we see from (14) that $\{x_k\}$ is a Cauchy sequence and hence it is convergent, to say x'', that is, $x'' := \lim_{k\to\infty} x_k$. Note that $\mathcal{F}$ has closed graph. Then, taking limit in (13), we get $y_2 \in f(x) + g(x'') + \nabla f(x)(x'' - x) + \mathcal{F}(x'')$, that is, $x'' \in \mathcal{G}_x^{-1}(y_2)$. Therefore, we obtain

$$\begin{aligned}
\|x' - x''\| \quad & \le \quad \limsup_{n\to\infty} \sum_{k=2}^{n} \|x_k - x_{k-1}\| \\
& \le \quad \limsup_{n\to\infty} \sum_{k=2}^{n} (M\varepsilon)^{k-2} M\|y_1 - y_2\| \\
& \le \quad \frac{M}{1 - M\varepsilon}\|y_1 - y_2\|.
\end{aligned}$$

That is,

$$e(\mathcal{G}_x^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}), \mathcal{G}_x^{-1}(y_2)) \le \frac{M}{1 - M\varepsilon}\|y_1 - y_2\|.$$

This completes the proof of the Lemma 3.2. □

Before going to prove our main results, we would like to introduce some notations. For our convenience, first define the mapping $J_x\colon \mathcal{X} \to \mathcal{Y}$, for each $x \in \mathcal{X}$, by

$$J_x(\cdot) := f(\bar{x}) + g(\cdot) + \nabla f(\bar{x})(\cdot - \bar{x}) - f(x) - g(x) - (\nabla f(x) + [\cdot, x; g])(\cdot - x).$$

and the set-valued mapping $\Phi_x\colon \mathcal{X} \rightrightarrows 2^{\mathcal{X}}$ by

$$\Phi_x(\cdot) := \mathcal{G}_{\bar{x}}^{-1}[J_x(\cdot)]. \tag{19}$$

Then for any $x',\ x'' \in \mathcal{X}$, we have

$$\begin{aligned}
\|J_x(x') - J_x(x'')\| &= \|g(x') - g(x'') - [x', x; g](x' - x) + [x'', x; g](x'' - x) \\
&\quad + (\nabla f(\bar{x}) - \nabla f(x))(x' - x'')\| \\
&\le \|g(x') - g(x'') - [x'', x; g](x' - x'')\| + \|([x'', x; g] \\
&\quad - [x', x; g])(x' - x)\| + \|\nabla f(\bar{x}) - \nabla f(x)\|\|x' - x''\| \\
&\le \Big(\|[x'', x'; g] - [x'', x; g]\| + \|\nabla f(\bar{x}) - \nabla f(x)\|\Big)\|x' - x''\| \\
&\quad + \|[x'', x; g] - [x', x; g]\|\|x' - x\|
\end{aligned} \tag{20}$$

3.1 Linear Convergence

The first main theorem of this study read as follows, which gives some sufficient conditions confirming the convergence of the extended Newton-type method with starting point x_0.

Theorem 3.1. *Suppose that $\eta > 1$ and that $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r_{\bar{y}})$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M. Let $\bar{r}$ be defined in* (10) *and let $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Suppose that $\varepsilon > 0$ be such that* (11) *is hold and ∇f is continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant ε. Let $\nu > 0$ and $\delta > 0$ be such that*

(a) $\delta \le \min\Big\{\dfrac{r_{\bar{x}}}{4},\ \dfrac{r_{\bar{y}}}{7(\varepsilon + 3\nu)},\ 1,\ \dfrac{3 - 5M\varepsilon}{30M\nu},\ \dfrac{\bar{r}}{3(\varepsilon + 3\nu)}\Big\}$,

(b) $6\eta M(\varepsilon + 3\nu) \le 1 - M\varepsilon$,

(c) $\|\bar{y}\| < (\varepsilon + 3\nu)\delta$.

Suppose that $f + g + \mathcal{F}$ is continuous at $\bar{x}$ for $\bar{y}$ i.e. (9) *is hold. Then there exists some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x}, \hat{\delta})$ converges to a solution x^* of (1), that is, x^* satisfies $0 \in f(x^*) + g(x^*) + \mathcal{F}(x^*)$.*

Proof. Letting that $q := \dfrac{\eta M(\varepsilon + 3\nu)}{1 - M\varepsilon}$. Then by the relation $6\eta M(\varepsilon + 3\nu) \le 1 - M\varepsilon$ from assumption (b), we obtain

$$q := \frac{\eta M(\varepsilon + 3\nu)}{1 - M\varepsilon} \le \frac{1}{6}.$$

Take $0 < \hat{\delta} \le \delta$ such that

$$\operatorname{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0)) \le (\varepsilon + 3\nu)\delta \quad \text{for each } x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \tag{21}$$

(Noting that such $\hat{\delta}$ exists by (9) and assumption (c)). Let $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta})$. We will proceed by mathematical induction to show that Algorithm 2 generates at least one sequence and any sequence $\{x_n\}$ generated by Algorithm 2 satisfies the following assertions:

$$\|x_n - \bar{x}\| \le 2\delta \tag{22}$$

and

$$\|x_{n+1} - x_n\| \le q^{n+1}\delta \tag{23}$$

hold for each $n = 0, 1, 2, \ldots$. For this purpose, define

$$r_x := \frac{5}{2}\left(M(\varepsilon + 3\nu\|x - \bar{x}\|)\|x - \bar{x}\| + M\|\bar{y}\|\right) \quad \text{for each } x \in X. \tag{24}$$

