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Convergence Theorems of Modified Proximal Algorithms for Asymptotical Quasi-nonexpansive Mappings in CAT(0) Spaces

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Abstract

In this paper, a new modified proximal point algorithm involving fixed point iterates of a finite number of asymptotically quasi-nonexpansive mappings in $CAT(0)$ spaces is proposed and been proved for the existence of a sequence generated by our iterative process converging to a minimizer of a convex function and a commen fixed point of a finite number of asymptotically quasi-nonexpansive mappings.

Keywords: convex minimization problem, $CAT(0)$ spaces, resolvent identity, asymptotically quasi-nonexpansive mapping, proximal point algorithm

1. Introduction

In recently years, many convergence results by the proximal point algorithm (shortly PPA) which was initiated by Martinet in 1970 for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces R^2, Hilbert spaces, and Banach spaces to the setting of some manifolds(for example, Riemannian manifolds, Hadamard manifolds).

A metric space (X, d) is called a $CAT(0)$ space (Ambrosio et al., 2008), if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. A complete $CAT(0)$ space is also called a *Hadamard* space. Especially, every real Hilbert space H is a complete $CAT(0)$ space. A subset K of a $CAT(0)$ space X is convex, if for any $x, y \in K$, we have $[x, y]$ in K, where $[x, y] := \{\lambda x \oplus (1-\lambda)y : 0 \le \lambda \le 1\}$ is the unique geodesic joining x and y. Let C be a nonempty closed subset of $CAT(0)$ space X and let $T : C \to C$ be a mapping. The set of fixed point of T is denote by $F(T)$, that is, denote by $F(T)$ the set of fixed point of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that T is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ and $p \in F(T)$ such that

$$d(T^n x, p) \le k_n d(x, p), \forall x \in C, n \ge 1.$$

It is well known that a geodesic space (X, d) is a $CAT(0)$ space, if and only if the inequality

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y) \tag{1.1}$$

is satisfied for all $x, y, z \in X$ and $t \in [0, 1]$. In particular, if x, y, z are points in a $CAT(0)$ space (X, d) and $t \in [0, 1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z). \tag{1.2}$$

We call that a function $f : C \to [-\infty, \infty)$ defined on a convex subset C of a $CAT(0)$ space is convex if, for any geodesic $[x, y] := \{\gamma_{x,y}(\lambda) : 0 \le \lambda \le 1\} = \{\lambda x \oplus (1-\lambda)y : 0 \le \lambda \le 1\}$ joining $x, y \in C$, the function $f \circ \gamma$ is convex, i.e. $f(\gamma_{x,y}(\lambda)) := f(\lambda x \oplus (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$. For all $\lambda \ge 0$, the $Moreau - Yosida$ resolvent of f is defined in a complete $CAT(0)$ space X as follows:

$$J_\lambda(x) = argmin_{y\in C}[f(y) + \frac{1}{2\lambda}d^2(y, x)].$$

Let $f : X \to (-\infty, \infty)$ be a proper convex and lower semi-continuous function. It was shown in (Agarwal, 2007) that the set $F(J_\lambda)$ of the fixed point of the resolvent J_λ associated with f coincides with the set $argmin_{y\in C} f(y)$ of minimizers of f. Also, for any $\lambda \ge 0$, the resolvent J_λ of f is nonexpansive (Jost, 1995). In 2013, *Bačák* (*Bačák*, 2013) introduced the

PPA in a $CAT(0)$ space (X, d) as follows: for any $x_1 \in X$ and

$$x_{n+1} = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)],$$

where $\lambda_n > 0, \forall n \in N$. It was shown that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ $\Delta - converges$ to its minimizer (Ariza-Ruiz, 2014).

Many mathematical researchers have continued their directions of the research work. In 2017, Nuttapol Pakkaranang.etc (Nuttapol et al, 2017), they introduced the following algorithm:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) \oplus \frac{1}{2\lambda_n} d^2(y, x_n)], \\ w_n = (1 - \alpha_n)z_n \oplus \alpha_n R^n z_n, \\ y_n = (1 - \beta_n)w_n \oplus \beta_n S^n w_n, \\ x_{n+1} = (1 - \gamma_n)y_n \oplus \gamma_n T^n y_n, \ n \geq 1, \end{cases}$$

where R, S, T are three asymptotically quasi-nonexpansive mappings. They proved some weakly convergence theorems of the sequence $\{x_n\}$ for the proposed algorithm to common fixed points of asymptotically quasi-nonexpansive mappings and to minimizers of a convex function in $CAT(0)$ spaces.

