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Regularity and Green's Relations for Generalized Semigroups of Transformations with Invariant Set

Lei Sun[1]

[1] School of Mathematics and Information Science, Henan Polytechnic University, P.R.China

Correspondence: Lei Sun, School of Mathematics and Information Science, Henan Polytechnic University, Henan, Jiaozuo, 454003, P.R.China. E-mail: sunlei97@163.com

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Abstract

Let $\mathcal{T}_X$ be the full transformation semigroup on a set X. For $Y \subseteq X$, the semigroup $S(X,Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}$ is a subsemigroup of $\mathcal{T}_X$. Fix an element $\theta \in S(X,Y)$ and for $f, g \in S(X,Y)$, define a new operation $*$ on $S(X,Y)$ by $f*g = f\theta g$ where $f\theta g$ denotes the produce of g, θ and f in the original sense. Under this operation, the semigroup $S(X,Y)$ forms a semigroup which is called generalized semigroup of $S(X,Y)$ with the sandwich function θ and denoted by $S(X,Y,*_\theta)$. In this paper we first characterize the regular elements and then describe Green's relations for the semigroup $S(X,Y,*_\theta)$.

Keywords: generalized transformation semigroups, regular elements, Green's relations

2010 Mathematics Subject Classification: 20M20

1.Introduction

Let S be a semigroup and $a, b \in S$. If $a = axa$ for some $x \in S$, then a is called a *regular* element of S. The semigroup S is called *regular* if all its elements are regular. If a and b generate the same left principle ideal, that is, $S^1a = S^1b$, then we say that a and b are $\mathcal{L}$ equivalent and write $(a,b) \in \mathcal{L}$ or $a\mathcal{L}b$. If a and b generate the same right principle ideal, that is, $aS^1 = bS^1$, then we say that a and b are $\mathcal{R}$ equivalent and write $(a,b) \in \mathcal{R}$ or $a\mathcal{R}b$. If a and b generate the same principle ideal, that is, $S^1aS^1 = S^1bS^1$, then we say that a and b are $\mathcal{J}$ equivalent and write $(a,b) \in \mathcal{J}$ or $a\mathcal{J}b$. It is not difficult to see that $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ are equivalence relations on S. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. Then $\mathcal{H}$ and $\mathcal{D}$ are also equivalences. These five equivalences are usually called Green's relations on S. They were introduced by J.A. Green and play an important role in the study of the algebraic structure of semigroups.

Let $\mathcal{T}_X$ be the full transformation semigroup on a set X. Given a subset Y of X, the authors in (Honyam, P. & Sanwong, J., 2011) observed a class of subsemigroup of $\mathcal{T}_X$ defined by

$$S(X,Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}.$$

It is clear that if $Y = X$ then $S(X,Y) = \mathcal{T}_X$. To this extent the semigroup $S(X,Y)$ is regarded as a generalization of $\mathcal{T}_X$. Regularity for the elements in $S(X,Y)$ and Green's relations on $S(X,Y)$ were described in (Honyam, P. & Sanwong, J., 2011).

We apply transformations on the left so that for $f, g \in S(X,Y)$, their product fg is the transformation obtained by first performing g and then f. Fix an element $\theta \in S(X,Y)$ and for $f, g \in S(X,Y)$, define a new operation $*$ on $S(X,Y)$ by $f * g = f\theta g$ where $f\theta g$ denotes the produce of g, θ and f in the original sense. Under this operation, the semigroup $S(X,Y)$ forms a semigroup which is called *generalized* semigroup of $S(X,Y)$ with the sandwich function θ and denoted by $S(X,Y,*_\theta)$. Then $S(X,Y,*_\theta) = S(X,Y)$ as sets. Moreover, if $\theta = \mathrm{id}_X$(the identity transformation on the set X), then $S(X,Y,*_\theta) = S(X,Y)$ as semigroups. The generalized transformation semigroups of the various subsemigroups of $\mathcal{T}_X$ were studied by many authors, see for example (Hickey, J. B., 1983; Kemprasit, Y. & Jaidee, S., 2005; Magill, K. D. Jr. & Subbiah, S., 1975; Pei, H. S., Sun, L. & Zhai, H. C., 2007; Symons, J. S., 1975; Tsyaputa, G. Y., 2004).