Then, thanks to the fact that $6\eta M(\varepsilon+3\nu) \le 1 - M\varepsilon < 1$ by assumption (b) and $\|\bar{y}\| < (\varepsilon+3\nu)\delta$ by assumption (c). Since $\eta > 1$, (24) yields that

$$\begin{aligned} r_x &< 5M(\varepsilon+6\nu\delta)\delta + M(\varepsilon+3\nu)\delta < 5M(\varepsilon+6\nu\delta) + M(\varepsilon+3\nu)\delta \\ &= 6M\varepsilon\delta + 33M\nu\delta < 11M\varepsilon\delta + 33M\nu\delta = 11M(\varepsilon+3\nu)\delta \le \frac{11}{6\eta}\delta \\ &\le 2\delta \quad \text{for each } x \in \mathbb{B}(\bar{x}, 2\delta). \end{aligned} \tag{25}$$

Note that (22) is trivial for $n = 0$. To show (23) holds for $n = 0$, firstly we need to show that x_1 exists. To complete this, we have to prove that $\mathcal{N}(x_0) \neq \emptyset$ by applying Lemma 2.2 to the map Φ_{x_0} with $\eta_0 = \bar{x}$. Let us check that both assertions (3) and (4) of Lemma 2.2 hold with $r := r_{x_0}$ and $\lambda := \dfrac{3}{5}$. Noting that $\bar{x} \in \mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta)$ by (7) and according to the definition of the excess e and the mapping Φ_{x_0} in (19), we obtain

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(\bar{x})) \le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(\bar{x})) \\ &\le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(\bar{x})]) \end{aligned} \tag{26}$$

(noting that $\mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, r_{\bar{x}})$). By the choice of ε, we have

$$\begin{aligned} \|J_{x_0}(x) - \bar{y}\| &= \|f(\bar{x}) + g(x) + \nabla f(\bar{x})(x-\bar{x}) - f(x_0) - g(x_0) \\ &\quad - (\nabla f(x_0) + [x, x_0; g])(x - x_0) - \bar{y}\| \\ &\le \|f(\bar{x}) - f(x_0) - \nabla f(x_0)(\bar{x} - x_0)\| + \|\nabla f(\bar{x} - \nabla f(x_0))(x - \bar{x})\| \\ &\quad + \|g(x) - g(x_0) - [x, x_0; g](x - x_0)\| + \|\bar{y}\| \\ &\le \varepsilon(\|\bar{x} - x_0\| + \|x - \bar{x}\|) + \|[x_0, x; g] - [x, x_0; g]\|\|x - x_0\| \\ &\quad + \|\bar{y}\| \\ &\le \varepsilon(\|\bar{x} - x_0\| + \|x - \bar{x}\|) + \nu(\|x_0 - x\| + \|x - x_0\|)\|x - x_0\| \\ &\quad + \|\bar{y}\|. \end{aligned} \tag{27}$$

Note that $\|x_0 - \bar{x}\| \le \hat{\delta} \le \delta$, $7(\varepsilon+3\nu)\delta \le r_{\bar{y}}$ by assumption (a) and $\|\bar{y}\| < (\varepsilon+3\nu)\delta$ by assumption (c), it follows from (27) that, for each $x \in \mathbb{B}(\bar{x}, 2\delta)$,

$$\begin{aligned} \|J_{x_0}(x) - \bar{y}\| &\le 3\varepsilon\delta + 18\nu\delta^2 + (\varepsilon+3\nu)\delta < 3\varepsilon\delta + 18\nu\delta + (\varepsilon+3\nu)\delta \\ &< 6\varepsilon\delta + 18\nu\delta + (\varepsilon+3\nu)\delta = 7(\varepsilon+3\nu)\delta \\ &\le r_{\bar{y}}. \end{aligned} \tag{28}$$

This implies that for all $x \in \mathbb{B}(\bar{x}, 2\delta)$, $J_{x_0}(x) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$. In particular, letting $x = \bar{x}$ in (27). Then we obtain that

$$\begin{aligned} \|J_{x_0}(\bar{x}) - \bar{y}\| &\le \varepsilon\|\bar{x} - x_0\| + \nu(2\|x_0 - \bar{x}\| + \|\bar{x} - x_0\|)\|\bar{x} - x_0\| + \|\bar{y}\| \\ &= (\varepsilon + 3\nu\|\bar{x} - x_0\|)\|\bar{x} - x_0\| + \|\bar{y}\| \\ &\le (\varepsilon + 3\nu\delta)\delta + \|\bar{y}\| < (\varepsilon+3\nu)\delta + \|\bar{y}\| \\ &\le 2(\varepsilon+3\nu)\delta \le r_{\bar{y}}; \end{aligned} \tag{29}$$

and hence $J_{x_0}(\bar{x}) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$.

Hence, by (24), (26), (29) and the assumed Lipschitz-like property, we have

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\le M\|\bar{y} - J_{x_0}(\bar{x})\| \\ &\le M(\varepsilon + 3\nu\|\bar{x} - x_0\|)\|\bar{x} - x_0\| + M\|\bar{y}\| \\ &= \Big(1 - \frac{3}{5}\Big) r_{x_0} = (1-\lambda)r; \end{aligned}$$

that is, the assertion (3) of Lemma 2.2 is satisfied.

