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A Deeper Analysis on a Generalization of Fermat ́s Last Theorem

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Abstract

In 1997, the following conjecture was considered by Mauldin as a generalization of Fermat's Last Theorem: “for X, Y, Z, n_1, n_2 and n_3 positive integers with n_1, n_2, n_3> 2, if $X^{n_1} + Y^{n_2} = Z^{n_3}$ then X, Y, Z must have a common prime factor”. The present work provides an investigation focusing in various aspects of this conjecture, exploring the problem ́s specificities with graphic resources and offering a complementary approach to the arguments presented in our previous paper. In fact, we recently discovered the general form of the counterexamples of this conjecture, what is explored in detail in this article. We also analyzed the domain in which the conjecture is valid, defined the situations in which it could fail and previewed some characteristics of its exceptions, in an analytical way.

Keywords: diophantine equations, Fermat ́s Last Theorem, Beal conjecture, Tijdeman-Zagier conjecture

1. Introduction

The Beal Conjecture (also referred by Elkies, 2007, as Tijdeman-Zagier conjecture) was considered by Mauldin (1997) as a generalization of Fermat's Last Theorem (FLT). It consists in a proposition within the number theory ́s field of work according to which, for X, Y, Z, n_1, n_2 and n_3 positive integers with n_1, n_2, n_3> 2, if $X^{n_1} + Y^{n_2} = Z^{n_3}$ then X, Y, Z must have a common prime factor. Stated another way, there is no solution in integers for $X^{n_1} + Y^{n_2} = Z^{n_3}$ in the case of X, Y, Z, n_1, n_2, n_3 positive integers and n_1, n_2, n_3 > 2 if X, Y and Z are coprime (Mauldin, 1997).

Beukers (1998) have also worked in this kind of problem, as well as Darmon and Granville (1995), who investigated integer solutions for the superelliptic equation $z^m = F(x, y)$, where F is a homogeneous polynomial with integer coefficients of the generalized Fermat equation $Ax^p + By^q = Cz^r$.

The Beal Conjecture is considered a generalization of Fermat ́s Last Theorem because FLT Equation ($X^n + Y^n = Z^n$) is a particular case of $X^{n_1} + Y^{n_2} = Z^{n_3}$. Concerning FLT, one of the most famous problems in history of mathematics, the mathematician Andrew Wiles proved the modularity theorem for semistable elliptic curves, which, together with Ribet's theorem, provides a proof for Fermat's Last Theorem. This FLT proof is based in the proof of Taniyama-Shimura ́s Conjecture (Rubin & Silverberg, 1995), which sustains that all elliptic curves are associated to a special class of elliptic functions, called modular functions (Ribet, 1993).

This paper aims to present a deeper approach on the Beal Conjecture, exploring the problem ́s specificities with graphic resources and offering a complementary material to our previous work, where we presented a partial proof of this conjecture. We also analyzed the domain in which the conjecture is valid, defined the situations in which it could fail and previewed some characteristics of the general form of its counterexamples, in an analytical way. Therefore, it is expected that the present text provides a novel and whole approach for the problem ́s overall understanding.

The search of numerical counterexamples via computational algorithms may lead to a treacherous trap and false counterexamples, due to overflow, rounding and truncation errors, especially when dealing with large numbers. This way, we believe that the analytical treatment offers a safe way (and perhaps the only reliable one) to enlighten this problem.

The analyses presented in this article were developed starting from the scope of real numbers and then evaluated under the scope of integer numbers.

2. Exploring the Problem

Based on the initial arguments presented by Di Gregorio (2013), let ́s start from an equation in the form

$$A^2 + B^2 = C^2 \tag{1}$$

in reals, here referred as Pythagoras ́equation. Multiplying (1) by A^{n-2} in order that the first term becomes the *n-th* power of A, there comes that:

$$A^n + B^2 A^{n-2} = C^2 A^{n-2} \tag{2}$$

Now let ́s take an equation in the general form

$$X^{n_1} + Y^{n_2} = Z^{n_3} \tag{3}$$

in reals, here referred as Beal ́s equation. Comparing Equation (2) to Equation (3) and by making $n = n_1$, one can have

$$X = A \tag{4}$$

$$Y = \sqrt[n_2]{B^2 A^{n_1-2}} = \sqrt[n_2]{B^2 X^{n_1-2}} \tag{5}$$

$$Z = \sqrt[n_3]{C^2 A^{n_1-2}} = \sqrt[n_3]{C^2 X^{n_1-2}} \tag{6}$$

Using Equations (4), (5) and (6) one can obtain real solutions X, Y, Z for Beal ́s equation starting from real solutions A, B, C for Pythagoras ́equation.

