Journal of Mathematics Research; Vol. 10, No. 3; June 2018
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Application of Extended Geometrical Criterion to Population Model with Two Time Delays

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Received: February 24, 2018 Accepted: March 12, 2018 Online Published: April 16, 2018

doi:10.5539/jmr.v10n3p63 URL: https://doi.org/10.5539/jmr.v10n3p63

Abstract

Geometrical criterion is a flexible method to be applied to a type of delay differential equations with delay dependent coefficient. The criterion is used to solve roots attribution of the related characteristic equation in complex plane effectively by introducing a new parameter skillfully. An extended geometrical criterion is developed to compute the stability of DDEs with two time delays. It is found that stability switching phenomena arise while equilibrium solution loses its stability and becomes unstable, then retrieve its stability again. Hopf bifurcation and the bifurcating periodic solution is analyzed by applying central manifold reduction method. The novel dynamical behaviors such as periodical solution bifurcating to chaos are discovered by using numerical simulation method.

Keywords: Hopf bifurcation, parameter delay dependent, geometrical criterion

1. Introduction

As is well known, the issue of delay differential equations has aroused a big attention from a rather diverse group of scientist since its application widely in many fields of science and engineering. The special feature of DDEs lies that system contains function which dependent on its past time history thus the corresponding phase space is infinite dimensional (Fowler, 1997). The stability analysis of system is ubiquitous since bifurcation behavior of equilibrium and periodic solution can change system dynamics dramatically as varying time delay (Gourley & Kuang, 2004; Wang & Hu, 1999; Cooke et al., 1999). In general, we describe DDEs with the following formula

$$x'(t) = f(x(t), x(t-\tau), \sigma, \tau) \tag{1.1}$$

where $x(t) : R \to R^n$ is the present state variable, while $x(t-\tau) : R \to R^n$ denotes the state variable at the past time τ. Parameter σ and τ are free parameters which affect system dynamics qualitatively. As a common example, population oscillating model always contains time delay as a physical parameter.

In reality and natural modelling of population with ”stage structure” (Ma et al., 2005; Aiello & Freedman, 1990; Aiello et al., 1992), the delay dependent coefficients appear in system expressed in an exponential formula with a decay rate σ. People conceive that the whole life stage survival of population is often a function of time delay τ but in distinct life stage population size is lessened by factor σ since death is unavoidable. A notable example was the work of Aiello and Freedman on a single species model with two growth stages and delay dependent coefficients. On another respect, population movement is the common habits that happened on some living beings, such as fishes or birds, for example, population partly migrate from the birth place to another habitat for finding plentiful foods, and then return back after a long period. We introduce state feedback control into system (1.1) which to be written as

$$x'(t) = f(x(t), x(t-\tau), \sigma, \tau) + K_1(x(t-T_1) - x(t)) + \cdots + K_n(x(t-T_n) - x(t)) \tag{1.2}$$

The feedback terms in Eq(1.2) express the migration movement of population in some designated life stages with $K_i(i = 1, \cdots, n)$ negative constants. The features of periodicity of migration for these living beings have knowledged people how to further protect species life safety effectively to facilitate human beings with big economic efficiency.

Recently, people have developed the stability criterion of the related characteristic equation of system (1.1) without feedback control (Cooke & Drissche, 1996, Boese, 1998, Beretta & Kuang, 2002). The stability of system is changed as varying parameter σ in its increased direction and the critical delay value can be solved effectively by geoemtrical method. Cooke mentioned that roots of characteristic equation in complex plane with zero real part can be curved by varying delay dependent coefficient freely to analyze system singularity. Beretta and Kuang have introduced a geometrical criterion by choosing τ as a bifurcation parameter (Beretta & Kuang, 2002). To get an insight into high codimension bifurcation analysis, We develop geometrical criterion to two parameter plane in (σ, τ) parameter space (Ma et al., 2008). This criterion is applied to some population models with time delay and computer graphics are produced to analyze Hopf bifurcations appearing in systems.

Considering periodical oscillating phenomena appearing in DDEs (Nisbet & Kurney, 1982; Kuang, 1993), authors in paper (Wang & Hu, 1999; Xu & Chung, 2003; Xu & Chung, 2009) have developed different analytical method to investigate stability of periodical solution efficiently. As is well known, to further investigate dynamical behavior for Hopf bifurcation, the normal form technique of delay dynamical systems is applied to reduce the infinite-dimensional DDEs to finite-dimensional ODEs (Hale & Lunel, 2003; Stech, 1985), and the fundamental theory of DDEs to see (Hale & Lunel, 2003) for reference.

To discuss the stability of system (1.2), the geometrical criterion for single time delay τ is extended further by given time delay T_i. Therefore, Hopf bifurcation may arise at the threshold value while equilibrium solution getting into instable. The whole paper is organized as follows. In section 2, we present geometrical analysis of Hopf bifurcation to system (1.2) as varying time delay. In section 3, we illustrate the geometrical method by applying the criterion to a logistic population model with delay density dependent and periodic migration feedback control; In section 4, by theory of center manifold reduction, the bifurcation direction of Hopf bifurcation and stability of bifurcating periodic solution is studied by normal form method.

2. Geometrical Criterion

We analyze the stability of system (1.2) by computing roots of its characteristic equation in complex plane. Without loss generity, it is supposed that Eq.(1.2) has trivial solution, and the linearized equation is listed

$$\begin{aligned} x'(t) = A(\sigma,\tau)x(t) + B(\sigma,\tau)x(t-\tau) + K_1(x(t-T_1) - x(t)) \\ +K_2(x(t-T_2) - x(t) + \cdots + K_n(x(t-T_n) - x(t)) \end{aligned} \tag{2.1}$$

For simplicity, it is assumed that $B(\sigma,\tau)$ has the form

$$B(\sigma,\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1}(\sigma,\tau) & b_{i2}(\sigma,\tau) & \cdots & b_{in}(\sigma,\tau) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$K_m = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ k^m_{i1}(\sigma,\tau) & k^m_{i2}(\sigma,\tau) & \cdots & k^m_{in}(\sigma,\tau) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

or

$$B(\sigma,\tau) = \begin{pmatrix} 0 & \cdots & b_{1j}(\sigma,\tau) & \cdots & 0 \\ 0 & \cdots & b_{2j}(\sigma,\tau) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & b_{n-1,j}(\sigma,\tau) & \cdots & 0 \\ 0 & \cdots & b_{nj}(\sigma,\tau) & \cdots & 0 \end{pmatrix},$$

and

$$K_m = \begin{pmatrix} 0 & \cdots & k^m_{1j}(\sigma,\tau) & \cdots & 0 \\ 0 & \cdots & k^m_{2j}(\sigma,\tau) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & k^m_{n-1,j}(\sigma,\tau) & \cdots & 0 \\ 0 & \cdots & k^m_{nj}(\sigma,\tau) & \cdots & 0 \end{pmatrix}.$$

for $m = 1, 2, \cdots, n$) The associated characteristic equation for Eq.(2.1) is written as

$$\Delta(\lambda,\sigma,\tau) = P(\lambda,\sigma,\tau) + Q(\lambda,\sigma,\tau)e^{-\lambda\tau} + \sum_{i=1}^{n} H_i(\lambda)(e^{-\lambda T_i} - 1) \tag{2.2}$$

for $i = 1, 2, \cdots, n$. and

$$P(\lambda, \sigma, \tau) = \sum_{k=0}^{n} p_k(\sigma, \tau)\lambda^k, \qquad Q(\lambda, \sigma, \tau) = \sum_{k=0}^{m} q_k(\sigma, \tau)\lambda^k \tag{2.3}$$

and $H_i(\lambda) = \sum_{k=0}^{m} H_i(\sigma, \tau)\lambda^k$ is a polynomial of λ.