Now, we show that the assertion (4) of Lemma 2.2 holds. To end this, let $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0})$. Then, it follows that $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0}) \subseteq \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, r_{\bar{x}})$ by (25) and assumption (a), and $J_{x_0}(x')$, $J_{x_0}(x'') \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ by (28). This together with the assumed Lipschitz-like property implies that

$$\begin{aligned} e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\le e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \Phi_{x_0}(x'')) \\ &= e(\mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(x')] \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(x'')]) \\ &\le M\|J_{x_0}(x') - J_{x_0}(x'')\|. \end{aligned} \tag{30}$$

Using (20) and the choice of x_0, we have

$$\begin{aligned}\|J_{x_0}(x') - J_{x_0}(x'')\| &\leq (\|[x'', x'; g] - [x'', x_0; g]\| + \|\nabla f(\bar{x}) - \nabla f(x_0)\|)\|x' - x''\| \\ &\quad + \|[x'', x_0; g] - [x', x_0; g]\|\|x' - x_0\| \\ &\leq \Big(\nu(\|x' - x_0\| + \|x' - x_0\|) + \varepsilon\Big)\|x' - x''\| \\ &\leq (\varepsilon + 6\nu\delta)\|x' - x''\|.\end{aligned}$$

It follows, from $30M\nu\delta \leq 3 - 5M\varepsilon$ as in assumption (a) together with (30) that

$$e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) \leq M(\varepsilon + 6\nu\delta)\|x' - x''\| \leq \frac{3}{5}\|x' - x''\| = \lambda\|x' - x''\|.$$

This yields that the assertion (4) of Lemma 2.2 is satisfied. Since both assertions of Lemma 2.2 are fulfilled, we can say that the Lemma 2.2 is applicable and hence we can conclude that there exists $\hat{x}_1 \in \mathbb{B}(\bar{x}, r_{x_0})$ such that $\hat{x}_1 \in \Phi_{x_0}(\hat{x}_1)$. This yields that $0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [\hat{x}_1, x_0; g])(\hat{x}_1 - x_0) + \mathcal{F}(\hat{x}_1)$ and thus we conclude that $\mathcal{N}(x_0) \neq \emptyset$. Since $\eta > 1$ and $\mathcal{N}(x_0) \neq \emptyset$, we can choose $d_0 \in \mathcal{N}(x_0)$ such that

$$\|d_0\| \leq \eta \operatorname{dist}(0, \mathcal{N}(x_0)).$$

By Algorithm 2, $x_1 := x_0 + d_0$ is defined. Furthermore, by the definition of $\mathcal{N}(x_0)$ and through (5), we can write

$$\begin{aligned}\mathcal{N}(x_0) &:= \Big\{d_0 \in \mathcal{X} : 0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [d_0 + x_0, x_0; g])d_0 + \mathcal{F}(x_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{X} : 0 \in f(x_0) + g(x_0 + d_0) + \nabla f(x_0)d_0 + \mathcal{F}(x_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{X} : x_0 + d_0 \in \mathcal{G}_{x_0}^{-1}(0)\Big\},\end{aligned}$$

and so

$$\operatorname{dist}(0, \mathcal{N}(x_0)) = \operatorname{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)). \tag{31}$$

Now, we show that (23) holds also for $n = 0$. The continuity property of ∇f implies that

$$\|\nabla f(x) - \nabla f(\bar{x})\| \leq \varepsilon, \text{ for all } x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$$

and note that $\bar{r} > 0$ by assumption (a). Therefore, (11) satisfies (10). Since $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like, it follows from Lemma 3.2 that the mapping $\mathcal{G}_x^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \bar{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\dfrac{M}{1 - M\varepsilon}$ for each $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. In particular, $\mathcal{G}_{x_0}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, \bar{r})$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\dfrac{M}{1 - M\varepsilon}$ as $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \subset \mathbb{B}(\bar{x}, \delta) \subset \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ by assumption (a) and by the choice of $\hat{\delta}$. Furthermore, by the relation $3(\varepsilon + 3\nu)\delta \leq \bar{r}$ in assumption (a) and assumption (c) imply that

$$\|\bar{y}\| < (\varepsilon + 3\nu)\delta \leq \frac{\bar{r}}{3} \tag{32}$$

and hence (21) implies that

$$\begin{aligned}\operatorname{dist}(0, \mathcal{G}_{x_0}(x_0)) &= \operatorname{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0)) \leq (\varepsilon + 3\nu)\delta \\ &\leq \frac{\bar{r}}{3}.\end{aligned} \tag{33}$$

It is noted earlier that $x_0 \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ and $0 \in \mathbb{B}(\bar{y}, \frac{\bar{r}}{3})$ by (32). Thus, applying Lemma 2.1 it can be shown that

$$\operatorname{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)) \leq \frac{M}{1 - M\varepsilon} \operatorname{dist}(0, \mathcal{G}_{x_0}(x_0)).$$

The above relation together with (31) yields that

$$\operatorname{dist}(0, \mathcal{N}(x_0)) = \operatorname{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)) \leq \frac{M}{1 - M\varepsilon} \operatorname{dist}(0, \mathcal{G}_{x_0}(x_0)). \tag{34}$$

According to Algorithm 2 and using (33) and (34), we obtain

$$\begin{aligned}\|d_0\| &\leq \eta\ \mathrm{dist}(0, \mathcal{N}(x_0)) \leq \frac{\eta M}{1 - M\varepsilon}\ \mathrm{dist}(0, \mathcal{G}_{x_0}(x_0)) \\ &\leq \frac{\eta M(\varepsilon + 3\nu)\delta}{1 - M\varepsilon} = q\delta.\end{aligned}$$

This implies that

$$\|x_1 - x_0\| = \|d_0\| \leq q\delta$$

and therefore, (23) is hold for $n = 0$.