It is also possible to obtain real solutions A, B, C for Pythagoras ́equation starting from real solutions X, Y, Z for Beal ́s equation, using the following transforms

$$A = X \tag{7}$$

$$B^2 = \frac{Y^{n_2}}{X^{n_1-2}} \tag{8}$$

$$C^2 = \frac{Z^{n_3}}{X^{n_1-2}} \tag{9}$$

From Equations (4), (5) and (6) one can note that, in principle, unless that $n_1 = 2$ (situation in which the power of A is zero, resulting in the unit) or $A = 0$ (trivial solution), the variable $A = X$ is always present in the transforms.

As we want to investigate integer solutions for Equation (3), let ́s assume X, Y, Z positive integers and $n_1 > 2$. This imply that B^2 and $C^2$are necessarily rational numbers (integers or non-integers), because they are written as quotients of integers (Niven, 1990) as shown in Equations (8) and (9).

Once Euclid proved that there are infinite primes (Euclid, trans. 2009), X^{n_1-2}, Y^{n_2} and Z^{n_3} can be written in the form of infinite products (Landau, trans. 2002):

$$X^{n_1-2} = \prod_{i=1}^{\infty} P_i^{(n_1-2)k_{X,i}} = P_1^{(n_1-2)k_{X,1}} \dots P_\infty^{(n_1-2)k_{X,\infty}} \tag{10}$$

$$Y^{n_2} = \prod_{i=1}^{\infty} P_i^{n_2 k_{Y,i}} = P_1^{n_2 k_{Y,1}} \dots P_\infty^{n_2 k_{Y,\infty}} \tag{11}$$

$$Z^{n_3} = \prod_{i=1}^{\infty} P_i^{n_3 k_{Z,i}} = P_1^{n_3 k_{Z,1}} \dots P_\infty^{n_3 k_{Z,\infty}} \tag{12}$$

in which the powers $k_{X,i}$, $k_{Y,i}$and $k_{Z,i}$ represent the number of times the *i-th* prime appears in the factorization of X, Y and Z, respectively. Therefore, B^2 and C^2can be written as:

$$B^2 = \frac{P_1^{n_2 k_{Y,1}} \dots P_\infty^{n_2 k_{Y,\infty}}}{P_1^{(n_1-2)k_{X,1}} \dots P_\infty^{(n_1-2)k_{X,\infty}}} \tag{13}$$

$$C^2 = \frac{P_1^{n_3 k_{Z,1}} \dots P_\infty^{n_3 k_{Z,\infty}}}{P_1^{(n_1-2)k_{X,1}} \dots P_\infty^{(n_1-2)k_{X,\infty}}} \tag{14}$$

or equivalently:

$$B^2 = P_1^{n_2 k_{Y,1} - (n_1-2)k_{X,1}} \dots P_\infty^{n_2 k_{Y,\infty} - (n_1-2)k_{X,\infty}} \tag{15}$$

$$C^2 = P_1^{n_3 k_{Z,1} - (n_1-2)k_{X,1}} \dots P_\infty^{n_3 k_{Z,\infty} - (n_1-2)k_{X,\infty}} \tag{16}$$

Since B^2 and $C^2$are necessarily rational numbers, three situations may occur:

- Situation 1: B^2 and C^2 are integers;
- Situation 2: B^2 or C^2 is integer and the other is a non-integer rational;
- Situation 3: B^2 and C^2 are non-integer rational numbers.

2.1 Situation 1 Analysis (B^2 and C^2 are Integers)

Assuming that B^2 and C^2 are integers, for all the powers of their prime factors P_i, $k_{B^2,i}, k_{C^2,i} \geq 0$, respectively, $\forall i$. Contrariwise, a prime factor could be raised to a negative power, going to the denominator and leading B^2 and C^2 to be non-integers, what conflicts with Situation 1 fundamental hypothesis. This condition can be expressed in the general form as

$$n_2 k_{Y,i} - (n_1 - 2)k_{X,i} = k_{B^2,i} \therefore k_{Y,i} = \frac{k_{B^2,i}}{n_2} + \frac{(n_1-2)}{n_2} k_{X,i} \tag{17}$$

$$n_3 k_{Z,i} - (n_1 - 2)k_{X,i} = k_{C^2,i} \therefore k_{Z,i} = \frac{k_{C^2,i}}{n_3} + \frac{(n_1-2)}{n_3} k_{X,i} \tag{18}$$

Equations (17) and (18) can be represented by the arbitrary straight lines in Figure 1. It is important to highlight that the graphic shows (17) and (18) as continuous functions, but in fact the target of the study focuses on $k_{X,i}, k_{Y,i}, k_{Z,i}$ integers, implying in a discrete behavior that would be difficult to represent graphically. However, this simplification does not affect the conclusions, because $k_{X,i}, k_{Y,i}, k_{Z,i}$ integers are a particular case of $k_{X,i}, k_{Y,i}, k_{Z,i}$ reals.