The purpose of this paper is to investigate the regularity of elements and Green's relations on generalized semigroup $S(X,Y,*_\theta)$. Accordingly, in Section 2, the condition under which an element $f \in S(X,Y,*_\theta)$ is regular is analyzed. In Section 3, Green's relations on $S(X,Y,*_\theta)$ are considered and the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are descried for arbitrary elements, respectively.

2. The Regular Elements of $S(X, Y, *_\theta)$

In this section we investigate the condition under which an element of $S(X, Y, *_\theta)$ is regular.

Theorem 2.1. Let $f \in S(X, Y, *_\theta)$. Then f is regular if and only if the following statements hold.

(1) $\theta|_{f(X)}$ is injective.

(2) $\theta f(X) = \theta f\theta(X)$ and $\theta f(X) \cap Y = \theta f\theta(Y)$.

Proof. Suppose that f is regular. Then $f = f * g * f = f\theta g\theta f$ for some $g \in S(X, Y, *_\theta)$. It follows that $(f\theta)(g\theta)|_{f(X)} = \mathrm{id}|_{f(X)}$ and $(g\theta)|_{f(X)}$ is injective. So $\theta|_{f(X)}$ is injective and (1) holds. Clearly, $\theta f\theta(X) \subseteq \theta f(X)$. For each $z \in \theta f(X)$, let $z = \theta f(x)$ for some $x \in X$. Write $y = g\theta f(x)$ and then $z = \theta f(x) = \theta f\theta g\theta f(x) = \theta f\theta(y)$ which implies that $\theta f(X) \subseteq \theta f\theta(X)$. Thus $\theta f(X) = \theta f\theta(X)$. Similarly, we have $\theta f(X) \cap Y = \theta f\theta(Y)$.

Conversely, assume that (1)-(2) hold. Then, for each $x \in \theta f(X) \cap Y$, let $x = \theta f\theta(y)$ for some $y \in Y$, and for each $x \in \theta f(X) - Y$, let $x = \theta f\theta(y')$ for some $y' \in X$. Arbitrarily fix $a \in Y$ and define $g : X \to X$ by

$$g(x) = \begin{cases} y & \text{if } x \in \theta f(X) \cap Y \\ y' & \text{if } x \in \theta f(X) - Y \\ a & \text{otherwise.} \end{cases}$$

Clearly, $g \in S(X, Y, *_\theta)$. To see $f = f\theta g\theta f$, we need only to show that $\theta f = \theta f\theta g\theta f$ since $\theta|_{f(X)}$ is injective. For each $x \in X$, if $\theta f(x) \in \theta f(X) \cap Y$, then let $\theta f(x) = \theta f\theta(y)$ for some $y \in Y$. If $\theta f(x) \in \theta f(X) - Y$, then let $\theta f(x) = \theta f\theta(y')$ for some $y' \in X$. So

$$\theta f\theta g\theta f(x) = \begin{cases} \theta f\theta(y) & \text{if } \theta f(x) \in \theta f(X) \cap Y \\ \theta f\theta(y') & \text{if } \theta f(x) \in \theta f(X) - Y \end{cases}$$

$$= \theta f(x)$$

which means that $\theta f = \theta f\theta g\theta f$ and so $f = f\theta g\theta f$. Therefore f is regular. □

Denote by *Reg*($S(X, Y, *_\theta)$) and *Reg*($S(X, Y)$) the sets of all regular elements in semigroups $S(X, Y, *_\theta)$ and $S(X, Y)$, respectively. It is clear that $\mathrm{Reg}(S(X, Y, *_\theta)) \subseteq \mathrm{Reg}(S(X, Y))$. In generally, an element $f \in \mathrm{Reg}(S(X, Y))$ may be not regular in $S(X, Y, *_\theta)$. The following theorem shows when $\mathrm{Reg}(S(X, Y, *_\theta)) = \mathrm{Reg}(S(X, Y))$.

Theorem 2.2. Let the sets $\mathrm{Reg}(S(X, Y, *_\theta))$ and $\mathrm{Reg}(S(X, Y))$ be defined as above. Then $\mathrm{Reg}(S(X, Y, *_\theta)) = \mathrm{Reg}(S(X, Y))$ if and only if θ is a bijection and $\theta(X - Y) \cap Y = \emptyset$.