Stimulated and inspired by the work of the above mathematics researchers, in this paper, we come up with the following algorithm:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, m = 1, \ n \geq 1, \\ x_1 \in C \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, \\ y_{1n} = (1 - \alpha_{2n})y_{2n} \oplus \alpha_{2n} T_2^n y_{2n}, \\ \cdots\cdots \\ y_{((m-2)n)} = (1 - \alpha_{(m-1)n})y_{(m-1)n} \oplus \alpha_{(m-1)n} T_{(m-1)}^n y_{(m-1)n}, \\ y_{((m-1)n)} = (1 - \alpha_{mn})z_n \oplus \alpha_{mn} T_{(m-1)}^n z_n, m \geq 2, n \geq 1, \end{cases} \tag{1}$$

where $\lambda_n > 0, \forall n \in N, T_i (i = 1, 2, ..., m)$ is a finite number of asymptotically quasi-nonexpansive mappings. Research its convergence, the results that we obtained improve and extend the results of reference (Nuttapol et al., 2017).

2. Preliminries

In this section, we will metion some basic concepts, and useful lemmas, which will be used in the next section.

Definition2.1 (Chang et al., 2012) Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space (X, d). For any $x \in X$, we put $r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$.

(1) The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$;

(2) The asymptotic center $A(\{x_n\})$ of x_n is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

It is well known that, in a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point (Kirk & Panyanak, 2008).

Definition2.2 (Chang, 2016) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be $\Delta - convergent$ to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} = x$ of $\{x_n\}$ and denote

$$\varpi_{\Delta}(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subset \Omega,$$

where the union is sum over all subsequences $\{u_n\}$ of $\{x_n\}$.

Definition 2.3 Let C be a nonempty closed convex subset of a $CAT(0)$ space (X, d). A family of mappings $\{T_1, T_2, ..., T_m, T_{m+1}\}$ is said to satisfy the condition (ω^*) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) \geq 0$ for all $r \in (0, \infty)$ such that

$$d(x, T_1 x) \geq f(d(x, F))$$

or

$$d(x, T_2x) \geq f(d(x, F))$$

$$...$$

or

$$d(x, T_mx) \geq f(d(x, F))$$

or

$$d(x, T_{m+1}x) \geq f(d(x, F))$$

for all $x \in X$, where $F = \bigcap_{i=1}^{m+1} F(T_i)$.

Definition 2.4 (Nuttapol et al., 2017) Let (X, d) be a metric space, and C is a nonempty subset of X. A mapping $T : C \to C$ is said to be *semi − compact* if any sequence $\{x_n\}$ in C satisfying $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ has a convergent subsequence, that is, it exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in C$.

Lemma2.5 (Chang, 2012) If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $d(x_n, u)$ converges, then $x = u$.

Lemma2.6 (Kirk & Panyanak, 2008) Every bounded sequence in a complete $CAT(0)$ space X has a $\Delta - convergent$ subsequence.

Lemma2.7 (Ambrosio, 2008) Let (X, d) be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous. Then, for all $x, y \in X$ and $\lambda \geq 0$, the following inequality holds:

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x, y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y), \tag{2.3}$$

where J_λ is a $Moreau - Yosida$ resolvent of f.

Lemma2.8 (Chang, 2012) Assume that C is a closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta{-}\lim x_n = p$ and $\lim_{n\to 0} d(x_n, Tx_n)$. Then $Tp = p$.

Lemma2.9 (Xu, 2003) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following conditions:

$$\alpha_{n+1} \leq (1 + b_n)\alpha_n,$$

where $b_n \geq 0$ and $\Sigma_{n=1}^{\infty} b_n < \infty$, then the $\lim_{n\to\infty} \alpha_n$ exists.

Lemma2.10 (Mayer,1998) Let (X, d) be a complete $CAT(0)$ space and $f : (-\infty, \infty] \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:

$$J_\lambda x = J_\mu(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x), \forall x \in X, \lambda \geq \mu > 0, \tag{2.6}$$

where J_λ is a $Moreau - Yosida$ resolvent of f.

3. Results

In this section, we prove our main results.

