Convergence Theorems of Modified Proximal Algorithms for Asymptotical Quasi-nonexpansive Mappings in CAT(0) Spaces

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Abstract

In this paper, a new modified proximal point algorithm involving fixed point iterates of a finite number of asymptotically quasi-nonexpansive mappings in CAT(0) spaces is proposed and been proved for the existence of a sequence generated by our iterative process converging to a minimizer of a convex function and a common fixed point of a finite number of asymptotically quasi-nonexpansive mappings.

Keywords: convex minimization problem, CAT(0) spaces, resolvent identity, asymptotically quasi-nonexpansive mapping, proximal point algorithm

1. Introduction

In recent years, many convergence results by the proximal point algorithm (shortly PPA) which was initiated by Martinet in 1970 for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces to the setting of some manifolds (for example, Riemannian manifolds, Hadamard manifolds).

A metric space $(X,d)$ is called a CAT(0) space (Ambrosio et al., 2008), if it is geodesically connected and if every geodesic triangle in $X$ is at least as 'thin' as its comparison triangle in the Euclidean plane. A complete CAT(0) space is also called a Hadamard space. Especially, every real Hilbert space $H$ is a complete CAT(0) space. A subset $K$ of a CAT(0) space $X$ is convex, if for any $x,y \in K$, we have $[x,y]$ in $K$, where $[x,y] := \{ \lambda x \oplus (1-\lambda)y : 0 \leq \lambda \leq 1 \}$ is the unique geodesic joining $x$ and $y$. Let $C$ be a nonempty closed subset of CAT(0) space $X$ and let $T : C \rightarrow C$ be a mapping. The set of fixed point of $T$ is denote by $F(T)$. Recall that $T$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $(k_n)$ in $[1,\infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ and $p \in F(T)$ such that

$$d(T^n x, p) \leq k_n d(x, p), \forall x \in C, n \geq 1.$$  

It is well known that a geodesic space $(X,d)$ is a CAT(0) space, if and only if the inequality

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$$  

(1.1)

is satisfied for all $x,y,z \in X$ and $t \in [0,1]$. In particular, if $x,y,z$ are points in a CAT(0) space $(X,d)$ and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$  

(1.2)

We call that a function $f : C \rightarrow [-\infty,\infty]$ defined on a convex subset $C$ of a CAT(0) space is convex if, for any geodesic $[x,y] := \{ \gamma\lambda : 0 \leq \lambda \leq 1 \} = \{ \lambda x \oplus (1-\lambda)y : 0 \leq \lambda \leq 1 \}$ joining $x,y \in C$, the function $f \circ \gamma$ is convex, i.e. $f(\gamma\lambda) := f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. For all $\lambda \geq 0$, the Moreau – Yosida resolvent of $f$ is defined in a complete CAT(0) space $X$ as follows:

$$J_\lambda(x) = \arg\min_{y \in C} \{ f(y) + \frac{1}{2\lambda}d^2(x,y) \}.$$  

Let $f : X \rightarrow (-\infty,\infty)$ be a proper convex and lower semi-continuous function. It was shown in (Agarwal, 2007) that the set $F(J_\lambda)$ of the fixed point of the resolvent $J_\lambda$ associated with $f$ coincides with the set $\arg\min_{y \in C} f(y)$ of minimizers of $f$. Also, for any $\lambda \geq 0$, the resolvent $J_\lambda$ of $f$ is nonexpansive (Jost, 1995). In 2013, Bačák (Bačák, 2013) introduced the
PPA in a CAT(0) space $(X, d)$ as follows: for any $x_1 \in X$ and

$$x_{n+1} = \text{argmin}_{y \in C} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)],$$

where $\lambda_n > 0, \forall n \in N$. It was shown that if $f$ has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ $\Delta -$ converges to its minimizer (Ariza-Ruiz, 2014).

Many mathematical researchers have continued their directions of the research work. In 2017, Nuttopol Pakkaranang, etc (Nuttapol et al., 2017), they introduced the following algorithm:

$$
\begin{align*}
&x_1 \in C \text{ chosen arbitrarily}, \\
&z_n = \text{argmin}_{y \in C} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\
&w_n = (1 - \alpha_n)z_n + \alpha_n R_1^m z_n, \\
&y_n = (1 - \beta_n)w_n + \beta_n S_n^m w_n, \\
&x_{n+1} = (1 - \gamma_n)y_n + \gamma_n T_2^n y_n, \quad n \geq 1,
\end{align*}
$$

where $R, S, T$ are three asymptotically quasi-nonexpansive mappings. They proved some weakly convergence theorems of the sequence $\{x_n\}$ for the proposed algorithm to common fixed points of asymptotically quasi-nonexpansive mappings and to minimizers of a convex function in CAT(0) spaces.