From Figure 1 one can note that both functions $k_{Y,i}$ and $k_{Z,i}$ are crescent, since $n_1 > 2$, $n_2 > 0$ and $n_3 > 0$. It is noteworthy that once $k_{B^2,i}, k_{C^2,i} \geq 0$, the linear coefficients $\frac{k_{B^2,i}}{n_2}, \frac{k_{C^2,i}}{n_3} \in [0, \infty)$, what implies that, for $k_{X,i} > 0$, $k_{Y,i}, k_{Z,i} > 0$.

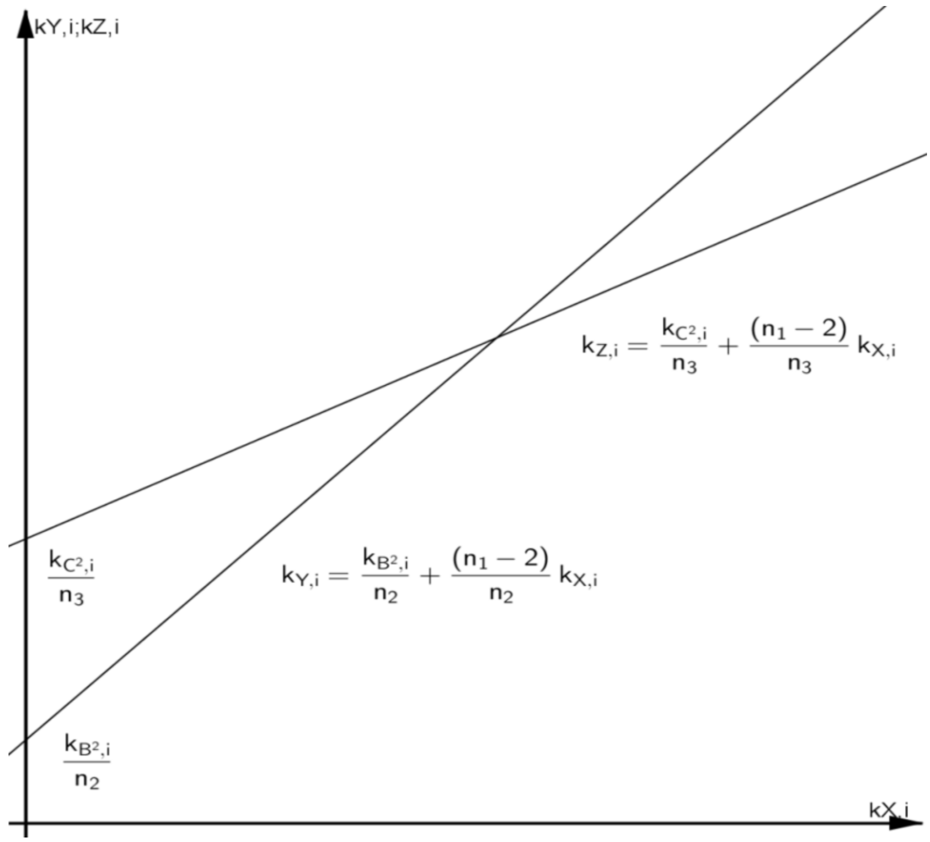

Figure 1. Graphic arbitrary representation of equations (17) and (18), which rule the non-negative powers of primes P_i in B^2 and C^2, respectively.

Studying Figure 1 in details, the behavior of the powers $k_{B^2,i}, k_{C^2,i}, k_{X,i}, k_{Y,i}, k_{Z,i}$ can be resumed in Table 1.

Table 1. Behavior of the powers of a prime P_i illustrated in Figure 1.

Powers of P_i		C1	C2	C3	C4
		$\boldsymbol{k_{B^2,i}=0}$	$\boldsymbol{k_{B^2,i}>0}$	$\boldsymbol{k_{C^2,i}=0}$	$\boldsymbol{k_{C^2,i}>0}$
L1	$k_{X,i}=0$	$k_{Y,i}=0$ $(\nexists P_i in\ X;\nexists P_i in\ Y)$	$k_{Y,i}>0$ $(\nexists P_i in\ X;\exists P_i in\ Y)$	$k_{Z,i}=0$ $(\nexists P_i in\ X;\nexists P_i in\ Z)$	$k_{Z,i}>0$ $(\nexists P_i in\ X;\exists P_i in\ Z)$
L2	$k_{X,i}>0$	$k_{Y,i}>0$ $(\exists P_i in\ X;\exists P_i in\ Y)$	$k_{Y,i}>0$ $(\exists P_i in\ X;\exists P_i in\ Y)$	$k_{Z,i}>0$ $(\exists P_i in\ X;\exists P_i in\ Z)$	$k_{Z,i}>0$ $(\exists P_i in\ X;\exists P_i in\ Z)$

Considering a hypothetical situation where $k_{X,i}, k_{Y,i}, k_{Z,i}$ are integers simultaneously (as always happens in integer solutions for Equation 3), if a prime factor P_i exists in X (and X must have at least one prime factor since it is an integer by initial hypothesis), then P_i necessarily is present in Y (see Table 1 cells (2;1) and (2;2) – the pair in brackets represents relative position of the cell at Table 1: line and column, respectively) and in Z (see Table 1 cells (2;3) and (2;4)). ***<u>This proves Beal Conjecture for Situation 1</u>*.**

As already mentioned, the exception occurs when $n_1=2$, when Y and Z do not depend on the variable X anymore. One can note that, in principle, it seems not to be necessary that $n_2, n_3>2$ for the rule to be considered valid, but only $n_1>2$. However, there are cases of known integer solutions of X, Y, Z coprime in which $n_1>2$ and $n_2=2$ or $n_3=2$, i.e. $7^3+13^2=2^9$ e $2^7+17^3=71^2$ (Darmon & Granville, 1995). This aspect will be clarified in section 3.