Proof. Suppose that $\mathrm{Reg}(S(X, Y, *_\theta)) = \mathrm{Reg}(S(X, Y))$. Since the identity transformation id_X on X is regular in $S(X, Y)$, we have that id_X is also regular in $S(X, Y, *_\theta)$, that is, $\mathrm{id}_X = \mathrm{id}_X\theta g\theta \mathrm{id}_X = \theta g\theta$ for some $g \in S(X, Y, *_\theta)$. So θ is bijective. Now we assert that $\theta(X - Y) \cap Y = \emptyset$. Indeed, if $\theta(x) \in Y$ for some $x \in X - Y$, then $x = \mathrm{id}_X(x) = \theta g\theta(x) \in Y$, a contradiction. Therefore, $\theta(X - Y) \cap Y = \emptyset$.

Conversely, we need to show that $\mathrm{Reg}(S(X, Y)) \subseteq \mathrm{Reg}(S(X, Y, *_\theta))$. For this purpose, let $f \in \mathrm{Reg}(S(X, Y))$. Then $f = fgf$ for some $g \in S(X, Y)$. Since θ is a bijection and $\theta(X - Y) \cap Y = \emptyset$, it follows that $\theta^{-1}(Y) \subseteq Y$. So $\theta^{-1} \in S(X, Y, *_\theta)$ and $g' = \theta^{-1}g\theta^{-1} \in S(X, Y, *_\theta)$. Thus $f = f\theta g'\theta f$ which implies that $f \in \mathrm{Reg}(S(X, Y, *_\theta))$. Hence $\mathrm{Reg}(S(X, Y)) \subseteq \mathrm{Reg}(S(X, Y, *_\theta))$ and $\mathrm{Reg}(S(X, Y)) = \mathrm{Reg}(S(X, Y, *_\theta))$. □

Theorem 2.3. The semigroup $S(X, Y, *_\theta)$ is regular if and only if the following statements hold.

(1) θ is a bijection and $\theta(X - Y) \cap Y = \emptyset$.

(2) $Y = X$ or $|Y| = 1$.

Proof. Suppose that $S(X, Y, *_\theta)$ is regular. Then

$$S(X, Y, *_\theta) = \mathrm{Reg}(S(X, Y, *_\theta)) \subseteq \mathrm{Reg}(S(X, Y)) \subseteq S(X, Y).$$

Since $S(X, Y, *_\theta) = S(X, Y)$ as sets, it follows that $\mathrm{Reg}(S(X, Y, *_\theta)) = \mathrm{Reg}(S(X, Y))$. By Theorem 2.2, θ is a bijection and $\theta(X - Y) \cap Y = \emptyset$. In the meantime, the semigroup $S(X, Y)$ is also regular. By [5, Corollary 2.4], $Y = X$ or $|Y| = 1$.

Conversely, by [5, Corollary 2.4] and Theorem 2.2, we have

$$S(X, Y) = \mathrm{Reg}(S(X, Y)) = \mathrm{Reg}(S(X, Y, *_\theta)) \subseteq S(X, Y, *_\theta).$$

Since $S(X, Y, *_\theta) = S(X, Y)$ as sets, it follows that $S(X, Y, *_\theta) = \mathrm{Reg}(S(X, Y, *_\theta))$, as required. □

3. Green's Relations on $S(X, Y, *_\theta)$

In this section we describe Green's relations on $S(X, Y, *_\theta)$.

Denote by $\pi(f)$ the partition of X induced by $f \in \mathcal{T}_X$, namely,

$$\pi(f) = \{f^{-1}(y) : y \in f(X)\}.$$

Also, let

$$\pi_Y(f) = \{P \in \pi(f) : P \cap Y \neq \emptyset\}.$$

Let $\psi : \pi(f) \to \pi(g)$ be a map. If $\theta(X) \cap \psi(P) \neq \emptyset$ for each $P \in \pi(f) - \pi_Y(f)$ and $\theta(Y) \cap \psi(P) \neq \emptyset$ for each $P \in \pi_Y(f)$, then ψ is said to be θ_Y-*admissible*. If ψ is bijective and both ψ and ψ^{-1} are θ_Y-admissible, then ψ is said to be θ_Y^*-*admissible*.