Stimulated and inspired by the work of the above mathematics researchers, in this paper, we come up with the following algorithm:

$$
\begin{align*}
&x_1 \in C \text{ chosen arbitrarily}, \\
&z_n = \text{argmin}_{y \in C} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\
&w_n = (1 - \alpha_n)z_n + \alpha_n T_1^m z_n, \\
&y_n = (1 - \beta_n)w_n + \beta_n T_1^n w_n, \\
&x_{n+1} = (1 - \gamma_n)y_1 + \gamma_n T_2^n y_1, \\
&\ldots \ldots \\
&y_{(m-2)n} = (1 - \alpha_{(m-1)n})y_{(m-1)n} + \alpha_{(m-1)n} T_1^n y_{(m-1)n}, \\
&y_{(m-1)n} = (1 - \gamma_{(m-1)n})y_{(m-1)n} + \gamma_{(m-1)n} T_2^n y_{(m-1)n}, \\
&\text{where } \lambda_n > 0, \forall n \in N, T_i(i = 1, 2, \ldots, m) \text{ is a finite number of asymptotically quasi-nonexpansive mappings. Research its convergence, the results that we obtained improve and extend the results of reference (Nuttapol et al., 2017).}
\end{align*}
$$

2. Preliminaries

In this section, we will metion some basic concepts, and useful lemmas, which will be used in the next section.

**Definition 2.1 (Chang et al., 2012)** Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $(X, d)$. For any $x \in X$, we put $r(x, \{x_n\}) = \lim_{n \to \infty} \sup d(x, x_n)$.

(1) The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$;

(2) The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

It is well known that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (Kirk & Panyanak, 2008).

**Definition 2.2 (Chang, 2016)** A sequence $\{x_n\}$ in a CAT(0) space $X$ is said to be $\Delta -$ convergent to a point $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n \to \infty} x = \{x_n\}$ and denote

$$\omega_{\Delta}(x_n) := \bigcup_{\{u_n\} \subset \{x_n\}} A(\{u_n\}) \subset \Omega,$$

where the union is sum over all subsequences $\{u_n\}$ of $\{x_n\}$.

**Definition 2.3** Let $C$ be a nonempty closed convex subset of a CAT(0) space $(X, d)$. A family of mappings $\{T_1, T_2, \ldots, T_m, T_{m+1}\}$ is said to satisfy the condition $(\omega^*)$ if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) \geq 0$ for all $r \in (0, \infty)$ such that

$$d(x, T_1x) \geq f(d(x, F)).$$
or
\[ d(x, T_2 x) \geq f(d(x, F)) \]
... or
\[ d(x, T_m x) \geq f(d(x, F)) \]
\[ d(x, T_{m+1} x) \geq f(d(x, F)) \]
for all \( x \in X \), where \( F = \bigcap_{i=1}^{m+1} F(T_i) \).

**Definition 2.4 (Nuttapal et al., 2017)** Let \((X, d)\) be a metric space, and \( C \) is a nonempty subset of \( X \). A mapping \( T : C \to C \) is said to be semi-compact if any sequence \( \{x_n\} \) in \( C \) satisfying \( \lim_{n \to \infty} d(x_n, T x_n) = 0 \) has a convergent subsequence, that is, it exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to p \in C \).

**Lemma 2.5 (Chang, 2012)** Let \((X, d)\) be a complete CAT(0) space with \( A([x_n]) = \{x\} \) is a subsequence of \( \{x_n\} \) with \( A([u_n]) = \{u\} \) and the sequence \( d(x_n, u) \) converges, then \( x = u \).

**Lemma 2.8 (Chang, 2012)** Assume that \( C \) is a closed convex subset of a complete CAT(0) space \( X \) and \( T : C \to C \) be an asymptotically nonexpansive mapping. Let \( \{x_n\} \) be a bounded sequence in \( C \) such that \( \Delta x_n = p \) and \( \lim_{n \to 0} d(x_n, T x_n) \). Then \( T p = p \).

**Lemma 2.9 (Xu, 2003)** Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following conditions:
\[ a_{n+1} \leq (1 + b_n) a_n, \]
where \( b_n \geq 0 \) and \( \sum_{n=1}^{\infty} b_n < \infty \), then the \( \lim_{n \to \infty} a_n \) exists.

**Lemma 2.10 (Mayer, 1998)** Let \((X, d)\) be a complete CAT(0) space and \( f : (-\infty, \infty] \to (-\infty, \infty] \) be a proper convex and lower semi-continuous function. Then the following identity holds:
\[ J_{\lambda x} = J_{\mu \lambda} \left( \frac{\lambda - \mu}{\lambda} J_{\lambda x} \oplus \frac{\mu}{\lambda} x \right), \forall x \in X, \lambda \geq \mu > 0, \]
where \( J_{\lambda} \) is a Moreau–Yosida resolvent of \( f \).

### 3. Results

In this section, we prove our main results.

**Theorem 3.1** Suppose that the following conditions are satisfied:

1. Let \((X, d)\) be a complete CAT(0) space and \( C \) be a nonempty closed convex subset of \( X \);
2. Let \( f : X \to (-\infty, \infty] \) be a proper convex and lower semi-continuous function;
3. \( T_i : C \to C, i = 1, 2, ..., m \) are a finite number of \( \{k_n\} \)-asymptotically quasi-nonexpansive mappings with \( k_n \in [1, \infty], \lim_{n \to \infty} k_n = 1, \sum_{n=1}^{\infty} k_n - 1 < \infty \) such that
\[ \Omega = \bigcap_{i=1}^{m} F(T_i) \cap \argmin_{y \in C} f(y) \neq \emptyset; \]
4. \( \{a_n\}_{i=1, 2, ..., m} \) are sequences in \([0, 1]\) with \( 0 < a \leq a_m \leq c < 1 \) for all \( n \in N \) and for some \( a, c \) are positive constants in \((0, 1)\);
5. \( \{\lambda_n\} \) be a sequence with \( \lambda_n \geq \lambda > 0 \) for all \( n \geq 1 \) and for some \( \lambda \).
Then, the sequence \( \{x_n\} \) defined by the algorithm (1) Α–converges to a point \( x^* \in \Omega \), which is a minimizer of \( f \) in \( C \) as well as a common fixed point of \( T_i, i = 1, 2, .., m. \)

Proof: The proof will be completed in five steps.