2.2 Situation 2 Analysis (B^2 or C^2 is Integer and the Other is a Non-integer Rational Number)

Supposing B^2 a non-integer rational and writing it in the form $I_{B^2}+\varepsilon_{B^2}$, in which I_{B^2} represents the whole part of B^2 and ε_{B^2} is the decimal part, and as $A^2+B^2=C^2$, it follows that:

$$A^2+I_{B^2}+\varepsilon_{B^2}=C^2 \tag{19}$$

$$(A^2+I_{B^2})+\varepsilon_{B^2}=C^2 \tag{20}$$

Equation (20) shows that, since A^2 and I_{B^2} are integers, then $(A^2+I_{B^2})=I_{C^2}$, in which I_{C^2} is the whole part of C^2. This results that $C^2=I_{C^2}+\varepsilon_{B^2}$, that is, if B^2 is a non-integer rational number, then C^2 also is, and the decimal part is common to both ($\varepsilon_{B^2}=\varepsilon_{C^2}$). ***<u>Therefore, Situation 2 is impossible to happen, leaving only Situation 3 to be analyzed, in order that Beal Conjecture was not contradicted by Situation 2</u>*.**

2.3 Situation 3 Analysis (B^2 and C^2 are Non-integer Rational Numbers)

Suppose the prime factors with negative powers in B^2 are $P_{N_1},\cdots,P_{N_f}$ (there should be one or more primes P_{N_j} in B^2 since it is a non-integer rational number by hypothesis of Situation 3). It is important to highlight that A is integer according to the initial hypothesis, resulting $k_{A^2,i}\geq 0, \forall i$.

Writing A^2 and B^2 in the notation of infinite products, then $A^2+B^2=C^2$ becomes

$$\left(P_1^{k_{A^2,1}}\dots P_{N_1}^{k_{A^2,N_1}}\dots P_{N_f}^{k_{A^2,N_f}}\dots P_\infty^{k_{A^2,\infty}}\right)+\left(P_1^{k_{B^2,1}}\dots P_{N_1}^{k_{B^2,N_1}}\dots P_{N_f}^{k_{B^2,N_f}}\dots P_\infty^{k_{B^2,\infty}}\right)=C^2 \tag{21}$$

The powers of $P_{N_1},\cdots,P_{N_f}$ in B^2 are necessarily negative by hypothesis ($k_{B^2,N_1},\cdots,k_{B^2,N_f}<0$) and can be put at evidence, resulting

$$\left[\left(P_1^{k_{A^2,1}}\dots P_{N_1}^{k_{A^2,N_1}-k_{B^2,N_1}}\dots P_{N_f}^{k_{A^2,N_f}-k_{B^2,N_f}}\dots P_\infty^{k_{A^2,\infty}}\right)+\left(P_1^{k_{B^2,1}}\dots P_{N_1}^{0}\dots P_{N_f}^{0}\dots P_\infty^{k_{B^2,\infty}}\right)\right]P_{N_1}^{k_{B^2,N_1}}\dots P_{N_f}^{k_{B^2,N_f}}=C^2 \tag{22}$$

Now, for all primes inside the brackets in Equation (22), their respective powers are not negative, resulting that the content of the brackets is an integer number (here named M), what brings to Equation (23).

$$[M]P_{N_1}^{k_{B^2,N_1}}\dots P_{N_f}^{k_{B^2,N_f}}=C^2 \tag{23}$$

As one can see, the only terms with negative powers in the left member of Equation (23) are $P_{N_1},\cdots,P_{N_f}$, resulting that these primes are necessarily responsible for the negative powers in C^2, since it is a non-integer rational number by hypothesis of Situation 3. It is important to highlight that the powers of $P_{N_1},\cdots,P_{N_f}$ in C^2 are not necessarily $k_{B^2,N_1},\cdots,k_{B^2,N_f}$, because if M have primes $P_{N_1},\cdots,P_{N_f}$ in its factorization, their powers would be altered. Thus, C^2 can be written in the form of Equation (24).