In formula (2.3), it is assumed $n > m$. Both P and Q are continuous and differentiable with respect to σ and τ, and analytical to variable λ. Authors in paper [1,2,3,4] discuss the occurrence of simple "Hopf bifurcation" of system (1.2) with $K_i = 0$ for $i = 1, 2, \cdots n$. It is supposed that the following condition are satisfied:
(i) For $\lambda = 0$, $P(0, \sigma, \tau) + Q(0, \sigma, \tau)$ for any σ, τ, that is, $\lambda = 0$ is not a characteristic root of Eq.(2.3);
(ii)For $\lambda = i\omega$,$P(i\omega, \sigma, \tau) + Q(i\omega, \sigma, \tau) \neq 0$, for any σ, τ;
(iii) $\limsup \{|Q(\lambda, \sigma, \tau)/P(\lambda, \sigma, \tau)| : \lambda \to \infty, \Re(\lambda) > 0\} < 1$;
(iv) $F(\omega, \sigma, \tau) = |P(i\omega, \sigma, \tau)|^2 - |Q(i\omega, \sigma, \tau)|^2$ is continuous and differentiable with respect to σ and τ and has at most finite real roots.

Condition (i) implies that imaginary axis cannot intersect with $\lambda = 0$; Condition (ii) is to express the solvability condition for imaginary roots $\lambda = i\omega$; Condition (iii) is necessary for the occurrence of Hopf bifurcation since no imaginary roots cross the imaginary axis from infinity direction; and with given condition (iv), there is at most finite "gates" for the occurrence of Hopf bifurcation.

Authors in paper [2] and [3] have discussed the occurrence of Hopf bifurcation for linearized equation (2.1) geometrically based on the above assumptions. In order to do this, suppose $\lambda = i\omega$ with $\omega > 0$ and substitute it into Eq.(2.2), then seperate the real part from the imaginary part to get

$$\begin{aligned} &P_R(i\omega, \sigma, \tau) + Q_R(i\omega, \sigma, \tau)\cos(\omega\tau) + Q_I(i\omega, \sigma, \tau)\sin(\omega\tau) = 0, \\ &P_I(i\omega, \sigma, \tau) - Q_R(i\omega, \sigma, \tau)\sin(\omega\tau) + Q_I(i\omega, \sigma, \tau)\cos(\omega\tau) = 0, \end{aligned} \tag{2.4}$$

It is easily seen that ω is also a positive root of $F(\omega, \sigma, \tau) = |P(i\omega, \sigma, \tau)|^2 - |Q(i\omega, \sigma, \tau)|^2$ Assume $I \subset R \times R_{+0}$ is the region where $\omega(\tau)$ is positive root of $F(\omega, \sigma, \tau)$, and for any $(\sigma, \tau) \in I$ with σ fixed parameter. Define $C = \omega(\tau)\tau$, then we have

$$\begin{aligned} \sin(C) &= \frac{-P_R(iC/\tau, \sigma, \tau)Q_I(iC/\tau, \sigma, \tau) + P_I(iC/\tau, \sigma, \tau)Q_R(iC/\tau, \sigma, \tau)}{|Q(iC/\tau, \sigma, \tau)|^2}, \\ \cos(C) &= -\frac{P_R(iC/\tau, \sigma, \tau)Q_R(iC/\tau, \sigma, \tau) + P_I(iC/\tau, \sigma, \tau)Q_I(iC/\tau, \sigma, \tau)}{|Q(iC/\tau, \sigma, \tau)|^2}, \end{aligned} \tag{2.5}$$

Set

$$C_n = C_0 + 2n\pi, \qquad n \in N_0, \qquad C_0 \in (0, 2\pi). \tag{2.6}$$

We define the maps $\tau_n : (0, 2\pi) \to R_{+0}$ given by $\tau_n(C_0) := \tau(C)$, with $(\sigma, \tau(C) \in I$, then refer to paper [2,3], we deduce the following technical Lemma:

Lemma 1 *For any $n \in N_0$, if $C_0^* \in (0, 2\pi)$ and $\tau_n(C_0^*)$ satisfies Eqs (2.5), then*

$$\omega(\sigma, \tau_n(C_0^*)) = \frac{C_0^* + 2n\pi}{\tau_n(C_0^*)}$$

is the positive root of $F(\omega, \sigma, \tau) = 0$. Therefore, on complex plane, there is a pair of imaginary roots $\lambda = \pm i\omega$ cross the imaginary axis from left to right as τ increases if $\delta(C_0^*) > 0$, whereas cross the imaginary axis from right to left as τ increases if $\delta(C_0^*) < 0$, where $\delta(C_0^*)$ is given by

$$\begin{aligned} \delta(C_0^*) &= sign\left\{\frac{d\Re(\lambda)}{d\tau}|_{\lambda=i\omega}\right\} \\ &= sign(\frac{1}{\omega'(C_0^*)}(-\Re(W)Y + \omega(C_0^*)\Re(W)|P(i\omega, \sigma, \tau)|^2 \\ &\quad -\Im(W)X + \Re(V)X - \Im(V)Y + \omega(C_0^*)\Im(V)|P(i\omega, \sigma, \tau)|^2) \\ &\quad -\tau|P(i\omega, \sigma, \tau)|^2X), \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} W &:= (P'_{C_0} - P'_\tau\tau'_n(C_0))P_R - (Q'_{C_0} - Q'_\tau\tau'_n(C_0))Q_R, \\ V &:= (P'_{C_0} - P'_\tau\tau'_n(C_0))P_I - (Q'_{C_0} - Q'_\tau\tau'_n(C_0))Q_I, \\ X &:= P'_{R_\tau}P_R + P'_{I_\tau}P_I - Q'_{R_\tau}Q_R - Q'_{I_\tau}Q_I, \\ Y &:= P'_{R_\tau}P_I + P'_{I_\tau}P_R - Q'_{R_\tau}Q_I - Q'_{I_\tau}Q_R, \end{aligned}$$

Substitute $\lambda = i\omega(\omega > 0)$ into Eq.(2.2), then seperate the real part from the imaginary part to get

$$\begin{array}{l} P_R(i\omega,\sigma,\tau) + Q_R(i\omega,\sigma,\tau)\cos(\omega\tau) + Q_I(i\omega,\sigma,\tau)\sin(\omega\tau) \\ + \sum_{l=1}^{n} H_{lR}(i\omega)\cos(\omega T_l) + \sum_{l=1}^{n} HlI(i\omega)\sin(\omega T_l) = 0, \\ P_I(i\omega,\sigma,\tau) - Q_R(i\omega,\sigma,\tau)\sin(\omega\tau) + Q_I(i\omega,\sigma,\tau)\cos(\omega\tau) \\ - \sum_{l=1}^{n} H_{lR}(i\omega)\sin(\omega T_l) + \sum_{l=1}^{n} H_{lI}(i\omega)\cos(\omega T_l) = 0, \end{array} \tag{2.8}$$