Let \( p \in \Omega \). Then \( p = T_1p = T_2p = ... = T_mp \) and \( f(p) \leq f(y), \forall y \in C \). Therefore, we have

\[
f(p) + \frac{1}{2\lambda_n} d^2(p, p) \leq f(y) + \frac{1}{2\lambda_n} d^2(y, p), \forall y \in C.
\] (3.2)

Hence, \( p = J_\lambda p, \forall n \geq 1. \)

The first step, we prove that the limit \( \lim_{n \to \infty} d(x_n, p) \) exists.

Since \( J_\lambda \) is nonexpansive and \( z_n = J_\lambda x_n \), so we have

\[
d(z_n, p) = d(J_\lambda x_n, J_\lambda p) \leq d(x_n, p).
\] (3.3)

While \( m = 1 \), we obtain that

\[
\begin{align*}
&x_1 \in C \text{ chosen arbitrarily}, \\
z_n = \arg\min_{y \in C} \{ f(y) + \frac{1}{\lambda_n} d^2(y, x_n) \}, \\
x_{n+1} = (1 - \alpha_n)z_n + \alpha_n T_1^n z_n, \ n \geq 1.
\end{align*}
\]

By (1.2), we get

\[
\begin{align*}
d(x_{n+1}, p) &= d((1 - \alpha_n)z_n + \alpha_n T_1^n z_n, p) \\
&\leq (1 - \alpha_n) d(z_n, p) + \alpha_n d(T_1^n z_n, p) \\
&\leq [1 + (k_1 - 1)\alpha_n] d(x_n, p).
\end{align*}
\]

This implies that there exists a \( Q_{1n} = (k_1 - 1)\alpha_n \) and \( Q_{1n} \geq 0 \) and \( \sum_{n=1}^{\infty} Q_{1n} < \infty \), such that

\[
d(x_{n+1}, p) \leq (1 + Q_{1n}) d(x_n, p).
\]

By lemma 2.9, we obtain that the limit \( \lim_{n \to \infty} d(x_n, p) \) exists. So, \( \{x_n\} \) is bounded. Thus, \( \{z_n\} \) and \( \{T_1^n z_n\} \) is bounded.

While \( m = 2 \), we have

\[
\begin{align*}
&x_1 \in C \text{ chosen arbitrarily}, \\
z_n = \arg\min_{y \in C} \{ f(y) + \frac{1}{\lambda_n} d^2(y, x_n) \}, \\
x_{n+1} = (1 - \alpha_n)y_{1n} + \alpha_n T_1^n y_{1n} \\
y_{1n} = (1 - \alpha_{2n})z_n + \alpha_{2n} T_2^n z_n, \ n \geq 1.
\end{align*}
\]

By virtue of (1.2) and (3.3), we get

\[
\begin{align*}
d(y_{1n}, p) &= d((1 - \alpha_{2n})z_n + \alpha_{2n} T_2^n z_n, p) \\
&\leq (1 - \alpha_{2n}) d(z_n, p) + \alpha_{2n} d(T_2^n z_n, p) \\
&\leq (1 - \alpha_{2n} + \alpha_{2n} + \alpha_{2n} k_2) d(x_n, p)
\end{align*}
\]

and

\[
\begin{align*}
d(x_{n+1}, p) &= d((1 - \alpha_n)y_{1n} + \alpha_n T_1^n y_{1n}, p) \\
&\leq (1 - \alpha_n) d(y_{1n}, p) + \alpha_n d(T_1^n y_{1n}, p) \\
&\leq (1 - \alpha_n) d(y_{1n}, p) + \alpha_n k_1 d(y_{1n}, p) \\
&\leq (1 - \alpha_n + \alpha_n k_2)(1 - \alpha_{2n} + \alpha_{2n} k_2) d(x_n, p) \\
&= [1 + (\alpha_{2n} + \alpha_{2n} + \alpha_{2n} k_2)(k_n - 1)](k_n - 1) d(x_n, p).
\end{align*}
\]

So, there is a \( Q_{2n} = (\alpha_{1n} + \alpha_{2n} + \alpha_{1n} \alpha_{2n}(k_n - 1)) \), and \( Q_{2n} \geq 0 \), and \( \sum_{n=1}^{\infty} Q_{2n} < \infty \) such that

\[
d(x_{n+1}, p) \leq (1 + Q_{2n}) d(x_n, p).
\]
Therefore, by lemma 2.9, we obtain that the limit \( \lim_{n \to \infty} d(x_n, p) \) exists. Thus, \( \{x_n\} \) is bounded, and so \( \{z_n\}, \{y_{1n}\}, \{T^n_1y_{1n}\} \) and \( \{T^n_1z_n\} \) also are bounded.