$$\frac{\left(P_1^{k_{M,1}=k_{C^2,1}}\ldots P_{N_1}^{k_{M,N_1}}\ldots P_{N_f}^{k_{M,N_f}}\ldots P_{\infty}^{k_{M,\infty}=k_{C^2,\infty}}\right)}{P_{N_1}^{-k_{B^2,N_1}}\ldots P_{N_f}^{-k_{B^2,N_f}}}=C^2 \tag{24}$$

Since the primes with negative powers in B^2 and C^2 are the same, the equations that rule the behavior of these powers can be represented in the same cartesian plane, as illustrated in Figure 2. In this situation, equations (17) and (18) become, for the primes P_{N_j}:

$$n_2k_{Y,N_j}-(n_1-2)k_{X,N_j}=k_{B^2,N_j} \therefore k_{Y,N_j}=\frac{k_{B^2,N_j}}{n_2}+\frac{(n_1-2)}{n_2}k_{X,N_j} \tag{25}$$

$$n_3k_{Z,N_j}-(n_1-2)k_{X,N_j}=k_{C^2,N_j} \therefore k_{Z,N_j}=\frac{k_{C^2,N_j}}{n_3}+\frac{(n_1-2)}{n_3}k_{X,N_j} \tag{26}$$

Equations (25) and (26) can be represented by the arbitrary straight lines in Figure 2. It is important to highlight that the graphic shows (25) and (26) as continuous functions, but in fact the target of the study focuses on $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ integers, implying in a discrete behavior that would be difficult to represent graphically. However, this simplification does not affect the conclusions, because $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ integers are a particular case of $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ reals.

From Figure 2 one can note that both functions k_{Y,N_j} and k_{Z,N_j} are crescent, since $n_1>2$, $n_2>0$ and $n_3>0$ and their angular coefficients have the same form of the linear functions presented in Figure 1. However, in Situation 3, $k_{B^2,N_j}, k_{C^2,N_j}<0$, leading to negative linear coefficients $\frac{k_{B^2,N_j}}{n_2}$ and $\frac{k_{C^2,N_j}}{n_3}$.

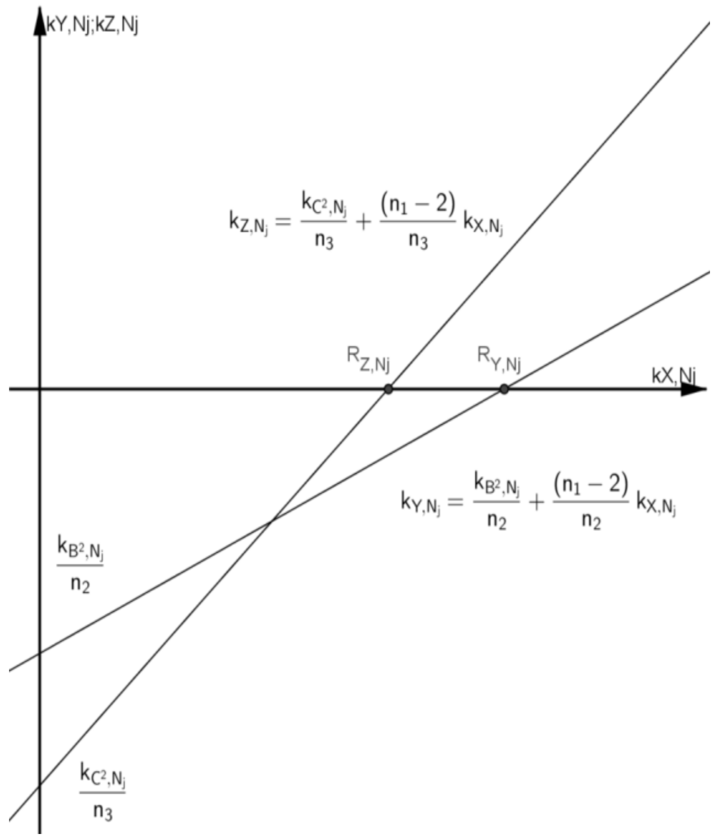

Figure 2. Graphic arbitrary representation of equations (25) and (26), which rule the negative powers of primes P_{N_j} in $B^2$and C^2, respectively.

Studying Figure 2 in details, the behavior of the powers $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ can be resumed in Table 2.

Table 2. Behavior of the powers of a prime P_{N_j} illustrated in Figure 2.