Theorem 3.1 Suppose that the following conditions are satisfied:

(1) Let (X, d) be a complete $CAT(0)$ space and C be a nonempty closed convex subset of X;

(2) Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function;

(3) $T_i : C \to C, i = 1, 2, ..., m$ are a finite number of $\{k_n\}$−asymptotically quasi-nonexpansive mappings with $k_n \in [1, \infty)$, $\lim_{n\to\infty} k_n = 1$, $\Sigma_{i=1}^{\infty}(k_n - 1) < \infty$ such that

$$\Omega = \bigcap_{i=1}^{m} F(T_i) \cap argmin_{y\in C} f(y) \neq \emptyset; \tag{3.1}$$

(4) $\{\alpha_{in}\}_{i=1,2,...,m}$ be sequences in $[0, 1]$ with $0 < a \leq \alpha_{in} \leq c < 1$ for all $n \in N$ and for some a, c are positive constants in $(0, 1)$;

(5) $\{\lambda_n\}$ be a sequence with $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ.

Then, the sequence $\{x_n\}$ defined by the algorithm (1) $\Delta-$converges to a point $x^* \in \Omega$, which is a minimizer of f in C as well as a commen fixed point of $T_i, i = 1, 2, .., m$.

Proof: The proof will be completed in five steps.

Let $p \in \Omega$. Then $p = T_1 p = T_2 p = ... = T_m p$ and $f(p) \le f(y), \forall y \in C$. Therefore, we have

$$f(p) + \frac{1}{2\lambda_n} d^2(p, p) \le f(y) + \frac{1}{2\lambda_n} d^2(y, p), \forall y \in C. \tag{3.2}$$

Hence, $p = J_\lambda p, \forall n \ge 1$.

The first step, we prove that the limit $\lim_{n\to\infty} d(x_n, p)$ exists.

Since J_{λ_n} is nonexpansive and $z_n = J_{\lambda_n} x_n$, so we have

$$d(z_n, p) = d(J_{\lambda_n} x_n, J_{\lambda_n} p) \le d(x_n, p). \tag{3.3}$$

While $m = 1$, we obtain that

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n,\ n \ge 1. \end{cases}$$

By (1.2), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, p) \\ &\le (1 - \alpha_{1n})d(z_n, p) + \alpha_{1n} d(T_1^n z_n, p) \\ &\le [1 + (k_n - 1)\alpha_{1n}]d(x_n, p). \end{aligned}$$

This implies that there exists a $Q_{1n} = (k_n - 1)\alpha_{1n}$ and $Q_{1n} \ge 0$ and $\Sigma_{n=1}^\infty Q_{1n} < \infty$, such that

$$d(x_{n+1}, p) \le (1 + Q_{1n})d(x_n, p).$$

By lemma 2.9, we obtain that the limit $\lim_{n\to\infty} d(x_n, p)$ exists. So, $\{x_n\}$ is bounded. Thus, $\{z_n\}$ and $\{T_1^n z_n\}$ is bounded.

While $m = 2$, we have

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n} \\ y_{1n} = (1 - \alpha_{2n})z_n \oplus \alpha_{2n} T_2^n z_n,\ n \ge 1. \end{cases}$$

By virtue of (1.2) and (3.3), we get

$$\begin{aligned} d(y_{1n}, p) &= d((1 - \alpha_{2n})z_n \oplus \alpha_{2n} T_2^n z_n, p) \\ &\le (1 - \alpha_{2n})d(z_n, p) + \alpha_{2n} d(T_2^n z_n, p) \\ &\le (1 - \alpha_{2n} + \alpha_{2n} k_n)d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, p) \\ &\le (1 - \alpha_{1n})d(y_{1n}, p) + \alpha_{1n} d(T_1^n y_{1n}, p) \\ &\le (1 - \alpha_{1n})d(y_{1n}, p) + \alpha_{1n} k_n d(y_{1n}, p) \\ &\le (1 - \alpha_{1n} + \alpha_{1n} k_n)(1 - \alpha_{2n} + \alpha_{2n} k_n)d(x_n, p) \\ &= [1 + (\alpha_{1n} + \alpha_{2n} + \alpha_{1n}\alpha_{2n}(k_n - 1))(k_n - 1)]d(x_n, p). \end{aligned}$$

So, there is a $Q_{2n} = (\alpha_{1n} + \alpha_{2n} + \alpha_{1n}\alpha_{2n}(k_n - 1))$, and $Q_{2n} \ge 0$, and $\Sigma_{n=1}^\infty Q_{2n} < \infty$ such that

$$d(x_{n+1}, p) \le (1 + Q_{2n})d(x_n, p).$$

Therefore, by lemma 2.9, we obtain that the limit $\lim_{n\to\infty} d(x_n, p)$ exists. Thus, $\{x_n\}$ is bounded, and so $\{z_n\}$, $\{y_{1n}\}$, $\{T_1^n y_{1n}\}$ and $\{T_1^n z_n\}$ also are bounded.