This analogues, it implies that there is a \( Q_m \geq 0, i = 1, 2, \ldots, m, \) and \( \sum_{m=1}^{\infty} Q_m < \infty \) such that

\[
d(x_{n+1}, p) \leq (1 + Q_m)d(x_n, p).
\]

Similarly, by lemma 2.9 we have that the limit \( \lim_{n \to \infty} d(x_n, p) \) exists.

Therefore, \( \{x_n\} \) is bounded, and so \( \{z_n\}, \{y_{1m}\}_{i=1,2,\ldots,m-1}, \{T^n_1y_{1m}\}_{i=1,2,\ldots,m-1} \) and \( \{T^n_1z_n\} \) are bounded.

The second step, we prove that \( \lim_{n \to \infty} d(x_n, z_n) = 0 \). Let

\[
\lim_{n \to \infty} d(x_n, p) = c \geq 0. \tag{3.4}
\]

By lemma 2.7, we get

\[
\frac{1}{2\lambda_n}d^2(z_n, p) = \frac{1}{2\lambda_n}d^2(x_n, p) + \frac{1}{2\lambda_n}d^2(x_n, z_n) \leq f(p) - f(z_n).
\]

That is,

\[
\frac{1}{2\lambda_n}(d^2(z_n, p) - d^2(x_n, p) + d^2(x_n, z_n)) \leq f(p) - f(z_n).
\]

Because of \( f(p) \leq f(z_n) \), \( \forall n \geq 1 \), we get

\[
d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(z_n, p). \tag{3.5}
\]

While \( m = 1 \), we have

\[
\begin{cases}
x_1 \in C \text{ chosen arbitrarily}, \\
z_n = \arg\min_{y \in C} [f(y) + \frac{1}{\lambda_n}d^2(y, x_n)], \\
x_{n+1} = (1 - \alpha_{1n})z_n + \alpha_{1n}T^n_1z_n, \ n \geq 1.
\end{cases}
\]

By (1.2), we get

\[
d(x_{n+1}, p) = d((1 - \alpha_{1n})z_n + \alpha_{1n}T^n_1z_n, p) \\
\leq (1 - \alpha_{1n})d(z_n, p) + \alpha_{1n}d(T^n_1z_n, p) \\
\leq (1 - \alpha_{1n} + \alpha_{1n}k_n)d(z_n, p),
\]

which can be rewritten as

\[
d(z_n, p) \geq \frac{1}{1 - \alpha_{1n} + \alpha_{1n}k_n}d(x_{n+1}, p).
\]

Therefore, this combines the above with (3.4) implies that

\[
\lim_{n \to \infty} d(z_n, p) \geq \lim_{n \to \infty} \frac{1}{1 - \alpha_{1n} + \alpha_{1n}k_n}d(x_{n+1}, p) = c.
\]

On the other hand, by (3.3), we also get

\[
\lim_{n \to \infty} \sup d(z_n, p) \leq \lim_{n \to \infty} \sup d(x_n, p) = c.
\]

This implies that

\[
\lim_{n \to \infty} d(z_n, p) = c. \tag{3.6}
\]

Also from (3.4), (3.5) and (3.6), we obtain that

\[
\lim_{n \to \infty} d(x_n, z_n) = 0. \tag{3.7}
\]

While \( m = 2 \), we have

\[
\begin{cases}
x_1 \in C \text{ chosen arbitrarily}, \\
z_n = \arg\min_{y \in C} [f(y) + \frac{1}{\lambda_n}d^2(y, x_n)], \\
x_{n+1} = (1 - \alpha_{1n})y_{1n} + \alpha_{1n}T^n_1y_{1n}, \\
y_{1n} = (1 - \alpha_{2n})z_n + \alpha_{2n}T^n_2z_n, \ n \geq 1.
\end{cases}
\]
By (1.2), we get
\[
d(x_{n+1}, p) = d((1 - \alpha_{1n})y_{1n} \oplus \alpha_{1n} T_1^n y_{1n}, p) \\
\leq (1 - \alpha_{1n})d(y_{1n}, p) + \alpha_{1n}d(T_1^n y_{1n}, p) \\
\leq (1 - \alpha_{1n} + \alpha_{1n}k_n)d(y_{1n}, p).
\]

Simplifying we have
\[
d(y_{1n}, p) \geq \frac{1}{1 - \alpha_{1n} + \alpha_{1n}k_n}d(x_{n+1}, p).
\]

So, we can get
\[
\liminf_{n \to \infty} d(y_{1n}, p) \geq \liminf_{n \to \infty} \left\{ \frac{1}{1 - \alpha_{1n} + \alpha_{1n}k_n}d(x_{n+1}, p) \right\} = c.
\]

On the other hand, it shows that
\[
\limsup_{n \to \infty} d(y_{1n}, p) \leq \limsup_{n \to \infty} ((1 - \alpha_{2n} + \alpha_{2n}k_n)d(x_n, p)) = c.
\]

Therefore, it implies that
\[
\lim_{n \to \infty} d(y_{1n}, p) = c. \quad (3.8)
\]

By (1.2) and (3.8), we get
\[
d(y_{1n}, p) = d((1 - \alpha_{2n})z_{2n} \oplus \alpha_{2n} T_2^n z_{2n}, p) \\
\leq (1 - \alpha_{2n})d(z_{2n}, p) + \alpha_{2n}d(T_2^n z_{2n}, p) \\
\leq (1 - \alpha_{2n} + \alpha_{2n}k_n)d(z_{2n}, p).
\]