Assume that $x_1, x_2, \ldots, x_k$ are constructed so that (22) and (23) are hold for $n = 0, 1, 2, \ldots, k - 1$. We will show that there exists x_{k+1} such that (22) and (23) are also hold for $n = k$. Since (22) and (23) are true for each $n \leq k - 1$, we have the following inequality

$$\|x_k - \bar{x}\| \leq \sum_{i=0}^{k-1} \|d_i\| + \|x_0 - \bar{x}\| \leq \delta \sum_{i=0}^{k-1} q^{i+1} + \delta \leq \frac{\delta q}{1 - q} + \delta \leq 2\delta.$$

This shows that (22) holds for $n = k$. Now with almost the same argument as we did for the case when $n = 0$, it can be shown that (23) hold for $n = k$. The proof is complete. □

When $\bar{y} = 0$, that is, $\bar{x}$ is a solution of (1), Theorem 3.1 is reduced to the following corollary, which gives the local convergent result for the extended Newton-type method.

Corollary 3.1. *Suppose that $\eta > 1$ and $\bar{x}$ is a solution of (1). Let $\mathcal{G}_{\bar{x}}^{-1}$ be pseudo-Lipschitz around $(0, \bar{x})$. Let $\tilde{r} > 0$, $\nu > 0$ and suppose that ∇f is continuous on $\mathbb{B}(\bar{x}, \tilde{r})$ and that*

$$\lim_{x \to \bar{x}} \mathrm{dist}(0, f(x) + g(x) + \mathcal{F}(x)) = 0.$$

Then there exists some $\hat{\delta}$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x}, \hat{\delta})$ converges to a solution x^* of (1).

Proof. Let $\mathcal{G}_{\bar{x}}^{-1}$ be pseudo-Lipschitz around $(0, \bar{x})$. Then there exist constants r_0, $\hat{r}_{\bar{x}}$ and M satisfy the following condition:

$$e(\mathcal{G}_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \hat{r}_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}(y_2)) \leq M\|y_1 - y_2\|, \quad \text{for every } y_1, y_2 \in \mathbb{B}(0, r_0). \tag{35}$$

Thus, by the definition of Lipschitz-like property we can say that $Q_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(0, r_0)$ relative to $\mathbb{B}(\bar{x}, \hat{r}_{\bar{x}})$ with constant M which satisfy (35). Then, for each $0 < \tilde{r} \leq \hat{r}_{\bar{x}}$, one has that

$$e(\mathcal{G}_{\bar{x}}^{-1}(y_1) \cap \mathbb{B}(\bar{x}, \tilde{r}), \mathcal{G}_{\bar{x}}^{-1}(y_2)) \leq M\|y_1 - y_2\|, \quad \text{for every } y_1, y_2 \in \mathbb{B}(0, r_0),$$

that is, $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(0, r_0)$ relative to $\mathbb{B}(\bar{x}, \tilde{r})$ with constant M. Let $\varepsilon \in (0, 1)$ be such that $M((6\eta+1)\varepsilon+3\nu) \leq 1$. By the continuity of ∇f we can choose $r_{\bar{x}} \in (0, \hat{r}_{\bar{x}})$ such that $\frac{r_{\bar{x}}}{2} \leq \tilde{r}$, $r_0 - 2\varepsilon r_{\bar{x}} > 0$ and

$$\|\nabla f(x) - \nabla f(x')\| \leq \varepsilon, \text{ for each } x,\ x' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2}).$$

Then

$$\bar{r} = \min\Big\{r_0 - 2\varepsilon r_{\bar{x}}, \frac{r_{\bar{x}}(1 - M\varepsilon)}{4M}\Big\} > 0,$$

and

$$\min\Big\{\frac{r_{\bar{x}}}{4}, \frac{\bar{r}}{3(\varepsilon + 3\nu)}, \frac{r_0}{7(\varepsilon + 3\nu)}, \frac{3 - 5M\varepsilon}{30M\nu}\Big\} > 0. \tag{36}$$

By (36), we can choose $0 < \delta \leq 1$ such that

$$\delta \leq \min\Big\{\frac{r_{\bar{x}}}{4}, \frac{\bar{r}}{3(\varepsilon + 3\nu)}, 1, \frac{r_0}{7(\varepsilon + 3\nu)}, \frac{3 - 5M\varepsilon}{30M\nu}\Big\}.$$

Thus it is routine to check that inequalities (a)-(c) of Theorem 3.1 are satisfied. Therefore, Theorem 3.1 is applicable to complete the proof. □

3.2 Quadratic Convergence

In this section we consider ∇f is Lipschitz continuous around $\bar{x}$ and show that the sequence generated by Algorithm 2 converges quadratically.