Define $C = \omega(\tau)\tau$, then we have

$$\begin{array}{rl} \sin(C) = & Q_I(iC/\tau,\sigma,\tau)(\frac{-P_R(iC/\tau,\sigma,\tau) + \sum_{l=1}^{n} H_{lR}(iC/\tau)(\cos(T_lC/\tau)-1)}{|Q(iC/\tau,\sigma,\tau)|^2} \\ & +\frac{\sum_{l=1}^{n} HlI(iC/\tau)\sin(T_lC/\tau)}{|Q(iC/\tau,\sigma,\tau)|^2}) \\ & +Q_R(iC/\tau,\sigma,\tau)(\frac{P_I(iC/\tau,\sigma,\tau) - \sum_{l=1}^{n} H_{lR}(iC/\tau)\sin(T_lC/\tau)}{|Q(iC/\tau,\sigma,\tau)|^2} \\ & +\frac{\sum_{l=1}^{n} H_{lI}(iC/\tau)(\cos(\omega T_l)-1)}{|Q(iC/\tau,\sigma,\tau)|^2}), \\ \cos(C) = & -Q_R(iC/\tau,\sigma,\tau)(\frac{P_R(iC/\tau,\sigma,\tau) + \sum_{l=1}^{n} H_{lR}(iC/\tau)(\cos(T_1C/\tau)-1)}{|Q(iC/\tau,\sigma,\tau)|^2} \\ & +\frac{\sum_{l=1}^{n} H_{lI}(iC/\tau)\sin(T_1C/\tau)}{|Q(iC/\tau,\sigma,\tau)|^2}) \\ & +Q_I(iC/\tau,\sigma,\tau)(\frac{P_I(iC/\tau,\sigma,\tau) - \sum_{l=1}^{n} H_{lR}(iC/\tau)\sin(\omega T_l)}{|Q(iC/\tau,\sigma,\tau)|^2} \\ & +\frac{\sum_{l=1}^{n} H_{lI}(iC/\tau)(\cos(T_lC/\tau)-1)}{|Q(iC/\tau,\sigma,\tau)|^2}), \end{array} \tag{2.9}$$

It is easily seen that ω is also a positive root of

$$F(\omega,\sigma,\tau) = \phi^2(\omega,\sigma,\tau) + \psi^2(\omega,\sigma,\tau) - 1 \tag{2.10}$$

with $\phi(\omega,\sigma,\tau) = \sin(C), \psi(\omega,\sigma,\tau) = \cos(C)$ as given in Eqs(2.9). Set $C_n = C_0 + 2n\pi$ for any $n \in N_0$, and$C_0 \in (0, 2\pi)$. We define the maps $\tau_n : (0, 2\pi) \to R_{+0}$ given by $\tau_n(C_0) := \tau(C)$, with $(\sigma, \tau(C)) \in I$.

By Lemma 1, we have

$$\begin{array}{l} \delta(C_0^*) = sign\left\{\frac{d\Re(\lambda)}{d\tau}|_{\lambda=i\omega}\right\} = sign\Re\left\{\frac{d\lambda}{d\tau}|_{\lambda=i\omega}\right\}^{-1} \\ = sign(\frac{1}{\omega'(C_0^*)}(-\Re(W)Y - \Im(W)X + \Re(V)X - \Im(V)Y \\ \quad +\omega(C_0^*)\Re(W)|P(i\omega,\sigma,\tau) + \sum_{l=1}^{n} H_l(i\omega T_l)(exp(i\omega T_l)-1)|^2 \\ \quad +\omega(C_0^*)\Im(V)|P(i\omega,\sigma,\tau) + \sum_{l=1}^{n} H_l(i\omega T_l)(exp(i\omega T_l)-1)|^2) \\ \quad -\tau|P(i\omega,\sigma,\tau) + \sum_{l=1}^{n} H_l(i\omega T_l)(exp(i\omega T_l)-1)|^2 X), \end{array} \tag{2.11}$$

with

$$\begin{array}{l} W := (P'_{C_0} + \sum_{l=1}^{n}(H_{lC_0}(exp(iT_lC/\tau)-1) + H_l e^{-i\omega T_l}(-T_l)) \\ \quad -P'_\tau\tau'_n(C_0))(P_R + \sum_{l=1}^{n} H_l(\cos(T_1C/\tau)-1)) + (Q'_{C_0} - Q'_\tau\tau'_n(C_0))Q_R, \\ V := (P'_{C_0} + \sum_{l=1}^{n}(H_{lC_0}(exp(iT_lC/\tau)-1) + H_l e^{-i\omega T}(-T_l)) \\ \quad -P'_\tau\tau'_n(C_0))(P_I - \sum_{l=1}^{n} H_l\sin(T_1C/\tau)) - (Q'_{C_0} - Q'_\tau\tau'_n(C_0))Q_I, \\ X := P'_{R_\tau}(P_R + \sum_{l=1}^{n} H_l(\cos(T_1C/\tau)-1)) + P'_{I_\tau}(P_I + \sum_{l=1}^{n} H_l\sin(T_1C/\tau)) \\ \quad -Q'_{R_\tau}Q_R - Q'_{I_\tau}Q_I, \\ Y := P'_{R_\tau}(P_I + \sum_{l=1}^{n} H_l\sin(T_1C/\tau)) + P'_{I_\tau}(P_R + \sum_{l=1}^{n} H_l(\cos(T_1C/\tau)-1)) \\ \quad -Q'_{R_\tau}Q_I - Q'_{I_\tau}Q_R, \end{array}$$

3. A Logistic Population Model with Feedback Control

As an attempt to explain the oscillating dynamical behavior observed both in ecology and biology, population model is set forth mathematically to describe the dynamics of the population of biological species in nature or laboratories qualitatively and quantitatively[1-4]. Among these models, the well known logistic population model[1] is introduced which obeys

$$\frac{dx}{dt} = bx(t)\left(1 - \frac{x(t))}{k}\right) - dx(t) \tag{3.1}$$

where state variable $x(t)$ to describe the density of population species, b is the birth rate and d is the death rate, the first term in Eq.(3.1) contains density restriction due to resource capacity which to give a description of population activities in real world.

Delays factor is ubiquitous in population activities since growing up from embryos to adult takes a long time period. Henceforth, people have introduced the delays factor into population model to comprehensively understand the population growth rules naturally (see[4-7] for reference). In their work, the rich dynamical behaviors of population model have been reported to identify the observed oscillating phenomena which is in coincidence with the results of theoretical analysis as varying time delay.

Based on equation (3.1), Logistic population model with density delay dependent[8] is put forward with the following formula

$$\frac{dx}{dt} = bx(t-r)e^{-pr}\left(1 - \frac{x(t-r)}{k}\right) - dx(t) \tag{3.2}$$

where r denotes the maturation time of the population species. be^{-pr} is exponential nonlinear birth rate with a birth decay coefficient p which always applied in so called "stage-structured" population model[9,10]. It explains the whole life stage of population survival dependent on time delay. As a comparison with system (3.1), decay coefficient p can dramatically changes the asymptotic stable behavior of system and oscillating behaviors occurs as system become instable as varying time delay simultaneously. However, it is conceived that population number varying in some designative life stage, for example population moving and biological statistics with some uncertain factors, which may change population activities complexly. Therefore, we put forward the following population model with state feedback control

$$\frac{dx}{dt} = bx(t-r)e^{-pr}\left(1 - \frac{x(t-r)}{k}\right) - dx(t) + f(x(t-T) - x(t)) \tag{3.3}$$

Time delays are ubiquitous factors which arose in population model naturally to affect system dynamics dramatically.