Similarly, simplifying we have
\[
d(z_{2n}, p) \geq \frac{1}{1 - \alpha_{2n} + \alpha_{2n}k_n}d(y_{1n}, p).
\]

So, we obtain that
\[
\liminf_{n \to \infty} d(z_{2n}, p) \geq \liminf_{n \to \infty} \left\{ \frac{1}{1 - \alpha_{2n} + \alpha_{2n}k_n}d(y_{1n}, p) \right\} = c.
\]

By (3.3) we have
\[
\limsup_{n \to \infty} d(z_{2n}, p) \leq \limsup_{n \to \infty} d(x_n, p) = c.
\]

So we get
\[
\lim_{n \to \infty} d(z_{2n}, p) = c.
\]

This together with (3.5) shows that
\[
\lim_{n \to \infty} d(x_n, z_{2n}) = 0.
\]

And it can be pushed that
\[
\lim_{n \to \infty} d(z_{2n}, p) = c, \quad \lim_{n \to \infty} d(y_{in}, p) = c; i = 1, 2, ..., m - 1.
\]

By (3.8) we also have
\[
\lim_{n \to \infty} d(x_n, z_{2n}) = 0.
\]

The third steps, we prove that
\[
\lim_{n \to \infty} d(x_n, T_1 x_n) = \lim_{n \to \infty} d(x_n, T_2 x_n) = ... = \lim_{n \to \infty} d(x_n, T_m x_n) = 0.
\]

While \( m = 1 \), we have
\[
\begin{align*}
x_1 & \in C \text{ chosen arbitrarily}, \\
z_n & = \arg\min_{y \in C} \left\{ f(y) + \frac{1}{\alpha_{1n}}d^2(y, x_n) \right\}, \\
x_{n+1} & = (1 - \alpha_{1n})z_n \oplus \alpha_{1n} T_1^n z_n, \quad n \geq 1.
\end{align*}
\]
By (1.1) we get

\[
d^2(x_{n+1}, p) = d^2((1 - \alpha_n)x_n \oplus \alpha_n T_{1}^{n} z_n, p) \\
\leq (1 - \alpha_n)d^2(z_n, p) + \alpha_n d^2(T_{1}^{n} z_n, p) - \alpha_n(1 - \alpha_n)d^2(z_n, T_{1}^{n} z_n) \\
\leq (1 - \alpha_n + \alpha_n k_2^2)d^2(z_n, p) - \alpha_n(1 - \alpha_n)d^2(z_n, T_{1}^{n} z_n).
\]

Simplifying above the inequality, we have

\[
d^2(z_n, T_{1}^{n} z_n) \leq \frac{1}{\alpha_n(1 - \alpha_n)}[(1 - \alpha_n + \alpha_n k_2^2)d^2(z_n, p) - d^2(x_{n+1}, p)] \\
\leq \frac{1}{a(1 - c)}[(1 - \alpha_n + \alpha_n k_2^2)d^2(z_n, p) - d^2(x_{n+1}, p)] \to 0(n \to \infty).
\]

Thus, we obtain that

\[
d(x_n, T_{1}^{n} x_n) \leq d(T_{1}^{n} x_n, T_{1}^{n} z_n) + d(x_n, z_n) \\
\leq k_0d(x_n, z_n) + d(T_{1}^{n} z_n, z_n) + d(x_n, z_n) \\
= (1 - k_0)d(x_n, z_n) + d(T_{1}^{n} z_n, z_n) \to 0(n \to \infty).
\]

It shows that

\[
d(x_{n+1}, x_n) = d((1 - \alpha_n)x_n \oplus \alpha_n T_{1}^{n} z_n, x_n) \\
\leq (1 - \alpha_n)d(z_n, x_n) + \alpha_n d(T_{1}^{n} z_n, x_n) \\
\leq (1 - \alpha_n)d(z_n, x_n) + \alpha_n[d(T_{1}^{n} z_n, z_n) + d(z_n, x_n)] \to 0(n \to \infty).
\]

So, we get

\[
d(x_n, T_{1}^{n} x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T_{1}^{n+1} x_{n+1}) + d(T_{1}^{n+1} x_{n+1}, T_{1}^{n} x_{n+1}) + d(x_{n+1}, x_n) + d(T_{1}^{n+1} x_{n+1}, T_{1}^{n} x_{n+1}) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_n) + d(x_{n+1}, T_{1}^{n+1} x_{n+1}) + k_{n+1}d(x_{n+1}, x_n) + k_1d(T_{1}^{n} x_n, x_n) \\
\leq (1 - \alpha_{n+1})d(z_n, x_n) + \alpha_{n+1}[d(T_{1}^{n} z_n, z_n) + d(z_n, x_n)] \to 0(n \to \infty).
\]

This implies that

\[
\lim_{n \to \infty} d(x_n, T_{1}^{n} x_n) = 0.
\]

While \(m = 2\), we have

\[
\begin{align*}
   x_1 &\in C \text{ chosen arbitrarily,} \\
   z_n &= \arg\min_{y \in C} \{f(y) + \frac{1}{x_n} d^2(y, x_n)\}, \\
   x_{n+1} &= (1 - \alpha_{n})y_{n} \oplus \alpha_{n} T_{1}^{n} y_{n}, \\
   y_{1} &= (1 - \alpha_{2n})z_{n} \oplus \alpha_{2n} T_{2}^{n} z_{n}, \ n \geq 1.
\end{align*}
\]