Now we begin with the relation $\mathcal{L}$ in $S(X, Y, *_\theta)$.

Theorem 3.1. Let $f, g \in S(X, Y, *_\theta)$. Then the following statements are equivalent.

(1) $(f, g) \in \mathcal{L}$.

(2) $f(X) = g\theta(X)$, $f(Y) = g\theta(Y)$ and $g(X) = f\theta(X)$, $g(Y) = f\theta(Y)$.

(3) There is a θ_Y^*-admissible bijection $\psi : \pi(f) \to \pi(g)$ such that $f = g\psi$.

Proof. (1)$\Longrightarrow$(2). Suppose that $(f, g) \in \mathcal{L}$. Then $f = g\theta h$ and $g = f\theta k$ for some $h, k \in S(X, Y, *_\theta)$ and

$$f(X) = g\theta h(X) \subseteq g\theta(X) = f\theta k\theta(X) \subseteq f(X),$$

which implies that $f(X) = g\theta(X)$. Moreover,

$$f(Y) = g\theta h(Y) \subseteq g\theta(Y) = f\theta k\theta(Y) \subseteq f(Y),$$

which implies that $f(Y) = g\theta(Y)$. Similarly, $g(X) = f\theta(X)$ and $g(Y) = f\theta(Y)$.

(2)$\Longrightarrow$(3). It is readily consequential on (2) that $f(X) = g(X)$. Now define $\psi : \pi(f) \to \pi(g)$ as follows. For each $P \in \pi(f)$, let $\psi(P) = g^{-1}(f(P))$. Then ψ is a well-defined bijection and $f = g\psi$. To see that $\psi : \pi(f) \to \pi(g)$ is θ_Y^*-admissible, let $\pi(f) = \{P_i : i \in I\}$ (where I is some index set) and $x_i = f(P_i)(i \in I)$. If $P_i \cap Y = \emptyset$, then $x_i \in f(X) = g\theta(X)$ and $x_i = g(y_i)$ for some $y_i \in \theta(X)$. So $y_i \in \theta(X) \cap g^{-1}(x_i)$ and

$$\theta(X) \cap \psi(P_i) = \theta(X) \cap g^{-1}(f(P_i)) = \theta(X) \cap g^{-1}(x_i) \neq \emptyset.$$

If $P_i \cap Y \neq \emptyset$, then $x_i \in f(Y) = g\theta(Y)$ and $x_i = g(y_i)$ for some $y_i \in \theta(Y)$. Thus $y_i \in \theta(Y) \cap g^{-1}(x_i)$ and

$$\theta(Y) \cap \psi(P_i) = \theta(Y) \cap g^{-1}(f(P_i)) = \theta(Y) \cap g^{-1}(x_i) \neq \emptyset.$$

Hence $\psi : \pi(f) \to \pi(g)$ is θ_Y-admissible. Similarly, ψ^{-1} is also θ_Y-admissible. Consequently, $\psi : \pi(f) \to \pi(g)$ is a θ_Y^*-admissible bijection.

(3)$\Longrightarrow$(1). Suppose that (3) holds. For each $x \in X$, if $P_x = f^{-1}(f(x)) \cap Y \neq \emptyset$, then take $z \in \theta(Y) \cap \psi(P_x)$ and let $z = \theta(y)$ for some $y \in Y$. Define $h(x) = y$. If $P_x = f^{-1}(f(x)) \cap Y = \emptyset$, then take $z \in \theta(X) \cap \psi(P_x)$ and let $z = \theta(y)$ for some $y \in X$. Define $h(x) = y$. Clearly, $h \in S(X, Y, *_\theta)$. To see that $f = g\theta h$, for each $x \in X$, let $P_x = f^{-1}(f(x))$ and $Q_z = g^{-1}(g(z))$ (where $z \in \theta(Y) \cap \psi(P_x)$ or $z \in \theta(X) \cap \psi(P_x)$), then

$$f(x) = f(P_x) = g\psi(P_x) = g(Q_z) = g(Q_{\theta(y)}) = g(Q_{\theta(h(x))}) = g\theta h(x)$$

and so $f = g\theta h$. Similarly, $g = f\theta k$ for some $k \in S(X, Y, *_\theta)$. Therefore, $(f, g) \in \mathcal{L}$. □