3.1 Linear System Analysis

It is easily calculated that system (3.2) or (3.3) has a unique positive equilibrium solution

$$x^* = \frac{k(be^{-pr} - d)}{be^{-pr}} \tag{3.4}$$

with the given condition $0 < p < \frac{1}{r} ln \frac{d}{b}$. If $r = 0$, the dynamics of Eq.(3.1) is simple and the unique positive equilibrium solution is globally asymptotically stable without delay effect.

In the case $r > 0$, the birth rate of the population species in system (3.2) is exponential nonlinear of time delay r with decay coefficient p. Eq.(3.2) is a single DDE with parameter delay dependent coefficient. Cooke have considered a infectious disease model with parameter delay dependent coefficient to affect system's stability quanitatively (Cooke et al., 1992; Cooke & van der Driessche, 1996). Thereafter, a general geometrical criterion for stability analysis of some delay differential equations with delay dependent coefficient is presented and developed (Beretta & Kuang, 2002).

By making transformation $\bar{x} = x - x^*$ and replacing $\bar{x}$ by x for simplicity, the linearized equation of system (3.3) is listed as

$$x'(t) = (2d - be^{-pr}x(t-r)) - dx + f(x(t-T) - x(t)) \tag{3.5}$$

The corresponding characteristic equation of system (3.5) is

$$\Delta(\lambda) = \lambda - (2d - be^{-pr})e^{-\lambda r} + d - f(e^{-\lambda T} - 1) = 0 \tag{3.6}$$

To analyze the stability and Hopf bifurcation of the equilibrium solution of Eq.(3.3), the properties of the characteristic roots of Eq.(3.5) are discussed. Set $\lambda = i\omega(\omega > 0)$ and submit it into Eq.(3.5), then separate the real part from the imaginary part to get

$$\begin{aligned} -(2d - be^{-pr})cos(wr) + d - f(cos(wT) - 1) = 0 \\ w + (2d - be^{-pr})sin(wr) + f sin(wT) = 0 \end{aligned} \tag{3.6}$$

It is difficult to solve w, T directly from Eq.(3.6), however, the geometrical analysis give an insight how w and T dependent indirectly, thus collect all the information of how Hopf bifurcation occurs. We denote $Y = (2d - be^{-pr})$ and $\beta = \omega r, S = \omega T$ to get the following formula

$$\begin{aligned} -Y\cos(\beta) + d - f(\cos(S) - 1) = 0 \\ \beta + rY\sin(\beta) + rf\sin(S) = 0 \end{aligned} \tag{3.7}$$

It is computed from Eq.(3.7) that

$$\cos(S) = -\frac{Y\cos(\beta) - d - f}{f}, \qquad \sin(S) = -\frac{rY\sin(\beta) + \beta}{rf}, \tag{3.8}$$

since $cos(S)^2 + sin(S)^2 = 1$, one gets

$$\begin{aligned} Y &= \frac{\cos(\beta)dr + \cos(\beta)fr - \sin(\beta)\beta \pm \sqrt{tt}}{r}, \\ T &= \frac{r}{\beta}(m\pi + arctan\frac{\beta + rY\sin(\beta)}{r(Y\cos(\beta) - d - f)}), \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} tt = {} & \cos(\beta)^2(d^2r^2 + 2dfr^2 + f^2r^2) - 2\cos(\beta)\sin(\beta)\beta(d+f)r + \sin(\beta)^2\beta^2 \\ & -d^2r^2 - 2dfr^2 - \beta^2. \end{aligned}$$

for $n = 0$ and $m = 1, 2, 3, \cdots$.

By Eq.(3.8) and the first equation in Eqs(3.9), we can plot curves in (S, Y) plane which is denoted by S_0 to determine the imaginary roots ω of the characteristic equation (3.5). Denote the line given by $Y = 2d - be^{-pr}$ as L which intersect with S_0 at point (S^*, Y^*), as shown in Figure 1(a). Other parameters are chosn as $b = 0.2, d = 0.025, k = .1, f = -0.002$. Furthermore, it is calculated that the imaginary root ω as $\omega = \frac{S^*}{T^*}$ with T^* given by the second equation in Eqs(3.9). Set

$$\begin{aligned} Y_1 &= \frac{\cos(\beta)dr + \cos(\beta)fr - \sin(\beta)\beta + \sqrt{tt}}{r}, \\ Y_2 &= \frac{\cos(\beta)dr + \cos(\beta)fr - \sin(\beta)\beta - \sqrt{tt}}{r}, \end{aligned} \tag{3.10}$$

differentiate $Y_{1,2}$ and $\cos(S)$ respectively with respect to the parameter β and making division of them, then solve zero solution of following equation

$$\frac{dY_{1,2}}{d\cos(S)} = \frac{f\left((d+f)r\sin(\beta) + \cos(\beta)\beta + \sin(\beta) \mp \frac{1}{2\sqrt{tt}}\frac{dtt}{d\beta}\right)}{-rY\sin(\beta) + \cos(\beta)\left((d+f)r\sin(\beta) + r\cos(\beta)\beta + r\sin(\beta) \mp \frac{1}{2\sqrt{tt}}\frac{dtt}{d\beta}\right)} \tag{3.11}$$

to get (S_{1*}, Y_{1*}) and (S_{2*}, Y_{2*}), which satisfy Eq.(3.11), where

$$\begin{aligned} \frac{dtt}{d\beta} = {} & -2\cos(\beta)\sin(\beta)(d^2r^2 + 2dfr^2 + f^2r^2) + 2(\sin(\beta)^2 - cos(\beta)^2)\beta(d+f)r \\ & -2\cos(\beta)\sin(\beta)(d+f)r + 2\sin(\beta)\cos(\beta)\beta^2 + 2\beta\sin(\beta)^2 - 2\beta. \end{aligned}$$

Obviously, the curve S_0 and the line L intersect tangentially at $(S_{1*,2*}, Y_{1*,2*})$ by choosing $p_{1,2} = -\frac{1}{r}ln(\frac{2d - Y_{1*,2*}}{b})$. We derive the following result: For fixed $p_1 < p < p_2$, stability switching phenomena of the equilibrium solution x^* of system (3.3) happens at the intersection points of the curve S_0 and the line L as varying time delay r (or time delay T) and it may lead to Hopf bifurcation since the transversity condition $\frac{d\Re\lambda}{dr}|_{\lambda=i\omega} \neq 0$ (or $\frac{d\Re\lambda}{dT}|_{\lambda=i\omega} \neq 0$) is satisfied .

Notice that $\beta = \omega r$, thus we have

$$\omega'(\beta^*)r + \omega(\beta^*)r'(\beta^*) = 1 \tag{3.12}$$

with $\omega(\beta^*) = \frac{S^*}{T}$.