<table>
<tr><th colspan="2" rowspan="2">Powers of P_{N_j}</th><th>C1</th><th>C2</th></tr>
<tr><th>k_{Y,N_j}</th><th>k_{Z,N_j}</th></tr>
<tr><td>L1</td><td>$k_{X,N_j} = 0$</td><td>$k_{Y,N_j} < 0$
$(\nexists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Y\ denominator;$
$Y\ non\ integer)$</td><td>$k_{Z,N_j} < 0$
$(\nexists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Z\ denominator;$
$Z\ non\ integer)$</td></tr>
<tr><td>L2</td><td>$0 < k_{X,N_j} < min\{R_{Y,N_j};R_{Z,N_j}\}$</td><td>$k_{Y,N_j} < 0$
$(\exists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Y\ denominator;$
$Y\ non\ integer)$</td><td>$k_{Z,N_j} < 0$
$(\exists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Z\ denominator;$
$Z\ non\ integer)$</td></tr>
<tr><td>L3</td><td rowspan="2">$k_{X,N_j} = min\{R_{Y,N_j};R_{Z,N_j}\}$</td><td>$If\ R_{Y,N_j} < R_{Z,N_j}$
$k_{Y,N_j} = 0$
$(\exists P_{N_j} in\ X; \nexists P_{N_j} in\ Y)$</td><td>$If\ R_{Y,N_j} < R_{Z,N_j}$
$k_{Z,N_j} < 0$
$(\exists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Z\ denominator;$
$Z\ non\ integer)$</td></tr>
<tr><td>L4</td><td>$If\ R_{Y,N_j} > R_{Z,N_j}$
$k_{Y,N_j} < 0$
$(\exists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Y\ denominator;$
$Y\ non\ integer)$</td><td>$If\ R_{Y,N_j} > R_{Z,N_j}$
$k_{Z,N_j} = 0$
$(\exists P_{N_j} in\ X; \nexists P_{N_j} in\ Z)$</td></tr>
<tr><td>L5</td><td>$min\{R_{Y,N_j};R_{Z,N_j}\} < k_{X,N_j}$
$< max\{R_{Y,N_j};R_{Z,N_j}\}$</td><td>$If\ R_{Y,N_j} < R_{Z,N_j}$
$k_{Y,N_j} > 0$
$(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Y)$</td><td>$If\ R_{Y,N_j} < R_{Z,N_j}$
$k_{Z,N_j} < 0$
$(\exists P_{N_j} in\ X;$
$\exists P_{N_j} in\ Z\ denominator;$
$Z\ non\ integer)$</td></tr>
</table>

L6		$If\ R_{Y,N_j}>R_{Z,N_j}$ $k_{Y,N_j}<0$ $(\exists P_{N_j} in\ X;$ $\exists P_{N_j} in\ Y\ denominator;$ $Y\ non\ integer)$	$If\ R_{Y,N_j}>R_{Z,N_j}$ $k_{Z,N_j}>0$ $(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Z)$
L7	$k_{X,N_j}=max\{R_{Y,N_j};R_{Z,N_j}\}$	$If\ R_{Y,N_j}<R_{Z,N_j}$ $k_{Y,N_j}>0$ $(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Y)$	$If\ R_{Y,N_j}<R_{Z,N_j}$ $k_{Z,N_j}=0$ $(\exists P_{N_j} in\ X; \nexists P_{N_j} in\ Z)$
L8		$If\ R_{Y,N_j}>R_{Z,N_j}$ $k_{Y,N_j}=0$ $(\exists P_{N_j} in\ X; \nexists P_{N_j} in\ Y)$	$If\ R_{Y,N_j}>R_{Z,N_j}$ $k_{Z,N_j}>0$ $(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Z)$
L9	$k_{X,N_j}=R_{Y,N_j}=R_{Z,N_j}$	$If\ R_{Y,N_j}=R_{Z,N_j}$ $k_{Y,N_j}=0\ \ and\ \ k_{Z,N_j}=0$ $(\exists P_{N_j} in\ X;\ \nexists P_{N_j} in\ Y; \nexists P_{N_j} in\ Z)$	
L10	$k_{X,N_j}>max\{R_{Y,N_j};R_{Z,N_j}\}$	$k_{Y,N_j}>0$ $(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Y)$	$k_{Z,N_j}>0$ $(\exists P_{N_j} in\ X; \exists P_{N_j} in\ Z)$

Considering a hypothetical situation where $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ are integers simultaneously (as always happens in integer solutions for Equation 3), from Table 2 one can conclude:

a) Lines 1 to 6 have *Y* and / or *Z* as non-integer rational numbers. *As Beal Conjecture is only about integer solutions for Equation (3), this conclusion does not contradict Beal Conjecture.*

b) Line 10 shows that if a prime factor P_{N_j} exists in *X*, then P_{N_j} necessarily is present in *Y* and in *Z*. *This is in accordance to Beal Conjecture*.

c) Lines 7, 8 and 9 reveal that, for primes P_{N_j}, there is a possibility that they exist in *X* and, do not exist in *Y* and/or do not exist in *Z*, for *X*, *Y* and *Z* integers. *This conclusion, in principle, could allow somehow a contradiction to Beal Conjecture and we will dedicate the following section to investigate it*.

It is important to highlight that an exception to Beal Conjecture may only occur in case each power k_{X,N_j} behave as one of the situations described in Table 2 cell (7;2), cell (8;1) or line 9 , what seems to be very rare, but not impossible. This also imply in X having no primes P_i different from $P_{N_1}, \cdots, P_{N_f}$, because if this doesnʼt happen, the primes P_i would be ruled by Situation 1, in which Beal Conjecture was already proved to be valid.