Let Z be a subset of X and $Z \cap Y \neq \emptyset$. Let $\phi : Z \to X$ be a map. If $\phi(Z \cap Y) \subseteq Y$, then ϕ is said to be Y-*variant*. Clearly, each transformation $f \in S(X, Y, *_\theta)$ is Y-variant. If ϕ is bijective and both ϕ and ϕ^{-1} are Y-variant, then ϕ is said to be Y^*-*variant*.

Now we consider the relation $\mathcal{R}$.

Theorem 3.2. Let $f, g \in S(X, Y, *_\theta)$. Then the following statements are equivalent.

(1) $(f, g) \in \mathcal{R}$.

(2) $\pi(\theta f) = \pi(f) = \pi(g) = \pi(\theta g)$ and $\pi_Y(\theta f) = \pi_Y(f) = \pi_Y(g) = \pi_Y(\theta g)$.

(3) There is a Y^*-variant bijection $\phi : f(X) \to g(X)$ such that $g = \phi f$, and $\theta|_{f(X)}$ and $\theta|_{g(X)}$ are injective. Moreover, $\theta f(x) \in Y \Rightarrow f(x) \in Y$ and $\theta g(x') \in Y \Rightarrow g(x') \in Y$ for some $x, x' \in X$.

Proof. (1)$\Longrightarrow$(2). Suppose that $(f, g) \in \mathcal{R}$. Then $f = h\theta g$ and $g = k\theta f$ for some $h, k \in S(X, Y, *_\theta)$. Immediately, $\pi(f) = \pi(g)$ and $\pi_Y(f) = \pi_Y(g)$. By $f = h\theta k\theta f$, $(h\theta)(k\theta)|_{f(X)} = \mathrm{id}|_{f(X)}$ and $(k\theta)|_{f(X)}$ is injective. It follows that $\theta|_{f(X)}$ is injective and $\pi(\theta f) = \pi(f)$. Similarly, $\pi(\theta g) = \pi(g)$. Thus $\pi(\theta f) = \pi(f) = \pi(g) = \pi(\theta g)$. Now we verify that $\pi_Y(\theta f) = \pi_Y(f)$. Clearly, $\pi_Y(f)$ refines $\pi_Y(\theta f)$. Let $\theta f(x) = \theta f(y) \in Y$ for some distinct $x, y \in X$. Then, by $\pi(\theta f) = \pi(f)$ and $f = h\theta k\theta f$, $f(x) = f(y) = h\theta k\theta f(y) \in Y$. So $\pi_Y(\theta f)$ refines $\pi_Y(f)$ and $\pi_Y(\theta f) = \pi_Y(f)$. Also, we have $\pi_Y(\theta g) = \pi_Y(g)$. Consequently, $\pi_Y(\theta f) = \pi_Y(f) = \pi_Y(g) = \pi_Y(\theta g)$.

(2)$\Longrightarrow$(3). By $\pi(f) = \pi(g)$, define $\phi : f(X) \to g(X)$ by $\phi(x) = g(f^{-1}(x))$ for each $x \in f(X)$. Then ϕ is a bijection and $g = \phi f$. Arbitrarily take $y \in f(X) \cap Y$. Then $f^{-1}(y) \in \pi_Y(f) = \pi_Y(g)$ and $\phi(y) = g(f^{-1}(y)) \in Y$ which implies that ϕ is Y-variant. Similarly, ϕ^{-1} is also Y-variant. Thus ϕ is Y^*-variant. In virtue of $\pi(\theta f) = \pi(f)$, $\theta|_{f(X)}$ is injective. Now assume that $\theta f(x) \in Y$ for some $x \in X$. Then there is some $P \in \pi_Y(\theta f)$ such that $x \in P$. It follows that from $\pi_Y(\theta f) = \pi_Y(f)$ that $f(x) = f(P) \in Y$. The argument for g is the same.