This analogies, it implies that there is a $Q_{in} \geq 0, i = 1, 2, ..., m$, and $\Sigma_{n=1}^{\infty} Q_{in} < \infty$ such that

$$d(x_{n+1}, p) \leq (1 + Q_{in})d(x_n, p).$$

Similarly, by lemma 2.9 we have that the limit $\lim_{n\to\infty} d(x_n, p)$ exists.

Therefore, $\{x_n\}$ is bounded, and so are $\{z_n\}$, $\{y_{in}\}_{i=1,2,...,m-1}$, $\{T_i^n y_{in}\}_{i=1,2,..,m-1}$, and $\{T_m^n z_n\}$ are bounded.

The second step, we prove that $\lim_{n\to\infty} d(x_n, z_n) = 0$. Let

$$\lim_{n\to\infty} d(x_n, p) = c \geq 0. \tag{3.4}$$

By lemma 2.7, we get

$$\frac{1}{2\lambda_n} d^2(z_n, p) - \frac{1}{2\lambda_n} d^2(x_n, p) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \leq f(p) - f(z_n).$$

That is,

$$\frac{1}{2\lambda_n}\{d^2(z_n, p) - d^2(x_n, p) + d^2(x_n, z_n)\} \leq f(p) - f(z_n).$$

Because of $f(p) \leq f(z_n), \forall n \geq 1$, we get

$$d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(z_n, p). \tag{3.5}$$

While $m = 1$, we have

$$\begin{cases} x_1 \in C \ \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, \ n \geq 1. \end{cases}$$

By (1.2), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, p) \\ &\leq (1 - \alpha_{1n})d(z_n, p) + \alpha_{1n} d(T_1^n z_n, p) \\ &\leq (1 - \alpha_{1n} + \alpha_{1n} k_n)d(z_n, p), \end{aligned}$$

which can be rewritten as

$$d(z_n, p) \geq \frac{1}{1 - \alpha_{1n} + \alpha_{1n} k_n} d(x_{n+1}, p).$$

Therefore, this combines the above with (3.4) implies that

$$\lim_{n\to\infty} \inf d(z_n, p) \geq \lim_{n\to\infty} \inf\{\frac{1}{1 - \alpha_{1n} + \alpha_{1n} k_n} d(x_{n+1}, p)\} = c.$$

On the other hand, by (3.3), we also get

$$\lim_{n\to\infty} \sup d(z_n, p) \leq \lim_{n\to\infty} \sup d(x_n, p) = c.$$

This implies that

$$\lim_{n\to\infty} d(z_n, p) = c. \tag{3.6}$$

Also from (3.4), (3.5) and (3.6), we obtain that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.7}$$

While $m = 2$, we have

$$\begin{cases} x_1 \in C \ \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, \\ y_{1n} = (1 - \alpha_{2n})z_n \oplus \alpha_{2n} T_2^n z_n, \ n \geq 1. \end{cases}$$

By (1.2), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, p) \\ &\leq (1-\alpha_{1n}) d(y_{1n}, p) + \alpha_{1n} d(T_1^n y_{1n}, p) \\ &\leq (1-\alpha_{1n} + \alpha_{1n} k_n) d(y_{1n}, p). \end{aligned}$$

Simplifying we have

$$d(y_{1n}, p) \geq \frac{1}{1-\alpha_{1n}+\alpha_{1n}k_n} d(x_{n+1}, p).$$

So, we can get

$$\lim_{n\to\infty} \inf d(y_{1n}, p) \geq \lim_{n\to\infty} \inf\{\frac{1}{1-\alpha_{1n}+\alpha_{1n}k_n} d(x_{n+1}, p)\} = c.$$

On the other hand, it shows that

$$\lim_{n\to\infty} \sup d(y_{1n}, p) \leq \lim_{n\to\infty} \sup\{(1-\alpha_{2n}+\alpha_{2n}k_n) d(x_n, p)\} = c.$$