Let $L > 0$ and define

$$r^* := \min\Big\{r_{\bar{y}} - 2Lr_{\bar{x}}^2, \frac{r_{\bar{x}}(1 - MLr_{\bar{x}})}{4M}\Big\}. \tag{37}$$

Now, our second main theorem can be read as follows:

Theorem 3.2. *Let $\eta > 1$ and suppose that $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r^*)$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$ with constant M and that ∇f is Lipschitz continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with Lipschitz constant L. Let $\nu > 0$, $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{x}}}{4}, \frac{10r^*}{3}, 1, \Big(\frac{r_{\bar{y}}}{3(L+4\nu)}\Big)^{\frac{1}{2}}\right\}$,

(b) $(M + 1)(L + 4\nu)(\eta\delta + r_{\bar{x}}) \le 1$,

(c) $\|\bar{y}\| < \dfrac{(L + 4\nu)\delta^2}{2}$.

Suppose that

$$\lim_{x\to\bar{x}} \operatorname{dist}(\bar{y}, f(x) + g(x) + F(x)) = 0. \tag{38}$$

Then there exist some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x}, \hat{\delta})$ converges quadratically to a solution x^* of (1).

Proof. Setting

$$s := \frac{\eta M(L + 4\nu)\delta}{1 - MLr_{\bar{x}}}. \tag{39}$$

Thanks to assumption (b). Since $\nu > 0$, it allows us to write the fact that

$$\begin{aligned} \eta M(L + 4\nu)\delta + MLr_{\bar{x}} &< (M + 1)(L + 4\nu)\eta\delta + (M + 1)(L + 4\nu)r_{\bar{x}} \\ &= (M + 1)(L + 4\nu)(\eta\delta + r_{\bar{x}}) \le 1. \end{aligned}$$

Thus, we have from (39) that

$$s := \frac{\eta M(L + 4\nu)\delta}{1 - MLr_{\bar{x}}} \le 1. \tag{40}$$

Pick $0 < \hat{\delta} \le \delta$ be such that

$$\operatorname{dist}(0, f(x_0) + g(x_0) + F(x_0)) \le \frac{(L + 4\nu)\delta^2}{2} \quad \text{for each } x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \tag{41}$$

Since (38) is hold and assumption (c) is true, we assume that such $\hat{\delta}$ exists, which satisfies (41). Let $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta})$. To complete the proof of this theorem we use almost similar argument that used for completing the proof of Theorem 3.1. We show that Algorithm 2 generates at least one sequence and such sequence $\{x_n\}$ generated by Algorithm 2 satisfies the following assertions:

$$\|x_n - \bar{x}\| \le 2\delta; \tag{42}$$

and

$$\|d_n\| \le s\Big(\frac{1}{2}\Big)^{2^n}\delta. \tag{43}$$

hold for each $n = 0, 1, 2, ...$ Let

$$r_x := \frac{5M}{8}\Big((L + 4\nu)\|x - \bar{x}\|^2 + 2\|\bar{y}\|\Big) \quad \text{for each } x \in X. \tag{44}$$

Owing to the fact $4\delta \le r_{\bar{x}}$ in assumption (a) and $\eta > 1$, by assumption (b) we can write as follows

$$\begin{aligned} 5(M + 1)(L + 4\nu)\delta &= (M + 1)(L + 4\nu)(\delta + 4\delta) \\ &\le (M + 1)(L + 4\nu)(\eta\delta + r_{\bar{x}}) \\ &\le 1. \end{aligned}$$

This gives

$$M(L+4\nu)\delta \le \frac{1}{5} \quad \text{and} \quad (L+4\nu)\delta \le \frac{1}{5}. \tag{45}$$

Hence by $3\delta \le 5r^*$ in assumption (a) together with second inequality of (45), we get

$$\|\bar{y}\| < \frac{(L+4\nu)\delta^2}{2} \le \frac{1}{5\cdot 2}\cdot\frac{10r^*}{3} = \frac{r^*}{3}. \tag{46}$$

Thanks to assumption (c). Utilizing the first inequality from (45) together with assumption (c), we obtain from (44) that

$$\begin{aligned} r_x &< \frac{5M}{8}\Big((L+4\nu)\delta^2 + (L+4\nu)\delta^2\Big) \\ &= \frac{10M}{8}(L+4\nu)\delta^2 \le \frac{10}{8\cdot 5}\delta \\ &= \frac{\delta}{4} < 2\delta \quad \text{for each } x \in \mathbb{B}(\bar{x}, 2\delta). \end{aligned} \tag{47}$$

Note that (42) is trivial for $n = 0$. In order to show that (43) is hold for $n = 0$, first we need to prove $\mathcal{N}(x_0) \neq \emptyset$. The nonemptyness of $\mathcal{N}(x_0)$ will ensure us to deduce the existence of the point x_1. To complete this, we will apply Lemma 2.2 to the map Φ_{x_0} with $\eta_0 = \bar{x}$. Let us check that both assertions (3) and (4) of Lemma 2.2 hold with $r := r_{x_0}$ and $\lambda := \frac{1}{5}$. Here we note by (7) that $\bar{x} \in \mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta)$. Then, according to the definition of the excess e and the mapping Φ_{x_0} defined by (19), we have that