(3)$\Longrightarrow$(1). Suppose that (3) holds. For each $x \in \theta f(X) \cap Y$, let $x = \theta f(x')$ for some $x' \in X$. Fix $a \in Y$ and define $k : X \to X$ by

$$k(x) = \begin{cases} \phi(f(x')) & \text{if } x \in \theta f(X) \cap Y \\ a & \text{otherwise.} \end{cases}$$

If $x = \theta f(x'')$ for some $x'' \in X$ and $x'' \neq x'$, then $f(x') = f(x'')$ since $\theta|_{f(X)}$ is injective and $\phi(f(x')) = \phi(f(x''))$. Thus k is well-defined. We now show that $k \in S(X, Y, *_\theta)$. For each $y \in Y$, either $y \notin \theta f(X)$ or $y \in \theta f(X)$. If $y \notin \theta f(X)$, then $k(y) = a \in Y$. If $y \in \theta f(X)$, let $y = \theta f(x) \in Y$ for some $x \in X$ and then $f(x) \in Y$. So $k(y) = \phi(f(x)) \in Y$ since the map ϕ is Y-variant. Thus $k \in S(X, Y, *_\theta)$. One can show $g = k\theta f$. Similarly, $f = h\theta g$ for some $h \in S(X, Y, *_\theta)$. Therefore, $(f, g) \in \mathcal{R}$. □

According to Theorems 3.1 and 3.2, we have the following conclusion readily.

Theorem 3.3. Let $f, g \in S(X, Y, *_\theta)$. Then the following statements are equivalent.

(1) $(f, g) \in \mathcal{H}$.

(2) $f(X) = g\theta(X)$, $f(Y) = g\theta(Y)$, $g(X) = f\theta(X)$, $g(Y) = f\theta(Y)$, and $\pi(\theta f) = \pi(f) = \pi(g) = \pi(\theta g), \pi_Y(\theta f) = \pi_Y(f) = \pi_Y(g) = \pi_Y(\theta g)$.

(3) There is a θ^*-admissible bijection $\psi : \pi(f) \to \pi(g)$ such that $f = g\psi$, and while there is a Y^*-variant bijection $\phi : f(X) \to g(X)$ such that $g = \phi f$, and $\theta|_{f(X)}$ and $\theta|_{g(X)}$ are injective. Moreover, $\theta f(x) \in Y \Rightarrow f(x) \in Y$ and $\theta g(x') \in Y \Rightarrow g(x') \in Y$ for some $x, x' \in X$.

In what follows we describe the relation $\mathcal{D}$.

Theorem 3.4. Let $f, g \in S(X, Y, *_\theta)$. Then the following statements are equivalent.

(1) $(f, g) \in \mathcal{D}$.

(2) There are a θ^*-admissible bijection $\psi : \pi(g) \to \pi(f)$ and a Y^*-variant bijection $\phi : g(X) \to f(X)$ such that $f\psi = \phi g$, $\theta|_{f(X)}$ and $\theta|_{g(X)}$ are injective. Moreover, $\theta f(x) \in Y \Rightarrow f(x) \in Y$ and $\theta g(x') \in Y \Rightarrow g(x') \in Y$ for some $x, x' \in X$.

Proof. (1)$\Longrightarrow$(2). Suppose that $(f, g) \in \mathcal{D}$. Then $(f, h) \in \mathcal{L}$ and $(h, g) \in \mathcal{R}$ for some $h \in S(X, Y, *_\theta)$. By $(f, h) \in \mathcal{L}$, $h(X) = f(X)$ and there is a θ^*-admissible bijection $\psi : \pi(h) \to \pi(f)$ such that $h = f\psi$. By $(h, g) \in \mathcal{R}$, $\pi(h) = \pi(g)$ and there is a Y^*-variant bijection $\phi : g(X) \to h(X)$ such that $h = \phi g, \theta|_{h(X)}$ and $\theta|_{g(X)}$ are injective, and $\theta h(x) \in Y \Rightarrow h(x) \in Y$ and $\theta g(x') \in Y \Rightarrow g(x') \in Y$. Replacing $\pi(h)$ by $\pi(g)$ and $h(X)$ by $f(X)$, the domain of ψ and the image of ϕ become respectively the required ones and $\theta|_{f(X)}$ is injective as well. Now let $\theta f(x) \in Y$, then $\theta f(x) = \theta h(x')$ for some $x' \in X$ and so $f(x) = h(x') \in Y$. From $\pi(h) = \pi(g)$ and $h = \phi g$, it follows that $h = \phi g$. Therefore, $f\psi = h = \phi g$.