By (1.1) we get

\[
d^2(y_{1n}, p) = d^2((1 - \alpha_{2n})z_n \oplus \alpha_{2n} T_{2}^{n} z_n, p) \\
\leq (1 - \alpha_{2n})d^2(z_n, p) + \alpha_{2n}d^2(T_{2}^{n} z_n, p) - \alpha_{2n}(1 - \alpha_{2n})d^2(z_n, T_{2}^{n} z_n).
\]

Simplifying the above and we get

\[
d^2(z_n, T_{2}^{n} z_n) \leq \frac{1}{\alpha_{2n}(1 - \alpha_{2n})}[(1 - \alpha_{2n} + \alpha_{2n} k_2^2)d^2(z_n, p) - d^2(y_{1n}, p)] \\
\leq \frac{1}{a(1 - c)}[(1 - \alpha_{2n} + \alpha_{2n} k_2^2)d^2(z_n, p) - d^2(y_{1n}, p)] \to 0(n \to \infty).
\]
Similarly we obtain that
\[
d(x_n, T^a_{2n} x_n) \leq d(T^a_{2n} x_n, T^a_{2n} z_n) + d(z_n, x_n) \\
\leq k_d d(x_n, z_n) + d(T^a_{2n} z_n, x_n) + d(z_n, x_n) \to 0(n \to \infty).
\]

This together with (1.2), we have
\[
d(y_{1n}, x_n) = d((1 - \alpha_{2n}) x_n \oplus \alpha_{2n} T^a_{2n} z_n, T^a_{2n} y_{1n}) \\
\leq (1 - \alpha_{2n}) d(z_n, x_n) + \alpha_{2n} d(T^a_{2n} z_n, x_n) \\
\leq (1 - \alpha_{2n}) d(z_n, x_n) + \alpha_{2n} [d(T^a_{2n} z_n, x_n) + d(z_n, x_n)] \\
= d(z_n, x_n) + \alpha_{2n} d(T^a_{2n} z_n, x_n) \to 0(n \to \infty).
\]

By (1.1) we get
\[
d^2(x_{n+1}, p) = d^2((1 - \alpha_{1n}) y_{1n} \oplus \alpha_{1n} T^a_{1} y_{1n}, p) \\
\leq (1 - \alpha_{1n}) d^2(y_{1n}, p) + \alpha_{1n} d^2(T^a_{1} y_{1n}, p) - \alpha_{1n} (1 - \alpha_{1n}) d^2(y_{1n}, T^a_{1} y_{1n}).
\]

Rearranging the above inequality, it implies that
\[
d^2(y_{1n}, T^a_{1} y_{1n}) \leq \frac{1}{\alpha_{1n}(1 - \alpha_{1n})} [d^2(y_{1n}, p) - d^2(x_{n+1}, p)] \\
\leq \frac{1}{\alpha_{1n}(1 - \alpha_{1n})} [d^2(y_{1n}, p) - d^2(x_{n+1}, p)] \to 0(n \to \infty).
\]

Thus, we obtain that
\[
d(x_n, T^a_{1} x_n) \leq d(T^a_{1} x_n, T^a_{1} y_{1n}) + d(y_{1n}, x_n) \\
\leq k_d d(x_n, y_{1n}) + d(y_{1n}, x_n) \to 0(n \to \infty).
\]

By (1.2), this together with the above some inequalities, we get
\[
d(x_{n+1}, x_n) = d((1 - \alpha_{1n}) y_{1n} \oplus \alpha_{1n} T^a_{1} y_{1n}, x_n) \\
\leq (1 - \alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n} d(T^a_{1} y_{1n}, x_n) \\
\leq (1 - \alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n} [d(T^a_{1} y_{1n}, T^a_{1} x_n) + d(T^a_{1} x_n, x_n)] \\
\leq (1 - \alpha_{1n}) d(y_{1n}, x_n) + \alpha_{1n} [k_d d(y_{1n}, x_n) + d(T^a_{1} x_n, x_n)] \\
= (1 - \alpha_{1n}) + \alpha_{1n} k_d d(y_{1n}, x_n) + \alpha_{1n} d(T^a_{1} x_n, x_n) \to 0(n \to \infty).
\]

Therefore, we have
\[
d(x_n, T^a_{1} x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^a_{1} x_{n+1}) + d(T^a_{1} x_{n+1}, T^a_{1} x_n) + d(T^a_{1} x_n, T^a_{1} x_n) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^a_{1} x_{n+1}) + d(T^a_{1} x_{n+1}, T^a_{1} x_n) + d(T^a_{1} x_n, T^a_{1} x_n) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^a_{1} x_{n+1}) + k_d d(x_{n+1}, x_n) + k_d d(T^a_{1} x_n, x_n) \to 0(n \to \infty).
\]

This implies that
\[
\lim_{n \to \infty} d(x_n, T^a_{1} x_n) = 0.
\]