Since $Y = 2d - be^{-pr}$, therefore, we get

$$Y'(\beta^*) = bpe^{-pr}r'(\beta^*) \tag{3.13}$$

On another respect, we also have

$$Y'(\beta^*) = \frac{dY}{d\cos(S)}(-\sin(S^*))(\frac{T}{r} - \frac{T\beta^*}{r^2}r'(\beta^*) \tag{3.14}$$

By computation with the aid of formula (3.13) and (3.14) to get

$$r'(\beta^*) = \frac{\dfrac{dY}{d\cos(S)}(-\sin(S^*)\dfrac{T}{r}}{bpe^{-pr} - \dfrac{dY}{d\cos(S)}\sin(S^*)\dfrac{T\beta^*}{r^2}} \tag{3.15}$$

Solve $\omega'(\beta^*)$ from Eq.(3.14) to obtain

$$\omega'(\beta^*) = \frac{\beta^{*2}(\beta^* T + S^{*2}\sin(S)\frac{dY}{d\cos(S)})}{\beta^{*3}Tbpe^{-pS^*/\beta^* T} - \frac{dY}{d\cos(S)}\sin(S^*)S^{*2}} \tag{3.16}$$

as $Y = Y_{1,2}$, respectively.

Since

$$\begin{aligned}
W &:= -i\omega'(\beta^*)(1 + fTe^{-i\omega T})(d + f - f\cos(\omega T));\\
V &:= -i\omega'(\beta^*)(1 + fTe^{-i\omega T})(\omega + f\sin(\omega T));\\
X &:= bpe^{-pr}(2d - be^{-pr}) = pY(\beta^*)(2d - Y(\beta^*))
\end{aligned}$$

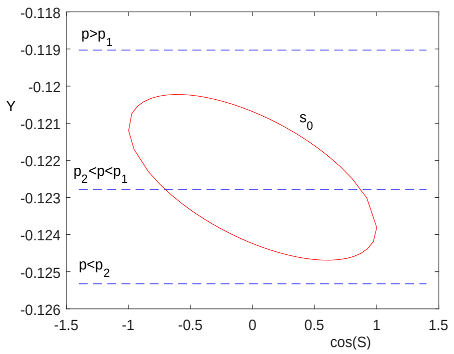
Therefore, by formula (2.11) we get

$$\begin{aligned}
\delta(\beta^*) &= sign\left\{\frac{d\Re(\lambda)}{dr}|_{\lambda=i\omega}\right\} = sign\Re\left\{\frac{d\lambda}{dr}|_{\lambda=i\omega}\right\}^{-1}\\
&= sign(\frac{1}{\omega'(\beta^*)}(-\Re(W)X + \Re(V)X + \omega(\beta^*)\Im(W)Y(\beta)^2 + \omega(\beta^*)\Im(V)Y(\beta)^2)\\
&\quad -\tfrac{S^*}{T\beta^*}Y(\beta^*)^2X),
\end{aligned} \tag{3.17}$$

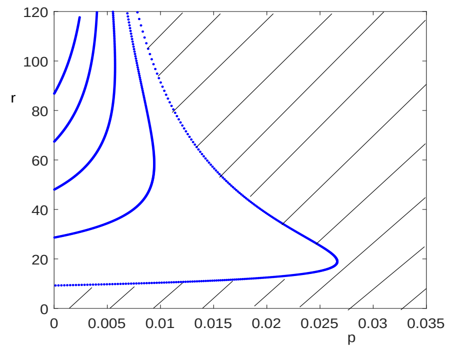
For fixed r, differentiate both sides of the characteristic equation to solve $\dfrac{d\lambda}{dT}$ and we have

$$\begin{aligned}
&sign\left\{\frac{d\Re(\lambda)}{dT}|_{\lambda=i\omega}\right\} = sign\Re\left\{\frac{d\lambda}{dT}|_{\lambda=i\omega}\right\}^{-1}\\
&= sign\Re(\frac{1 - Ye^{-i\omega r}(-r) + fTe^{-i\omega T}}{i\omega fe^{-i\omega T}})\\
&= sign(-(1 - Ycos(\beta))(Y\sin(\beta) + \tfrac{\beta}{r}) + r\sin(\beta)(Y\cos(\beta) - d - f))
\end{aligned} \tag{3.18}$$

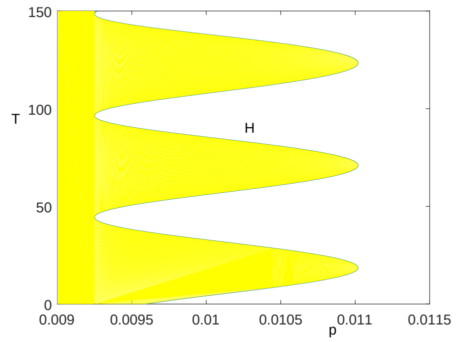
Furthermore, Hopf bifurcation lines which given by Eq.(3.9) are drawn on $(p - r)$ plane or on $(p - T)$ parameter plane. As shown in Figure 1(b), the shaded region denotes unstable regime of the positive equilibrium solution x^*. It can be seen, Hopf bifurcation lines form the margin of the unstable region of the equlibrium solution, as shown in Figure 1(c). Therefore, as increasing feedback delay T, the switching phenomena of the equlibrium solution x^* of Eq.(1.3) happens again and again, which bifurcates the oscillating periodic solution with small amplitudes near the margin since ”simple” Hopf bifurcation happens. Here ”simple” means ” codimension one” which is non-degenrate Hopf bifurcation.



(a)



(b)



(c)

Figure 1. The stability switching phenomena of the equilibrium solution x^* of system (1.3) when other fixed parameters chosen to be $b = 0.2, d = 0.25e - 1, k = .1, f = -0.002$. (a) The intersection points of the curve S_n and the line L for $n = 0$;(b) The Hopf lines in $(p - r)$ plane with $T = 15.61$; (c) The unstable region formed by Hopf bifurcation lines given by Eq.(2.7) with $r = 14.63$.

4. Center Manifold Reduction

By the above analysis, the "simple" Hopf bifurcation of the equilibrium solution happens at critical delay value of $T = T^*$. In this section, Both of the bifurcating direction of periodical solution and its stability are discussed by using center manifold method and norm form technique.

By the linearized method, system (3.3) can be written as the following version

$$\dot{x} = -(d + f)x(t) + (2d - be^{-pr})x(t - r) + fx(t - T) - \frac{b}{k}e^{-pr}x^2(t - r) \tag{4.1}$$

Suppose $\tau = \max\{r, T\}$, the phase space of system (4.1) is Banach space $C([-\tau, 0], R)$ with norm definition $||\phi|| = \sup_{\theta\leq\tau\leq 0} |\phi(\theta)|$. Set the linear operator $L(0)$ to be

$$L(0)\phi = -(d + f)\phi(0) + (2d - be^{-pr})\phi(-r) + f\phi(-T) \tag{4.2}$$

By Rieze representation theorem, there is a bounded variation function $\eta(\theta)$ which satisfies

$$L(0)\phi = \int_{-\tau}^{0} d\eta(\theta)\phi(\theta) \tag{4.3}$$

with

$$d\eta(\theta) = [-(d + f)\delta(\theta) + (2d - be^{-pr})\delta(\theta + r) + f\delta(\theta + T)].$$

Correspondingly, the characteristic equation of the linearized version (4.1) is written as

$$\Delta(\lambda) = \int_{-T}^{0} d\eta(\theta)e^{\lambda\theta} \tag{4.4}$$

which is verified to have a pair of imaginary roots $\pm i\omega_0$ at $T = T^*$, and all other characteristic roots with negative real parts.