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(\bar{x})) \le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(\bar{x})) \\ &\le e(\mathcal{G}_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(\bar{x})]). \end{aligned} \tag{48}$$

For each $x \in \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ and Lipschitz continuous property of ∇f, we obtain

$$\begin{aligned} \|J_{x_0}(x) - \bar{y}\| &= \|f(\bar{x}) + g(x) + \nabla f(\bar{x})(x-\bar{x}) - f(x_0) - g(x_0) \\ &\quad - (\nabla f(x_0) + [x, x_0; g])(x - x_0) - \bar{y}\| \\ &\le \|f(\bar{x}) - f(x_0) - \nabla f(x_0)(\bar{x} - x_0)\| + \|(\nabla f(x_0) - \nabla f(\bar{x}))(\bar{x} - x)\| \\ &\quad + \|g(x) - g(x_0) - [x, x_0; g](x - x_0)\| + \|\bar{y}\| \\ &\le \frac{L}{2}\|\bar{x} - x_0\|^2 + L\|x_0 - \bar{x}\|\|\bar{x} - x\| + \|[x_0, x; g] - [x, x_0; g]\|\|x - x_0\| \\ &\quad + \|\bar{y}\| \\ &\le \frac{L}{2}\|\bar{x} - x_0\|^2 + L\|x_0 - \bar{x}\|\|\bar{x} - x\| + \nu(\|x_0 - x\| + \|x - x_0\|)\|x - x_0\| \\ &\quad + \|\bar{y}\| \\ &\le \frac{L}{2}(\delta^2 + 4\delta^2) + 2\nu(2\delta)^2 + \|\bar{y}\| = \frac{5L\delta^2}{2} + 8\nu\delta^2 + \|\bar{y}\| \\ &< \frac{5}{2}(L+4\nu)\delta^2 + \|\bar{y}\|. \end{aligned} \tag{49}$$

It follows, from the facts $3(L+4\nu)\delta^2 \le r_{\bar{y}}$ and $2\|\bar{y}\| < (L+4\nu)\delta^2$ respectively in assumptions (a) and (c), that

$$\begin{aligned} \|J_{x_0}(x) - \bar{y}\| &\le \frac{5}{2}(L+4\nu)\delta^2 + \frac{(L+4\nu)\delta^2}{2} \\ &= 3(L+4\nu)\delta^2 \le r_{\bar{y}}. \end{aligned} \tag{50}$$

This shows that $J_{x_0}(x) \in \mathbb{B}(\bar{y}, r_{\bar{y}})$. In particular, let $x = \bar{x}$ in (49). Then it is easily shown that

$$J_{x_0}(\bar{x}) \in \mathbb{B}(\bar{y}, r_{\bar{y}}) \quad \text{and} \quad \|J_{x_0}(\bar{x}) - \bar{y}\| \le \frac{(L+4\nu)}{2}\|\bar{x} - x_0\|^2 + \|\bar{y}\|. \tag{51}$$

Using the Lipschitz-like property of $\mathcal{G}_{\bar{x}}^{-1}$ and (51) in (48), we have

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi_{x_0}(\bar{x})) &\le M\|\bar{y} - J_{x_0}(\bar{x})\| \le \frac{M(L+4\nu)}{2}\|\bar{x} - x_0\|^2 + M\|\bar{y}\| \\ &= (1 - \frac{1}{5})r_{x_0} = (1-\lambda)r; \end{aligned}$$

that is, the assertion (3) of Lemma 2.2 is satisfied.

Now, we show that assertion (4) of Lemma 2.2 holds. To end this, let $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0})$. Then we have that $x', x'' \in \mathbb{B}(\bar{x}, r_{x_0}) \subseteq \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, r_{\bar{x}})$ by (47) and $J_{x_0}(x'), J_{x_0}(x'') \in \mathbb{B}(\bar{y}, r_{\bar{y}})$ by (50). This together with Lipschitz-like property of $\mathcal{G}_{\bar{x}}^{-1}$ implies that

$$\begin{aligned} e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, 2\delta), \Phi_{x_0}(x'')) \\ &\leq e(\mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(x')] \cap \mathbb{B}(\bar{x}, r_{\bar{x}}), \mathcal{G}_{\bar{x}}^{-1}[J_{x_0}(x'')]) \\ &\leq M\|J_{x_0}(x') - J_{x_0}(x'')\|. \end{aligned}$$

Now, we have from (20) that

$$\begin{aligned} \|J_{x_0}(x') - J_{x_0}(x'')\| &\leq (\|[x'', x'; g] - [x'', x_0; g]\| + \|\nabla f(\bar{x}) - \nabla f(x_0)\|)\|x' - x''\| \\ &\quad + \|[x'', x_0; g] - [x', x_0; g]\|\|x' - x_0\| \\ &\leq \Big(\nu(\|x_0 - x'\| + \|x' - x_0\|) + L\|\bar{x} - x_0\|\Big)\|x' - x''\| \\ &\leq (L + 4\nu)\delta\|x' - x''\|. \end{aligned}$$

Combining above two inequalities and first inequality from (45), we obtain that

$$\begin{aligned} e(\Phi_{x_0}(x') \cap \mathbb{B}(\bar{x}, r_{x_0}), \Phi_{x_0}(x'')) &\leq M(L + 4\nu)\delta\|x' - x''\| \\ &\leq \frac{1}{5}\|x' - x''\| = \lambda\|x' - x''\|. \end{aligned}$$