Similarly, we obtain
\[
d(x_n, T^a_{2} x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^a_{2} x_{n+1}) + d(T^a_{2} x_{n+1}, T^a_{2} x_n) + d(T^a_{2} x_n, T^a_{2} x_n) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^a_{2} x_{n+1}) + k_d d(x_{n+1}, x_n) + d(T^a_{2} x_n, x_n) \to 0(n \to \infty).
\]
Therefore, we have 
\[ \lim_{n \to \infty} d(x_n, T_2 x_n) = 0. \]
Thus, we get
\[ \lim_{n \to \infty} d(x_n, T_1 x_n) = \lim_{n \to \infty} d(x_n, T_2 x_n) = 0. \]
This analogies, we obtain that
\[
\begin{align*}
    \lim_{n \to \infty} d(z_n, T_m^n z_n) &= 0; \quad m = 1, 2, \ldots; \quad m \geq 1, \\
    \lim_{n \to \infty} d(y_{in}, x_n) &= 0; \quad i = 1, 2, \ldots, m - 1; \quad m \geq 2, \\
    \lim_{n \to \infty} d(y_{in}, T_m^n y_{in}) &= 0; \quad i = 1, 2, \ldots, m - 1; \quad m \geq 2, \\
    \lim_{n \to \infty} d(x_n, T_m^n x_n) &= 0; \quad i = 1, 2, \ldots, m - 1; \quad m \geq 1, \\
    \lim_{n \to \infty} d(x_{n+1}, x_n) &= 0; \quad \forall n \geq 1.
\end{align*}
\]
It implies that
\[ \lim_{n \to \infty} d(x_n, T_1 x_n) = \lim_{n \to \infty} d(x_n, T_2 x_n) = \ldots = \lim_{n \to \infty} d(x_n, T_m x_n) = 0. \quad (3.9) \]
The fourth steps, we prove that
\[ \lim_{n \to \infty} d(J_1 x_n, x_n) = 0, \quad A_n \geq \lambda > 0. \]
Because of \( A_n \geq \lambda > 0 \), by lemma 2.10 and (3.7), we get
\[
\begin{align*}
    d(J_1 x_n, x_n) &\leq d(J_1 x_n, z_n) + d(z_n, x_n) \\
    &= d(J_1 x_n, J_1 x_n) + d(z_n, x_n) \\
    &\leq d(J_1 x_n, J_1 (\frac{A_n}{\lambda_n} J_2 x_n + \frac{\lambda}{A_n} x_n)) + d(z_n, x_n) \\
    &\leq d(x_n, \frac{A_n}{\lambda_n} J_2 x_n + \frac{\lambda}{A_n} x_n) + d(z_n, x_n) \\
    &= \frac{A_n}{\lambda_n} J_2 d(x_n, J_2 x_n) + d(z_n, x_n) \\
    &\leq \frac{A_n - \lambda}{\lambda_n} J_2 d(x_n, z_n) + d(z_n, x_n) \to 0(n \to \infty).
\end{align*}
\]
This shows that
\[ \lim_{n \to \infty} d(J_1 x_n, x_n) = 0. \quad (3.10) \]
By the first step, It follows that the limit \( \lim_{n \to \infty} d(x_n, p) \) exists. By (3.9) and (3.10), we have
\[ \lim_{n \to \infty} d(x_n, T_1 x_n) = \ldots = \lim_{n \to \infty} d(x_n, T_m x_n) = \lim_{n \to \infty} d(J_1 x_n, x_n) = 0. \quad (3.11) \]
The fifth steps, we prove that
\[ \sigma_\Delta(x_n) := \bigcup_{[u_n] \subset \{x_n\}} A([u_n]) \subset \Omega, \]
where \( A([u_n]) \) is the asymptotic center of \( \{u_n\} \).
Let \( u \in \sigma_\Delta(x_n) \). Then, by lemma 2.6, there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A([u_n]) = \{u\} \).
Therefore, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_{n \to \infty} v_n = v \) for some \( v \in C \).
This together with (3.11) and it shows that
\[ \lim_{n \to \infty} d(v_n, T_1 v_n) = 0, i = 1, 2, \ldots, m; \lim_{n \to \infty} d(J_1 v_n, v_n) = 0. \]
By lemma 2.8, it shows that \( v \in \Omega \). So, by lemma 2.5 we obtain that \( u = v \). This implies that \( \sigma_\Delta(x_n) \subset \Omega \).
Finally, we will prove that the sequence \( \{x_n\} \Delta \) converges to a point \( x^* \in \Omega \).
It will suffice to prove that \( \sigma_\Delta(x_n) \) consists of exactly one point in the end.
Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \) with \( A([u_n]) = \{u\} \) and let \( A([x_n]) = \{x\} \).
Since \( u \in \sigma_\Lambda(x_n) \subset \Omega \) and \( \{d(x_n,\mu)\} \) converges, so, by virtue of lemma 2.5, we obtain that \( x = u \). Thus, \( \sigma_\Lambda(x_n) = \{x^*\} \).