As in Situation 3 B^2 is a non-integer rational number, it can be written in the form of Equation (27). In fact, in the case of an exception for Beal Conjecture, the primes of X must be exactly $P_{N_1}, \cdots, P_{N_f}$, because if one of them (supposedly P_{N_θ}) is absent from X, then Y will never be integer, once the resultant power of P_{N_θ} inside the radical symbol "$\sqrt{}$" is

$\left(0-\left(-k_{B^2,N_\theta}\right)\right)<0$, leading to a non-integer radicand, as shown in Equation (28).

$$B^2=\frac{P_1^{k_{B^2,1}}\dots P_{N_1}^0\dots P_{N_f}^0\dots P_\infty^{k_{B^2,\infty}}}{P_{N_1}^{-k_{B^2,N_1}}\dots P_{N_f}^{-k_{B^2,N_f}}} \tag{27}$$

$$Y=\sqrt[n_2]{B^2X^{n_1-2}}=\sqrt[n_2]{\frac{P_1^{k_{B^2,1}}\dots P_{N_1}^0\dots P_{N_\theta}^0\dots P_{N_f}^0\dots P_\infty^{k_{B^2,\infty}}}{P_{N_1}^{-k_{B^2,N_1}}\dots P_{N_\theta}^{-k_{B^2,N_\theta}}\dots P_{N_f}^{-k_{B^2,N_f}}}P_{N_1}^{(n_1-2)k_{X,N_1}}\dots P_{N_\theta}^0\dots P_{N_f}^{(n_1-2)k_{X,N_f}}} \tag{28}$$

This way, A=X can be written as

$$A=X=P_{N_1}^{k_{X,N_1}}\dots P_{N_f}^{k_{X,N_f}} \tag{29}$$

From Equation (28) one can note that, in an exception for Beal Conjecture, it is also necessary that the non-negative powers of primes P_i in B^2 are multiple of n_2 ($k_{B^2,i}=\alpha_{B^2,i}n_2$; where $\alpha_{B^2,i}$ is a non-negative coefficient), in order that they can go out of Y ́s radical. A similar behavior happens with the non-negative powers of primes P_i in C^2 , that must be multiple of n_3 ($k_{C^2,i}=\alpha_{C^2,i}n_3$; where $\alpha_{C^2,i}$ is a non-negative coefficient), in order that they can go out of Z ́s radical.

2.3.1 Analysis for the Exceptions (Table 2 lines 7, 8 and 9)

Taking Equations (25) and (26) and making $k_{Y,N_j}=0$ and $k_{Z,N_j}=0$, one can obtain R_{Y,N_j} and R_{Z,N_j}, respectively (Equations (30) and (31)).

$$R_{Y,N_j}=\frac{-k_{B^2,N_j}}{(n_1-2)} \tag{30}$$

$$R_{Z,N_j}=\frac{-k_{C^2,N_j}}{(n_1-2)} \tag{31}$$

Isolating the term k_{X,N_j} in Equations (25) and (26) one can also obtain Equation (32), which can be graphically represented by Figure 3.