Therefore, it implies that

$$\lim_{n\to\infty} d(y_{1n}, p) = c. \tag{3.8}$$

By (1.2) and (3.8), we get

$$\begin{aligned} d(y_{1n}, p) &= d((1-\alpha_{2n})z_n \oplus \alpha_{2n} T_2^n z_n, p) \\ &\leq (1-\alpha_{2n}) d(z_n, p) + \alpha_{2n} d(T_2^n z_n, p) \\ &\leq (1-\alpha_{2n} + \alpha_{2n} k_n) d(z_n, p). \end{aligned}$$

Similarly, simplifying we have

$$d(z_n, p) \geq \frac{1}{1-\alpha_{2n}+\alpha_{2n}k_n} d(y_{1n}, p).$$

So, we obtain that

$$\lim_{n\to\infty} \inf d(z_n, p) \geq \lim_{n\to\infty} \inf\{\frac{1}{1-\alpha_{2n}+\alpha_{2n}k_n} d(y_{1n}, p)\} = c.$$

By (3.3) we have

$$\lim_{n\to\infty} \sup d(z_n, p) \leq \lim_{n\to\infty} \sup d(x_n, p) = c.$$

So we get $\lim_{n\to\infty} d(z_n, p) = c.$

This togethers the above with (3.5) shows that $\lim_{n\to\infty} d(x_n, z_n) = 0.$

And it can be pushed that

$$\lim_{n\to\infty} d(z_n, p) = c, \lim_{n\to\infty} d(y_{in}, p) = c; i = 1, 2, ..., m-1.$$

By (3.8) we also have $\lim_{n\to\infty} d(x_n, z_n) = 0.$

The third steps, we prove that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = ... = \lim_{n\to\infty} d(x_n, T_m x_n) = 0.$$

While $m = 1$, we have

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1-\alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, \ n \geq 1. \end{cases}$$

By (1.1) we get

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_{1n})z_n \oplus \alpha_{1n}T_1^n z_n, p) \\ &\le (1-\alpha_{1n})d^2(z_n, p) + \alpha_{1n}d^2(T_1^n z_n, p) - \alpha_{1n}(1-\alpha_{1n})d^2(z_n, T_1^n z_n) \\ &\le (1-\alpha_{1n}+\alpha_{1n}k_n^2)d^2(z_n, p) - \alpha_{1n}(1-\alpha_{1n})d^2(z_n, T_1^n z_n). \end{aligned}$$

Simplifying above the inequality, we have

$$\begin{aligned} d^2(z_n, T_1^n z_n) &\le \frac{1}{\alpha_{1n}(1-\alpha_{1n})}\{(1-\alpha_{1n}+\alpha_{1n}k_n^2)d^2(z_n, p) - d^2(x_{n+1}, p)\} \\ &\le \frac{1}{a(1-c)}\{(1-\alpha_{1n}+\alpha_{1n}k_n^2)d^2(z_n, p) - d^2(x_{n+1}, p)\} \to 0 (n\to\infty). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} d(x_n, T_1^n x_n) &\le d(T_1^n x_n, T_1^n z_n) + d(T_1^n z_n, z_n) + d(x_n, z_n) \\ &\le k_n d(x_n, z_n) + d(T_1^n z_n, z_n) + d(x_n, z_n) \\ &= (1-k_n)d(x_n, z_n) + d(T_1^n z_n, z_n) \to 0 (n\to\infty). \end{aligned}$$

It shows that

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\alpha_{1n})z_n \oplus \alpha_{1n}T_1^n z_n, x_n) \\ &\le (1-\alpha_{1n})d(z_n, x_n) + \alpha_{1n}d(T_1^n z_n, x_n) \\ &\le (1-\alpha_{1n})d(z_n, x_n) + \alpha_{1n}\{d(T_1^n z_n, z_n) + d(z_n, x_n)\} \to 0 (n\to\infty). \end{aligned}$$

So, we get

$$\begin{aligned} d(x_n, T_1 x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_1^{n+1}x_{n+1}) + d(T_1^{n+1}, T_1^{n+1}x_n) + d(T_1^{n+1}x_n, T_1 x_n) \\ &\le d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, T_1^{n+1}x_{n+1}) + k_{n+1}d(x_{n+1}, x_n) + k_1 d(T_1^n x_n, x_n) \\ &\le (1-\alpha_{1n})d(z_n, x_n) + \alpha_{1n}\{d(T_1^n z_n, z_n) + d(z_n, x_n)\} \to 0 (n\to\infty). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0.$$