This completes the proof. \( \square \)

**Corollary 3.2** Suppose that the following conditions are satisfied:

1. Let \((X,d)\) be a real Hilbert space and \(C\) be a nonempty closed convex subset of \(X\);
2. Let \( f : X \rightarrow (-\infty, \infty] \) be a proper convex and lower semi-continuous function;
3. \( T_i : C \rightarrow C, i = 1, 2, \ldots, m \) are a finite number of \( \{k_i\} \)-asymptotically quasi-nonexpansive mappings with \( k_i \in [1, \infty) \), \( \lim_{n \rightarrow \infty} k_i = 1 \), \( \Sigma_{i=1}^{\infty} (k_i - 1) < \infty \) such that
   \[
   \Omega = \bigcap_{i=1}^{m} F(T_i) \cap \arg\min_{y \in C} f(y) \neq \emptyset;
   \]
4. \( \{a_m\}, i = 1, 2, \ldots, m \) be sequences in \([0, 1]\) with \( 0 < a \leq a_m \leq c < 1 \) for all \( n \in N \) and for some \( a, c \) are positive constants in \((0, 1)\);
5. \( \{\lambda_n\} \) be a sequence with \( \lambda_n \geq \lambda > 0 \) for all \( n \geq 1 \) and for some \( \lambda \). Let \( \{x_n\} \) be the sequence generated by the following manner:

\[
\begin{align*}
    x_1 & \in C \text{ chosen arbitrarily,} \\
    z_n & = \arg\min_{y \in C} \left\{ f(y) + \frac{1}{a_m} \|y - x_n\|^2 \right\}, \\
    x_{n+1} & = (1 - a_{1n}) x_n \oplus a_{1n} T^1_{n} x_n, \quad m = 1, n \geq 1, \\
    x_1 & \in C \text{ chosen arbitrarily,} \\
    z_n & = \arg\min_{y \in C} \left\{ f(y) + \frac{1}{a_m} \|y - x_n\|^2 \right\}, \\
    x_{n+1} & = (1 - a_{1n}) y_{1n} \oplus a_{1n} T^1_{n} y_{1n}, \\
    y_{1n} & = (1 - a_{2n}) y_{2n} \oplus a_{2n} T^1_{2n} y_{2n}, \\
    \ldots \ldots \\
    y_{(m-2)n} & = (1 - a_{(m-1)n}) y_{(m-1)n} \oplus a_{(m-1)n} T^{m-1}_{(m-1)n} y_{(m-1)n}, \\
    y_{(m-1)n} & = (1 - a_{mn}) z_{2n} \oplus a_{mn} T^{m-1}_{(m-1)n} z_{2n}, \quad m \geq 2, \forall n \geq 1,
\end{align*}
\]

for each \( n \in N \), then the sequence \( \{x_n\} \) weakly converges to a common element \( x^* \in \Omega \).

**Theorem 3.3** Under the hypothesis of Theorem 3.1, suppose that the family of the mappings \( \{T_1, T_2, \ldots, T_m, J_1\} \) satisfies the condition \((\omega')\). Then, the sequence \( \{x_n\} \) defined by (1) strongly converges to a common element \( x^* \in \Omega \).

**Proof.** From the first step of theorem 3.1, we get that \( \lim_{n \rightarrow \infty} d(x_n, x^*) \) exists for \( x^* \in \Omega \). Also, it follows that \( \lim_{n \rightarrow \infty} d(x_n, \Omega) \) exists. On the other hand, because a finite number of asymptotically quasi-nonexpansive mappings \( \{T_1, T_2, \ldots, T_m, J_1\} \) satisfied the conditions \( \omega^* \), we have

\[
\lim_{n \rightarrow \infty} f(d(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, T_1 x_n) = 0
\]

or

\[
\lim_{n \rightarrow \infty} f(d(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, T_2 x_n) = 0
\]

or

\[
\lim_{n \rightarrow \infty} f(d(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, T_m x_n) = 0
\]

or

\[
\lim_{n \rightarrow \infty} f(d(x_n, \Omega)) \leq \lim_{n \rightarrow \infty} d(x_n, J_1 x_n) = 0.
\]

Thus, we have \( \lim_{n \rightarrow \infty} f(d(x_n, \Omega)) = 0 \). By using the property of \( f \), we have \( \lim_{n \rightarrow \infty} d(x_n, \Omega) = 0 \). Thus, following the proof of Theorem 3.3 of the reference (Pakkaranangnetal., 2017), which implies that \( \{x_n\} \) is a Cauchy sequence in \( X \) and so \( \{x_n\} \) converges to a point \( x^* \in X \) and hence \( d(x^*, \Omega) = 0 \). Since \( \Omega \) is closed, we have \( x^* \in \Omega \). This completes the proof. \( \square \)

**Theorem 3.4** Under the hypothesis of Theorem 3.1, suppose that \( T_1 \) or \( T_2 \) or \( \cdots \) or \( T_m \) or \( T_1 \) is semi-compact. Then the sequence \( \{x_n\} \) defined by (3.1) strongly converges to a common element \( p \in \Omega \).
Proof. Suppose that $T_1$ is semi-compact. By the third step of Theorem 3.1, we have $\lim_{n \to \infty} d(x_n, T_1 x_n) = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in X$. Since

$$\lim_{n \to \infty} d(x_n, T_2 x_n) = \lim_{n \to \infty} d(x_n, T_3 x_n) = \lim_{n \to \infty} d(x_n, J_A x_n) = 0.$$  

We have $d(p, T_2 p) = \ldots = d(p, T_m p) = 0$ and $d(p, J_p) = 0$, which shows that $p \in \Omega$. For other mappings, we also prove that the sequence $\{x_n\}$ strongly converges to a common element of $\Omega$. This completes the proof. □

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References


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