By the fundamental theory of functional differential equations (Xu & Chung, 2009; Hale & Lunel, 1993), the linear operator $L(0)$ generates a strong continuous semigroup of bounded linear operator on Banach space C with the infinitesimal generator

$$\mathcal{A}\phi = \begin{cases} \dfrac{d\phi}{d\theta} & -\tau \leq \theta < 0, \\ L(0)\phi, & \theta = 0, \end{cases} \tag{4.5}$$

and its conjugate operator A^* on the adjoint space $C^* = C([0, \tau], R)$ is

$$\mathcal{A}^*\psi = \begin{cases} -\dfrac{d\psi}{ds} & 0 < s \leq \tau, \\ \int_{-\tau}^{0} d\eta(-s)\psi(s), & s = 0 \end{cases} \tag{4.6}$$

with $\psi \in C^*$ and set

$$R\phi = \begin{cases} 0, & \tau \leq \theta < 0, \\ F(\phi(0), \phi(-r)), & \theta = 0, \end{cases}$$

with

$$F(\phi) = -\frac{be^{-pr}}{k}\phi^2(-r)$$

For any $\phi \in C, \psi \in C^*$, we define the inner product $< \cdot >$ as

$$< \phi, \psi >= \bar{\psi}(0)\phi(0) + \int_{-r}^{0} \bar{\psi}(\xi - r)(2d - be^{-pr})\phi(\xi)d\xi + \int_{-T}^{0} \bar{\psi}(\xi - T) f \phi(\xi) \tag{4.7}$$

Eq.(4.1) can be written as its operator form which is listed as

$$\dot{x}_t = L(0)x_t + F(x_t) \tag{4.8}$$

with $x_t = x(t + \theta), -\tau \leq \theta \leq 0$.

Suppose the collection of characteristic roots with zero real part as $\Lambda = \{-i\omega_0, i\omega_0\}$, and the related eigenspace of Λ is denoted as P_Λ. The complementary subspace of P_Λ is represented by Q_Λ, then the phase space C is decomposed as $C = P_\Lambda \oplus Q_\Lambda$. Furthermore, we suppose the eigenvector of operator A associated with the characteristic root $i\omega_0$ being $q(\theta)$, and $p(s)$ is the eigenvector of the adjoint operator A^* associated with characteristic root $-i\omega_0$, that is

$$\mathcal{A}q(\theta) = i\omega_0 q(\theta), \qquad \mathcal{A}^* p(s) = -i\omega_0 p(s) \tag{4.9}$$

with $-\tau \leq \theta \leq 0$ and $0 \leq s \leq \tau$. Furthermore, suppose

$$< p, q >= 1 \qquad < \bar{p}, q >= 0$$

It is easily to calculate that

$$\begin{aligned} q(\theta) &= e^{i\omega_0\theta}, \qquad -\tau \leq \theta \leq 0, \\ p(s) &= \frac{1}{1 + (2d - be^{-pr} re^{i\omega_0 r} + fTe^{i\omega_0 T}} e^{i\omega_0 s}, \qquad 0 \leq s \leq \tau. \end{aligned} \tag{4.10}$$

For every $x_t \in C$, set $x_t = zq(\theta) + \bar{z}\bar{q}(\theta) + v_t$ with $v_t \in Q_\Lambda$, then we have

$$z =< p, x_t > \tag{4.11}$$

Differentiating both sides of Eq.(4.11) and with the aid of Eq.(4.9) to obtain

$$z' =< q^*, x_t' >= i\omega_0 z + \bar{p}(0)F(zq + \bar{z}\bar{q} + v_t), \tag{4.12}$$

4.1 The Reduction of System

To decide the bifurcating direction and stability of periodic solution arose from Hopf point, it is necessary to compute v_t too. Expand $v_t = w(z,\bar{z})$ into its Taylor series

$$v_t = w(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + \cdots \tag{4.13}$$

with coefficients w_{ij} dependent only on θ. Further set $g(z,\bar{z}) = \bar{p}(0)\hat{f}(z,\bar{z})$, and rewritten Eq.(4.9) as

$$z' = i\omega_0 z + g(z,\bar{z}) \tag{4.14}$$

Expand $g(z,\bar{z})$ into its series as

$$g(z,\bar{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \cdots \tag{4.15}$$

By Eq(3.8) and (3.9), it is easily to compute

$$v_t' = \begin{cases} \mathcal{A}v_t - 2\Re\bar{p}(0)\hat{f}(z,\bar{z})q(\theta), & -\tau \le \theta < 0, \\ \mathcal{A}v_t + \hat{f} - 2\Re\bar{p}(0)\hat{f}(z,\bar{z})q(\theta), & \theta = 0 \end{cases} \tag{4.16}$$

We further suppose that

$$v_t' = \mathcal{A}v_t + H(z,\bar{z}), \qquad -\tau \le \theta < 0 \tag{4.17}$$

with

$$H(z,\bar{z}) = -2\Re\bar{p}(0)\hat{f}(z,\bar{z})q(\theta) = \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \cdots \tag{4.18}$$

We compute the coefficients g_{ij} in Eq.(4.15) to get

$$\begin{aligned} g_{20} &= -2\bar{N}\frac{be^{-pr}}{k}e^{-2i\omega\tau}, \\ g_{11} &= -2\bar{N}\frac{be^{-pr}}{k}, \\ g_{02} &= -2\bar{N}\frac{be^{-pr}}{k}e^{2i\omega\tau}, \\ g_{21} &= -2\bar{N}\frac{be^{-pr}}{k}(w_{20}(-\tau)e^{i\omega\tau} + 2w_{11}(-\tau)e^{-i\omega\tau}) \end{aligned} \tag{4.19}$$

The coefficients H_{ij} are determined as

$$\begin{aligned} H_{20} &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11} &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), \\ H_{02} &= -g_{02}q(\theta) - \bar{g}_{20}\bar{q}(\theta), \end{aligned} \tag{4.20}$$

By a comparison of the coefficients between Eq(4.16) and Eq(4.17), for $-\tau \le \theta < 0$, we have

$$\begin{aligned} &\mathcal{A}w_{20}(\theta) + 2i\omega_0 w_{20}(\theta) = H_{20}(\theta), \\ &\mathcal{A}w_{11}(\theta) = H_{11}(\theta), \\ &\mathcal{A}w_{02}(\theta) - 2i\omega_0 w_{02}(\theta) = H_{02}(\theta), \end{aligned} \tag{4.21}$$

and for $\theta = 0$, we have the following formula

$$\begin{aligned} (2d - be^{-pr})w_{20}(-r) - dw_{20}(0) + K(w_{20}(-T) - w_{20}(0)) &= 2i\omega_0 w_{20}(0) \\ &\quad + g_{20} + \bar{g}_{20} + \frac{be^{-pr}}{k}e^{-2i\omega\tau}, \\ (2d - be^{-pr})w_{11}(-r) - dw_{11}(0) + K(w_{11}(-T) - w_{20}(0)) &= g_{11} + \bar{g}_{11} \\ &\quad + 2\frac{be^{-pr}}{k}, \end{aligned} \tag{4.22}$$