It seems that the assertion (4) of Lemma 2.2 is also satisfied. Thus, we have seen that both assertions (3) and (4) of Lemma 2.2 are fulfilled. So, we can conclude that Lemma 2.2 is applicable to deduce the existence of a point $\hat{x}_1 \in \mathbb{B}(\bar{x}, r_{x_0})$ such that $\hat{x}_1 \in \Phi_{x_0}(\hat{x}_1)$. This implies that $0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [\hat{x}_1, x_0; g])(\hat{x}_1 - x_0) + \mathcal{F}(\hat{x}_1)$ and thus $\mathcal{N}(x_0) \neq \emptyset$. Since $\eta > 1$ and $\mathcal{N}(x_0) \neq \emptyset$, we can choose $d_0 \in \mathcal{N}(x_0)$ such that

$$\|d_0\| \leq \eta \operatorname{dist}(0, \mathcal{N}(x_0)).$$

By Algorithm 2, $x_1 := x_0 + d_0$ is defined. Furthermore, by the construction of $\mathcal{N}(x_0)$ and (5), we have that

$$\begin{aligned} \mathcal{N}(x_0) &:= \Big\{d_0 \in \mathcal{X} : 0 \in f(x_0) + g(x_0) + (\nabla f(x_0) + [d_0 + x_0, x_0; g])d_0 + \mathcal{F}(x_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{X} : 0 \in f(x_0) + g(x_0 + d_0) + \nabla f(x_0)d_0 + \mathcal{F}(x_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{X} : x_0 + d_0 \in \mathcal{G}_{x_0}^{-1}(0)\Big\}, \end{aligned}$$

and so

$$\operatorname{dist}(0, \mathcal{N}(x_0)) = \operatorname{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)). \tag{52}$$

Now we are ready to show that (43) is hold for $n = 0$.

Note by assumption (a) that $r^* > 0$. Then, from (37) we conclude that

$$L < \Big\{\frac{r_{\bar{y}}}{2r_{\bar{x}}^2}, \frac{1}{Mr_{\bar{x}}}\Big\}.$$

Since ∇f is Lipschitz continuous on $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with Lipschitz constant L, we have for all $x', x'' \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, that

$$\|\nabla f(x') - \nabla f(x'')\| \leq L\|x' - x''\| \leq Lr_{\bar{x}}.$$

This shows that Lemma 3.2 is applicable with $\varepsilon := Lr_{\bar{x}}$. According to our assumption $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r^*)$ relative to $\mathbb{B}(\bar{x}, r_{\bar{x}})$. Then, it follows from Lemma 3.2 that for each $x \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$, the mapping $\mathcal{G}_x^{-1})$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r^*)$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-MLr_{\bar{x}}}$. Specifically, $\mathcal{G}_{x_0}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r^*)$ relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-MLr_{\bar{x}}}$ as $x_0 \in \mathbb{B}(\bar{x}, \hat{\delta}) \subseteq \mathbb{B}(\bar{x}, 2\delta) \subseteq \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ by assumption (a). On the other hand, (41) implies that

$$\begin{aligned} \operatorname{dist}(0, \mathcal{G}_{x_0}(x_0)) &= \operatorname{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0)) \\ &\leq \frac{r^*}{3}. \end{aligned}$$

We have shown by (46) that $0 \in \mathbb{B}(\bar{y}, \frac{r^*}{3})$ and it is noted earlier that $x_0 \in \mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$. Thus by appying Lemma 2.1, we get the following inequality:

$$\mathrm{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)) \le \frac{M\,\mathrm{dist}(0, \mathcal{G}_{x_0}(x_0))}{1 - MLr_{\bar{x}}} = \frac{M\,\mathrm{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0))}{1 - MLr_{\bar{x}}}.$$

But, by (52), we can obtain

$$\mathrm{dist}(0, \mathcal{N}(x_0)) = \mathrm{dist}(x_0, \mathcal{G}_{x_0}^{-1}(0)) \le \frac{M\,\mathrm{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0))}{1 - MLr_{\bar{x}}}. \tag{53}$$

According to Algorithm 2 and using (39), (41) and (53), we have

$$\begin{aligned} \|d_0\| &\le \eta\,\mathrm{dist}(0, \mathcal{N}(x_0)) \\ &\le \frac{\eta M\,\mathrm{dist}(0, f(x_0) + g(x_0) + \mathcal{F}(x_0))}{(1 - MLr_{\bar{x}})} \\ &\le \frac{\eta M(L + 4\nu)\delta^2}{2(1 - MLr_{\bar{x}})} = s\left(\frac{1}{2}\right)\delta. \end{aligned}$$

This means that

$$\|x_1 - x_0\| = \|d_0\| \le s\left(\frac{1}{2}\right)\delta,$$

and therefore, (43) is true for $n = 0$.We assume that $x_1, x_2, \ldots, x_k$ are constructed and (42), and (43) are true for $n = 0, 1, 2, \ldots, k-1$. We show that there exists x_{k+1} such that (42) and (43) are also hold for $n = k$. Since (42) and (43) are true for each $n \le k-1$, we have the following inequality:

$$\|x_k - \bar{x}\| \le \sum_{i=0}^{k-1} \|d_i\| + \|x_0 - \bar{x}\| \le s\delta \sum_{i=0}^{k-1} \left(\frac{1}{2}\right)^{2^i} + \delta \le 2\delta.$$

This shows that (42) holds for $n = k$.