While $m = 2$, we have

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ x_{n+1} = (1-\alpha_{1n})y_{1n} \oplus \alpha_{1n}T_1^n y_{1n}, \\ y_{1n} = (1-\alpha_{2n})z_n \oplus \alpha_{2n}T_2^n z_n,\ n \ge 1. \end{cases}$$

By (1.1) we get

$$\begin{aligned} d^2(y_{1n}, p) &= d^2((1-\alpha_{2n})z_n \oplus \alpha_{2n}T_2^n z_n, p) \\ &\le (1-\alpha_{2n})d^2(z_n, p) + \alpha_{2n}d^2(T_2^n z_n, p) - \alpha_{2n}(1-\alpha_{2n})d^2(z_n, T_2^n z_n). \end{aligned}$$

Simplifying the above and we get

$$\begin{aligned} d^2(z_n, T_2^n z_n) &\le \frac{1}{\alpha_{2n}(1-\alpha_{2n})}\{(1-\alpha_{2n}+\alpha_{2n}k_n^2)d^2(z_n, p) - d^2(y_{1n}, p)\} \\ &\le \frac{1}{a(1-c)}\{(1-\alpha_{2n}+\alpha_{2n}k_n^2)d^2(z_n, p) - d^2(y_{1n}, p)\} \to 0 (n\to\infty). \end{aligned}$$

Similarly we obtain that

$$\begin{aligned} d(x_n, T_2^n x_n) &\leq d(T_2^n x_n, T_2^n z_n) + d(T_2^n z_n, z_n) + d(z_n, x_n) \\ &\leq k_n d(x_n, z_n) + d(T_2^n z_n, z_n) + d(z_n, x_n) \to 0 (n \to \infty). \end{aligned}$$

This together with (1.2), we have

$$\begin{aligned} d(y_{1n}, x_n) &= d((1-\alpha_{2n})z_n \oplus \alpha_{2n} T_2^n z_n, x_n) \\ &\leq (1-\alpha_{2n}) d(z_n, x_n) + \alpha_{2n} d(T_2^n z_n, x_n) \\ &\leq (1-\alpha_{2n}) d(z_n, x_n) + \alpha_{2n}\{d(T_2^n z_n, z_n) + d(z_n, x_n)\} \\ &= d(z_n, x_n) + \alpha_{2n} d(T_2^n z_n, z_n) \to 0 (n \to \infty). \end{aligned}$$

By (1.1) we get

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, p) \\ &\leq (1-\alpha_{1n}) d^2(y_{1n}, p) + \alpha_{1n} d^2(T_1^n y_{1n}, p) - \alpha_{1n}(1-\alpha_{1n}) d^2(y_{1n}, T_1^n y_{1n}). \end{aligned}$$

Rearranging the above inequality, it implies that

$$\begin{aligned} d^2(y_{1n}, T_1^n y_{1n}) &\leq \frac{1}{\alpha_{1n}(1-\alpha_{1n})}\{(1-\alpha_{1n}+\alpha_{1n}k_n^2) d^2(y_{1n}, p) - d^2(x_{n+1}, p)\} \\ &\leq \frac{1}{a(1-c)}\{(1-\alpha_{1n}+\alpha_{1n}k_n^2) d^2(y_{1n}, p) - d^2(x_{n+1}, p)\} \to 0 (n \to \infty). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} d(x_n, T_1^n x_n) &\leq d(T_1^n x_n, T_1^n y_{1n}) + d(T_1^n y_{1n}, y_{1n}) + d(y_{1n}, x_n) \\ &\leq k_n d(x_n, y_{1n}) + d(T_1^n y_{1n}, y_{1n}) + d(y_{1n}, x_n) \to 0 (n \to \infty). \end{aligned}$$

By (1.2), this together with the above some inequalities, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, x_n) \\ &\leq (1-\alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n} d(T_1^n y_{1n}, x_n) \\ &\leq (1-\alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n}\{d(T_1^n y_{1n}, T_1^n x_n) + d(T_1^n x_n, x_n)\} \\ &\leq (1-\alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n}\{k_n d(y_{1n}, x_n) + d(T_1^n x_n, x_n)\} \\ &= (1-\alpha_{1n}+\alpha_{1n}k_n) d(y_{1n}, x_n) + \alpha_{1n} d(T_1^n x_n, x_n) \to 0 (n \to \infty). \end{aligned}$$