Integrating Eqs(4.21) underlying the given initial boundary condition (4.22) to obtain

$$\begin{aligned} w_{20}(\theta) &= exp2i\omega_0\theta E_1 - \frac{i}{3\omega_0}(3g_{20}e^{i\omega_0\theta} + \bar{g}_{02}e^{-i\omega_0\theta}), \\ w_{11}(\theta) &= \frac{ig_{11}}{\omega_0}e^{i\omega_0\theta} - \frac{i\bar{g}_1 1}{\omega_0}e^{-i\omega_0\theta} + E_2, \end{aligned} \tag{4.23}$$

where

$$\begin{aligned}
E_1 = & \frac{1}{3\omega_0 k(fe^{-2i\omega_0 T} - 2i\omega_0 + 2de^{-2i\omega_0 r} - e^{-2i\omega_0 r - pr} - f - d)}(3ife^{-i\omega_0 T} g_{20}k \\
& + ife^{i\omega_0 T}\bar{g}_{02}k - 3ie^{-pr-i\omega_0 r}g_{20}k - ie^{-pr+i\omega_0 r}\bar{g}_{02}k \\
& + 6ie^{-i\omega_0 r}dg_{20}k + 2ie^{i\omega_0 r}d\bar{g}_{02}k - if\bar{g}_{02}k - 3ifg_{20}k \\
& + 3be^{-pr-2i\omega_0\tau}\omega_0 - id\bar{g}_{02}k - 3idg_{20}k + 3g_{02}\omega_0 k + 2\bar{g}_{02}k\omega_0 + 9g_{20}\omega_0 k), \\
E_2 = & \frac{1}{\omega_0 ke^{-pr} - d}(ife^{-i\omega_0 T}g_{11}k - ife^{i\omega_0 T}\bar{g}_{11}k \\
& - ie^{-pr-i\omega_0 r}g_{11}k + ie^{-pr+i\omega_0 r}\bar{g}_{11}k + 2ie^{-i\omega_0 r}dg_{11}k - 2ie^{i\omega_0 r}\bar{g}_{11}dk \\
& + if\bar{g}_{11}k - ifg_{11}k + i\bar{g}_{11}dk - idg_{11}k - 2be^{-pr}\omega_0 \\
& - \bar{g}_{11}\omega_0 k - g_{11}\omega_0 k.
\end{aligned} \tag{4.24}$$

Therefore by Eqs(4.20),Eqs(4.23) and Eqs(4.24),we compute the following quantity directly

$$\begin{aligned}
& C_1(T^*) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\
& \mu = -\frac{\Re C_1(T^*)}{\Re\lambda'(T^*)}, \\
& \sigma = \Re C_1(T^*)
\end{aligned} \tag{4.25}$$

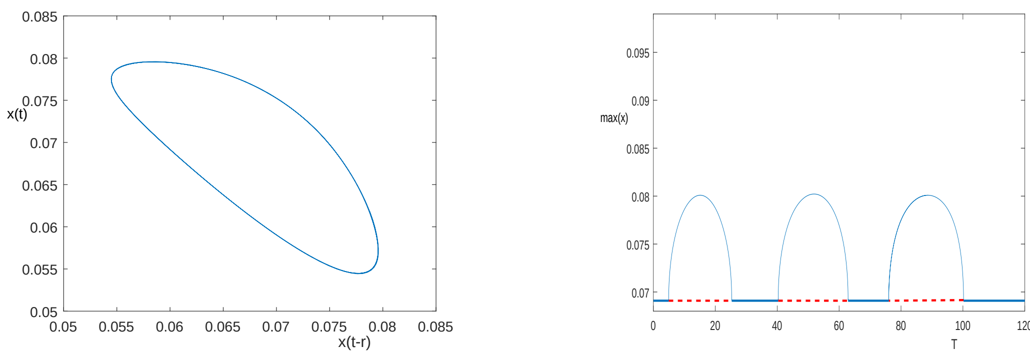
where μ determines the bifurcating direction of Hopf bifurcation, and σ determines the stability of the bifurcating periodical solution. Through the general theorem of Hopf bifurcation, the following results are derived:

Theorem 4.1 *If $\mu < 0$ at $T = T^*$, then the simple Hopf bifurcation of the eqilibrium solution x^* of Eq(3.3) is subcritical; whereas it is supercritical. Moreover, the bifurcating periodical solution is stable near the Hopf point if $\sigma < 0$ but unstable if $\sigma > 0$. Furthermore, we transform the studied population system with state feedback control into its 3-order Normal Form which is listed as[18]*

$$z' = i\omega_0 z + C_1(T^*)z|z|^2 + O(|z|^4) \tag{4.26}$$

and its universal unfolding is

$$z' = (\alpha(T) + i\beta(T))z + C_1(T)z|z|^2 + O(|z|^4) \tag{4.27}$$



(a) (b)

Figure 2. The bifurcating periodic solutions due to stability switching phenomena as Hopf bifurcation occurs. (a) Periodical solution near Hopf point is unique and globally asymptotically stable; (b) The continuously computation of the bifurcating periodical solution while varying delay T.

By the above discussion, the periodical oscillating phenomena appears since the instability switching of equilibrium solution x^* of population system (3.3) as Hopf bifurcation occurs. If $\mu > 0$, Hopf bifurcation is supercritical, the bifurcating periodical solution is asymptotically stable and it goes to extinction when varying parameter continuously to arrive at instable margin, since subcritical Hopf bifurcation happens again which satisfies $\mu < 0$. By choosing parameters $b = 0.7, d = 0.21, k = 0.1, r = 14.9688, f = -0.02$ and $p = 0.0021$ fixed, we compute the bifurcating periodical solutions continuously as varying time delay T as shown in Figure 2. The bifurcating periodical solution is locally asymptotically stable since $\sigma < 0$. It is easily observed that population size is limited within the resource capacity since the boundness character of Logistic population system. The bifurcating periodic solution is unique thus globally asymptotically stable.

4.2 The Observed Quasi-periodical Solution

The instability of population system brings out the complex dynamical behaviors as varying parameter p and time delay. For example, the quasi-periodic solution is observed in unstable region after Hopf bifurcation. The bifurcating scenario from periodic solution to chaos via double period bifurcation is numerically simulated.

For example, choosing $b = 0.2, d = 0.025, k = 0.1, r = 20.63, f = -0.002$, for fixed $p = 0.0006$, period-1,period-2, period-4 solution and chaos are detected with time delay $T = 2, 6, 12, 12.8$ respectively. The bifurcating periodical solutions are shown in Figure3 (a)-(d). The bifurcating periodical solutions to chaos with $T = 39.18$ are numerical simulated by choosing delay $r = 19.03, 19.63, 20.03, 20.83$ respectively as shown in Figure 4(a)-(d).

5. Discussion

The geometrical criterion of ”simple” Hopf bifurcation for multi-delay differential equations is extended as regarding time delay as a physical parameter since introducing delay dependent coefficients. As a peculiar example, Hopf bifurcation and stability switching of the unique positive equilibrium solution of a logistic population model with state feedback were studied. It is assumed that population movements in a designated time stage have influence on whole system dynamical behavior, such as periodic solution of system produced due to overwinter migration of fishes or birds. The method of geometrical analysis was used to detect the stability switching and track Hopf bifurcation when varying time delay. The oscillating periodic solution appear in the unstable regions was globally asymptotically stable. The routes to quasi-periodical solution via period doubling bifurcation was also explored as varying time delay.