(2)$\Longrightarrow$(1). Define $h(x) = \phi(g(x))$ for each $x \in X$. Clearly, $h \in S(X, Y, *_\theta)$ and $h = \phi g$. Then $\pi(h) = \pi(g)$ and $h = \phi g = f\psi$. By Theorem 3.1, $(h, f) \in \mathcal{L}$ and $h(X) = f(X)$. So $\phi : g(X) \to h(X)$ is a Y^*-variant bijection such that $h = \phi g, \theta|_{h(X)}$ is injective, and $\theta h(x) \in Y \Rightarrow h(x) \in Y$. By Theorem 3.2, $(h, g) \in \mathcal{R}$. Consequently, $(f, g) \in \mathcal{D}$. □

Finally, we investigate the relation $\mathcal{J}$.

Lemma 3.5. Let $f, g \in S(X, Y, *_\theta)$. Then $f = h\theta g\theta k$ for some $h, k \in S(X, Y, *_\theta)$ if and only if there is a Y-variant map $\phi : \theta g\theta(X) \to f(X)$ such that $f(X) = \phi(\theta g\theta(X))$ and $f(Y) = \phi(\theta g\theta(Y))$.

Proof. Suppose that $f = h\theta g\theta k$. Arbitrarily fix $a \in h\theta g\theta k(Y)$ and then define $\phi : \theta g\theta(X) \to f(X)$ by

$$\phi(x) = \begin{cases} h(x) & \text{if } x \in \theta g\theta k(X) \\ a & \text{if } x \in \theta g\theta(X) - \theta g\theta k(X). \end{cases}$$

It is clear that $\phi(\theta g\theta(X)) \subseteq f(X)$. Now take $y \in f(X)$ such that $y = f(x)$ for some $x \in X$. Write $k(x) = x' \in X$. Then

$$y = f(x) = h\theta g\theta k(x) = \phi(\theta g\theta k(x)) = \phi(\theta g\theta(x')).$$

So $f(X) \subseteq \phi(\theta g\theta(X))$ and $f(X) = \phi(\theta g\theta(X))$. Similarly, we have $f(Y) = \phi(\theta g\theta(Y))$. In what follows we show that ϕ is Y-variant. Let $y \in \theta g\theta(X) \cap Y$. If $y \in \theta g\theta k(X) \cap Y$, then $\phi(y) = h(y) \in Y$. If $y \in (\theta g\theta(X) - \theta g\theta k(X)) \cap Y$, then $\phi(x) = a \in h\theta g\theta k(Y) \subseteq Y$. Therefore, ϕ is Y-variant.

Conversely, suppose that (1)-(2) hold. Arbitrarily fix $a \in Y$ and define $h : X \to X$ as follows.

$$h(x) = \begin{cases} \phi(x) & \text{if } x \in \theta g\theta(X) \\ a & \text{otherwise.} \end{cases}$$

It is clear that $h \in S(X, Y, *_\theta)$. By the hypothesis, for each $x \in Y$, there is some $y \in Y$ such that $f(x) = \phi(\theta g\theta(y))$ and each $x \in X - Y$, there is some $z \in X$ such that $f(x) = \phi(\theta g\theta(z))$. Define

$$k(x) = \begin{cases} y & \text{if } x \in Y \\ z & \text{if } x \in X - Y. \end{cases}$$

It is routine to show that $k \in S(X, Y, *_\theta)$ and $f = h\theta g\theta k$. □

Theorem 3.6. Let $f, g \in S(X, Y, *_\theta)$. Then $(f, g) \in \mathcal{J}$ if and only if there are Y-variant maps $\phi : \theta g\theta(X) \to f(X)$ and $\psi : \theta f\theta(X) \to g(X)$ such that $f(X) = \phi(\theta g\theta(X))$, $f(Y) = \phi(\theta g\theta(Y))$ and $g(X) = \psi(\theta f\theta(X))$, $g(Y) = \psi(\theta f\theta(Y))$.

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