Finally, we will show that the assertion (43) holds for $n = k$. For doing this, we will apply again the contraction mapping principle to Φ_{x_k} with $r := r_{x_k}$ and $\lambda := \dfrac{1}{5}$. Then we can deduce the existence of a fixed point $\hat{x}_{k+1} \in \mathbb{B}(\bar{x}, r_{x_k})$ satisfying $\hat{x}_{k+1} \in \Phi_{x_k}(\hat{x}_{k+1})$, which translates to $J_{x_k}(\hat{x}_{k+1}) \in \mathcal{G}_{\bar{x}}(\hat{x}_{k+1})$. This means that $0 \in f(x_k) + g(x_k) + (\nabla f(x_k) + [\hat{x}_{k+1}, x_k; g])(\hat{x}_{k+1} - x_k) + \mathcal{F}(\hat{x}_{k+1})$, that is, $\mathcal{N}(x_k) \ne \emptyset$. Choose $d_k \in \mathcal{N}(x_k)$ such that

$$\|d_k\| \le \eta\,\mathrm{dist}(0, \mathcal{N}(x_k)).$$

Then by Algorithm 2, set $x_{k+1} := x_k + d_k$. Moreover, applying Lemma 3.2 we infer that $\mathcal{G}_{x_k}^{-1}$ is Lipschitz-like on $\mathbb{B}(\bar{y}, r^*)$

relative to $\mathbb{B}(\bar{x}, \frac{r_{\bar{x}}}{2})$ with constant $\frac{M}{1-MLr_{\bar{x}}}$. Therefore, we can obtain the following inequality:

$$\begin{aligned}
\|x_{k+1} - x_k\| &= \|d_k\| \leq \eta \operatorname{dist}(0, \mathcal{N}(x_k)) \\
&\leq \eta \operatorname{dist}(x_k, \mathcal{G}_{x_k}^{-1}(0)) \\
&= \frac{\eta M}{1 - MLr_{\bar{x}}} \operatorname{dist}(0, f(x_k) + g(x_k) + \mathcal{F}(x_k)) \\
&\leq \frac{\eta M}{1 - MLr_{\bar{x}}} \|f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1}) \\
&\quad -(\nabla f(x_{k-1}) + [x_k, x_{k-1}; g])(x_k - x_{k-1})\| \\
&\leq \frac{\eta M}{1 - MLr_{\bar{x}}} \Big(\|f(x_k) - f(x_{k-1}) - \nabla f(x_{k-1})(x_k - x_{k-1})\| \\
&\quad + \|g(x_k) - g(x_{k-1}) - [x_k, x_{k-1}; g](x_k - x_{k-1})\| \Big) \\
&\leq \frac{\eta M}{2(1 - MLr_{\bar{x}})} \Big(L\|x_k - x_{k-1}\|^2 + \\
&\quad 2\|[x_{k-1}, x_k; g] - [x_k, x_{k-1}; g]\|\|x_k - x_{k-1}\| \Big) \\
&\leq \frac{\eta M}{2(1 - MLr_{\bar{x}})} \Big(L\|x_k - x_{k-1}\|^2 + \\
&\quad 2\nu(\|x_{k-1} - x_k\| + \|x_k - x_{k-1}\|)\|x_k - x_{k-1}\| \Big) \\
&= \frac{\eta M(L + 4\nu)}{2(1 - MLr_{\bar{x}})} \|x_k - x_{k-1}\|^2 \\
&\leq \frac{s}{2}\left(s\left(\frac{1}{2}\right)^{2^{k-1}} \delta \right)^2 \leq s\left(\frac{1}{2}\right)^{2^k} \delta.
\end{aligned}$$

This implies that (43) holds for $n = k$ and therefore the proof is complete. □

Consider the special case when $\bar{x}$ is a solution of (1)(that is, $\bar{y} = 0$) in Theorem 3.2. Then the following corollary, which gives the local quadratic convergence result for the extended Newton-type method. The proof of this corollary is similar to that we did for Corollary 3.1.

Corollary 3.2. *Suppose that $\bar{x}$ is solution of (1) and that $\mathcal{G}_{\bar{x}}^{-1}$ is pseudo-Lipschitz around $(0, \bar{x})$. Let $\eta > 1$, $\nu > 0$, $\tilde{r} > 0$ and suppose that ∇f is Lipschitz continuous on $\mathbb{B}(\bar{x}, \tilde{r})$ with Lipschitz constant L. Suppose that*

$$\lim_{x \to \bar{x}} \operatorname{dist}(0, f(x) + g(x) + \mathcal{F}(x)) = 0.$$

Then there exist some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{x}, \hat{\delta})$ converges quadratically to a solution x^* of (1).

4. Concluding Remarks

The semilocal and local convergence results for the extended Newton-type method are established when $\eta > 1$, ∇f is continuous and Lipschitz continuous, g admits first order divided difference as well as $\mathcal{G}_{\bar{x}}^{-1}$ is Lipschitz-like. This work extends and improves the result corresponding to (Argyros & Hilout (2008); Rashid (2016)).

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