Therefore, we have

$$\begin{aligned} d(x_n, T_1 x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_1^{n+1} x_{n+1}) + d(T_1^{n+1} x_{n+1}, T_1^{n+1} x_n) + d(T_1^{n+1} x_n, T_1 x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_1^{n+1} x_{n+1}) + d(T_1^{n+1} x_{n+1}, T_1^{n+1} x_n) + d(T_1^{n+1} x_n, T_1 x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_1^{n+1} x_{n+1}) + k_n d(x_{n+1}, x_n) + k_1 d(T_1^n x_n, x_n) \to 0 (n \to \infty). \end{aligned}$$

This implies that $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0.$

Similarly, we obtain

$$\begin{aligned} d(x_n, T_2 x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_2^{n+1} x_{n+1}) + d(T_2^{n+1} x_{n+1}, T_2^{n+1} x_n) + d(T_2^{n+1} x_n, T_2 x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_2^{n+1} x_{n+1}) + k_n d(x_{n+1}, x_n) + d(T_2^{n+1} x_n, x_n) \to 0 (n \to \infty). \end{aligned}$$

Therefore, we have $\lim_{n\to\infty} d(x_n, T_2x_n) = 0.$

Thus, we get

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0.$$

This analogies, we obtain that

$$\begin{cases} \lim_{n\to\infty} d(z_n, T_m^n z_n) = 0;\ m = 1, 2, ...;\ m \geq 1, \\ \lim_{n\to\infty} d(y_{in}, x_n) = 0;\ i = 1, 2, ..., m-1;\ m \geq 2, \\ \lim_{n\to\infty} d(y_{in}, T_i^n y_{in}) = 0;\ i = 1, 2, ..., m-1;\ m \geq 2, \\ \lim_{n\to\infty} d(x_n, T_i^n x_n) = 0;\ i = 1, 2, ...m-1;\ m \geq 1, \\ \lim_{n\to\infty} d(x_{n+1}, x_n) = 0;\ \forall n \geq 1. \end{cases}$$

It implies that

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = ... = \lim_{n\to\infty} d(x_n, T_mx_n) = 0. \tag{3.9}$$

The fourth steps, we prove that

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0,\ \lambda_n \geq \lambda > 0.$$

Because of $\lambda_n \geq \lambda > 0$, by lemma 2.10 and (3.7), we get

$$\begin{aligned} d(J_\lambda x_n, x_n) &\leq d(J_\lambda x_n, z_n) + d(z_n, x_n) \\ &= d(J_\lambda x_n, J_{\lambda_n} x_n) + d(z_n, x_n) \\ &\leq d(J_\lambda x_n, J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n)) + d(z_n, x_n) \\ &\leq d(x_n, \frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n) + d(z_n, x_n) \\ &= \frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} d(x_n, J_{\lambda_n} x_n) + d(z_n, x_n) \\ &= \frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} d(x_n, z_n) + d(z_n, x_n) \to 0 (n \to \infty). \end{aligned}$$

This shows that

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0. \tag{3.10}$$

By the first step, It follows that the limit $\lim_{n\to\infty} d(x_n, p)$ exists. By (3.9) and (3.10) , we have

$$\lim_{n\to\infty} d(x_n, T_1x_n) = ... = \lim_{n\to\infty} d(x_n, T_mx_n) = lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0. \tag{3.11}$$

The fifth steps, we prove that

$$\varpi_\Delta(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subset \Omega,$$

where $A(\{u_n\})$ is the asymptotic center of $\{u_n\}$.

Let $u \in \varpi_\Delta(x_n)$. Then, by lemma 2.6, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$.

Therefore, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in C$.

This together with (3.11) and it shows that

$$\lim_{n\to\infty} d(v_n, T_iv_n) = 0, i = 1, 2, ..., m; lim_{n\to\infty} d(J_\lambda v_n, v_n) = 0.$$

By lemma 2.8, it shows that $v \in \Omega$. So, by lemma 2.5 we obtain that $u = v$. This implies that $\varpi_\Delta(x_n) \subset \Omega$.

Finally, we will prove that the sequence $\{x_n\}$ $\Delta - converges$ to a point $x^* \in \Omega$.

It will suffice to prove that $\varpi_\Delta(x_n)$ consists of exactly one point in the end.

Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$.