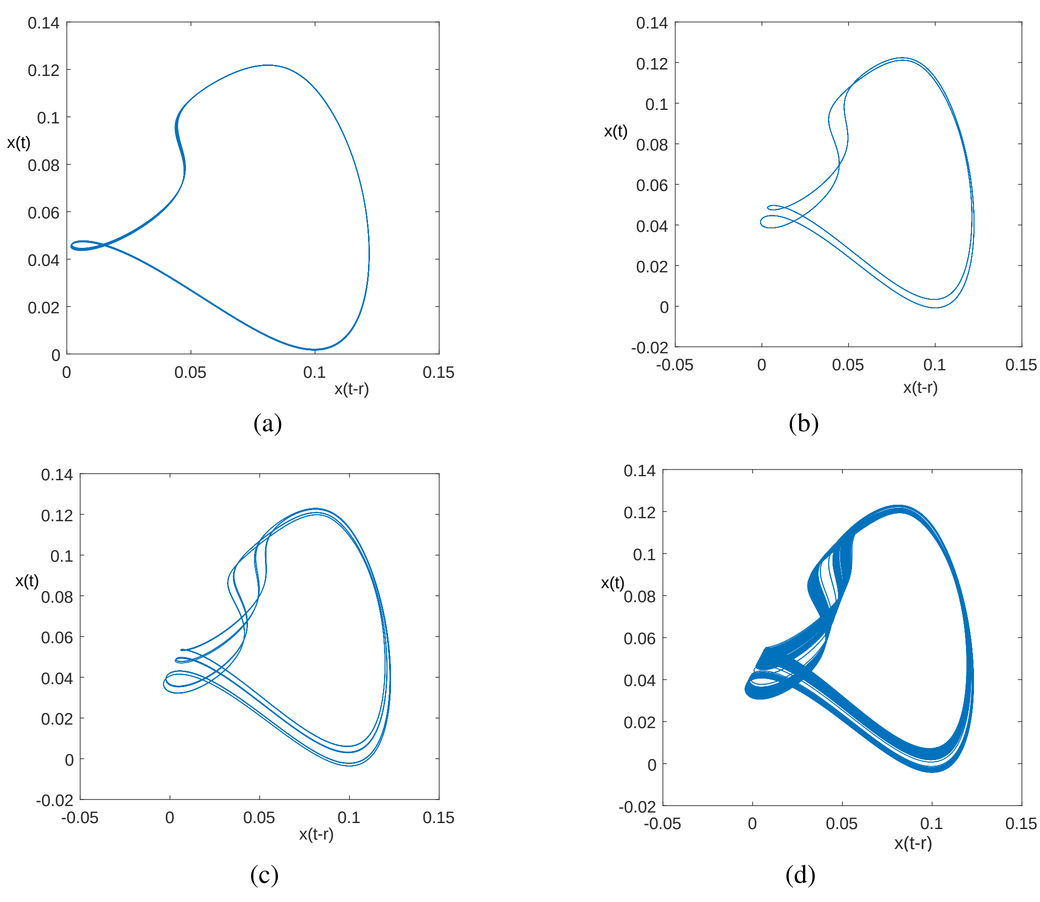


Figure 3. The bifurcating periodic solutions via period-doubling bifurcation with $b = 0.2, d = 0.025, k = 0.1, r = 20.63, f = -0.002$ and $p = 0.0006$, (a)the observed period-1 solution at $T = 2$; (b) the observed period-2 solution at $T = 6$; (c)the observed period-4 solution at $T = 12$; (d)the quasi-periodic solution at $T = 12.8$.

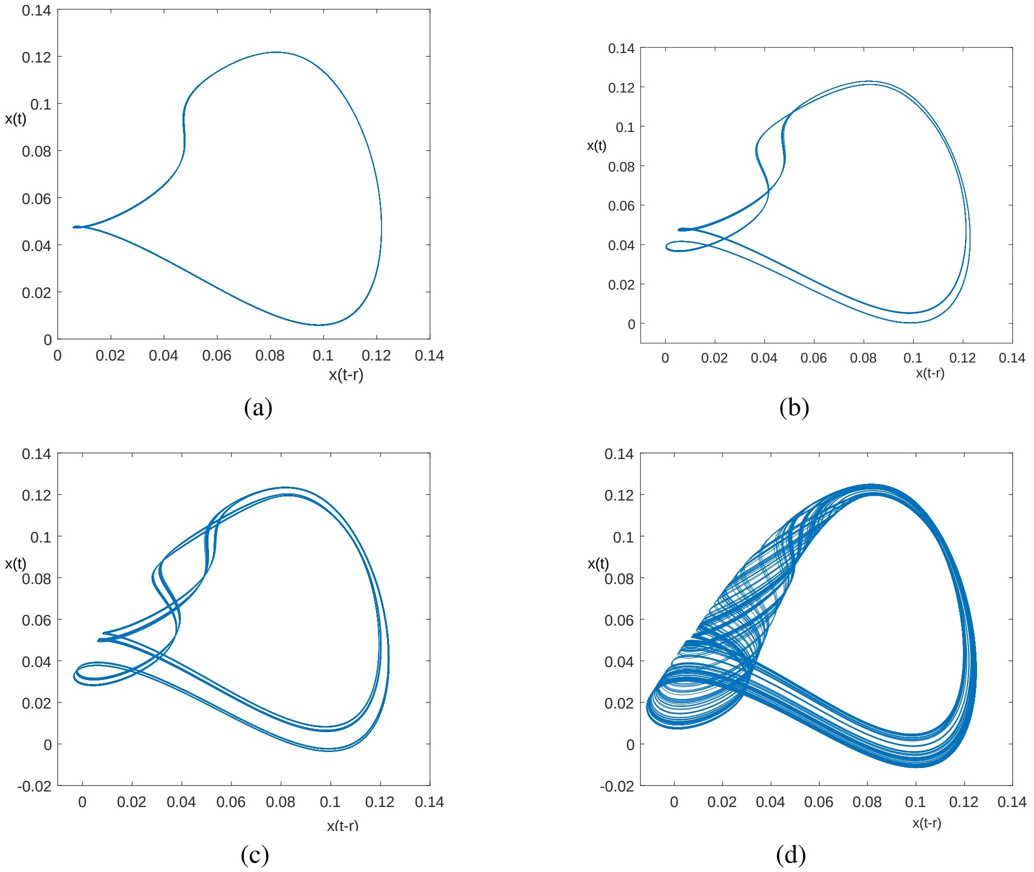


Figure 4. The bifurcating periodic solutions via period-doubling bifurcation with $b = 0.2, d = 0.025, k = 0.1, T = 39.18, f = -0.002$ and $p = 0.0006$, (a)the observed period-1 solution at $r = 19.03$; (b) the observed period-2 solution at $r = 19.63$; (c)the observed period-4 solution at $r = 20.03$; (d)the quasi-periodic solution at $r = 20.83$.

Acknowledgements

The auhtor are pleased to acknowledge the referees for their valuable suggestions to improve the presentation of this paper.

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