Since $u \in \varpi_\Delta(x_n) \subset \Omega$ and $\{d(x_n, \mu)\}$ converges, so, by virtue of lemma 2.5, we obtain that $x = u$. Thus, $\varpi_\Delta(x_n) = \{x^*\}$. This completes the proof. □

Corollary 3.2 Suppose that the following conditions are satisfied:

(1) Let (X, d) be a real Hilbert space and C be a nonempty closed convex subset of X;

(2) Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function;

(3) $T_i : C \to C, i = 1, 2, ..., m$ are a finite number of $\{k_n\}$−asymptotically quasi-nonexpansive mappings with $k_n \in [1, \infty), \lim_{n\to\infty} k_n = 1, \Sigma_{i=1}^{\infty}(k_n - 1) < \infty$ such that

$$\Omega = \bigcap_{i=1}^{m} F(T_i) \cap argmin_{y\in C} f(y) \neq \emptyset;$$

(4) $\{\alpha_{in}\}_{,i=1,2,...,m}$ be sequences in $[0, 1]$ with $0 < a \leq \alpha_{in} \leq c < 1$ for all $n \in N$ and for some a, c are positive constants in $(0, 1)$;

(5) $\{\lambda_n\}$ be a sequence with $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ. Let $\{x_n\}$ be the sequence generated by the following manner:

$$\begin{cases} x_1 \in C \ \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n}\|(y, x_n)\|^2], \\ x_{n+1} = (1 - \alpha_{1n})z_n \oplus \alpha_{1n}T_1^n z_n, m = 1, \ n \geq 1, \\ x_1 \in C \ \text{ chosen arbitrarily}, \\ z_n = argmin_{y\in C}[f(y) + \frac{1}{2\lambda_n}\|(y, x_n)\|^2], \\ x_{n+1} = (1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n}T_1^n y_{1n}, \\ y_{1n} = (1 - \alpha_{2n})y_{2n} \oplus \alpha_{2n}T_2^n y_{2n}, \\ \cdots\cdots \\ y_{((m-2)n)} = (1 - \alpha_{(m-1)n})y_{(m-1)n} \oplus \alpha_{(m-1)n}T_{(m-1)}^n y_{(m-1)n}, \\ y_{((m-1)n)} = (1 - \alpha_{mn})z_n \oplus \alpha_{mn}T_{(m-1)}^n z_n, m \geq 2, \forall n \geq 1, \end{cases}$$

for each $n \in N$, then the sequence $\{x_n\}$ weakly converges to a common element $x^* \in \Omega$.

Theorem 3.3 Under the hypothesis of Theorem 3.1, suppose that the family of the mappings $\{T_1, T_2, ..., T_m, J_\lambda\}$ satisfies the condition (ω^*). Then, the sequence $\{x_n\}$ defined by (1) strongly converges to a common element $x^* \in \Omega$.

Proof. From the first step of theorem 3.1 , we get that $\lim_{n\to\infty} d(x_n, x^*)$ exists for $x^* \in \Omega$. Also, it follows that $\lim_{n\to\infty} d(x_n, \Omega)$ exists. On the other hand, because a finite number of asymptotically quasi-nonexpansive mappings $\{T_1, T_2, ..., T_m, J_\lambda\}$ satisfied the conditions ω^*, we have

$$\lim_{n\to\infty} f(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_1x_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_2x_n) = 0$$

$$\cdots$$

or

$$\lim_{n\to\infty} f(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, T_mx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, \Omega)) \leq \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

Thus, we have $\lim_{n\to\infty} f(d(x_n, \Omega)) = 0$. By using the property of f, we have $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Thus, following the proof of Theorem 3.3 of the reference (*Pakkaranangetal.*, 2017), which implies that $\{x_n\}$ is a Cauchy sequence in X and so $\{x_n\}$ converges to a point $x^* \in X$ and hence $d(x^*, \Omega) = 0$. Since Ω is closed, we have $x^* \in \Omega$. This completes the proof. □

Theorem 3.4 Under the hypothesis of Theorem 3.1, suppose that T_1 or T_2 or $\cdots$ or T_m or T_λ is semi-compact. Then the sequence $\{x_n\}$ defined by (3.1) strongly converges to a common element $p \in \Omega$.

Proof. Suppose that T_1 is semi-compact. By the third step of Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in X$. Since

$$\lim_{n\to\infty} d(x_n, T_2x_n) = ... = \lim_{n\to\infty} d(x_n, T_mx_n) = lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

We have $d(p, T_2p) = ... = d(p, T_mp) = 0$ and $d(p, J_\lambda p) = 0$, which shows that $p \in \Omega$. For other mappings, we also prove that the sequence $\{x_n\}$ strongly converges to a common element of Ω. This completes the proof. □

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