$$n_3k_{Z,N_j}-n_2k_{Y,N_j}=k_{C^2,N_j}-k_{B^2,N_j} \tag{32}$$

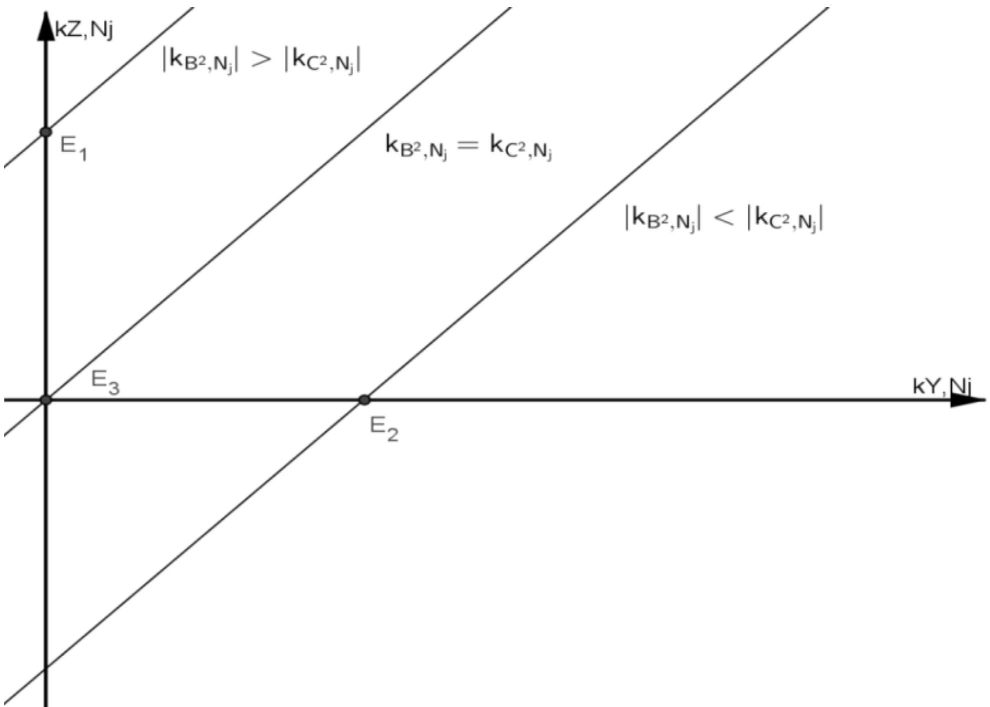

Figure 3. Graphic arbitrary representation of equation (32) according to the possibilities of the negative powers of primes P_{N_j} in $B^2$and C^2.

In Figure 3 we selected three points (E_1- correspondent to cell (8;1), E_2- correspondent to cell (7;2) and E_3- correspondent to line 9, in Table 2), which represent the exceptions to Beal Conjecture that are being investigated. At these points, primes P_{N_j} with positive powers exist in X and, do not exist in Y and/or do not exist in Z. Table 3 can enlighten these situations.

It is important to highlight that, for each prime P_{N_j}, only one of the situations E_1, E_2 or E_3 can happen, but each P_{N_j} must necessarily fit one of this three options to compose an exception to Beal Conjecture.

In order to distinguish them let ́s adopt the following convention for the primes P_{N_j}:

- primes in situation correspondent to point E_1 will be referred as $P_{N_{j,1}}$;
- primes in situation correspondent to point E_2 will be referred as $P_{N_{j,2}}$;
- primes in situation correspondent to point E_3 will be referred as $P_{N_{j,3}}$.

Table 3. Description of the points of exceptions to Beal Conjecture, presented in Figure 3.

POINT	$\boldsymbol{k_{Y,N_j}}$	$\boldsymbol{k_{Z,N_j}}$	$\boldsymbol{k_{X,N_j}}$	**CONDITION**
E_1	0	$\frac{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}}{n_3}$	$R_{Y,N_{j,1}}=\frac{-k_{B^2,N_{j,1}}}{(n_1-2)}$	$\lvert k_{B^2,N_{j,1}}\rvert > \lvert k_{C^2,N_{j,1}}\rvert$
E_2	$\frac{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}}{n_2}$	0	$R_{Z,N_{j,2}}=\frac{-k_{C^2,N_{j,2}}}{(n_1-2)}$	$\lvert k_{B^2,N_{j,2}}\rvert < \lvert k_{C^2,N_{j,2}}\rvert$
E_3	0	0	$R_{Y,N_{j,3}}=R_{Z,N_{j,3}}$	$k_{B^2,N_{j,3}}=k_{C^2,N_{j,3}}$

From Table 3 one can write X, Y and Z as the following generic expressions:

$$X=\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}}{(n_1-2)}}\prod P_{N_{j,2}}^{\frac{-k_{C^2,N_{j,2}}}{(n_1-2)}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}=R_{Z,N_{j,3}}} \tag{33}$$

$$Y=\prod P_i^{\frac{k_{B^2,i}}{n_2}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^{\frac{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}}{n_2}}\prod P_{N_{j,3}}^0 \tag{34}$$

$$Z=\prod P_i^{\frac{k_{C^2,i}}{n_3}}\prod P_{N_{j,1}}^{\frac{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}}{n_3}}\prod P_{N_{j,2}}^0\prod P_{N_{j,3}}^0 \tag{35}$$

Applying Equations (33) and (34) in Equation (3), there comes that:

$$\begin{gathered}X^{n_1}+Y^{n_2}=\\ \left(\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)}}\prod P_{N_{j,2}}^{\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_1=R_{Z,N_{j,3}}.n_1}\right)+\left(\prod P_i^{k_{B^2,i}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}}\prod P_{N_{j,3}}^0\right)=\\ \prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}}\left[\left(\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)}}\prod P_{N_{j,2}}^{\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)}-k_{B^2,N_{j,2}}+k_{C^2,N_{j,2}}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_1=R_{Z,N_{j,3}}.n_1}\right)+\right.\\ \left.\left(\prod P_i^{k_{B^2,i}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^0\prod P_{N_{j,3}}^0\right)\right]\end{gathered} \tag{36}$$

As one can note, as the powers of $P_{N_{j,2}}$ in X^{n_1} and Y^{n_2} are different from 0, then the term $\prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}}$ can be put at evidence. As for point E_2, $\lvert k_{B^2,N_{j,2}}\rvert<\lvert k_{C^2,N_{j,2}}\rvert$, then $k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}>0$ (it is important to remind that $k_{C^2,N_{j,2}}$ and $k_{B^2,N_{j,2}}$ are negative), and the term inside the bracket in Equation (36) is necessarily an integer number (because once $\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)} > -k_{C^2,N_{j,2}}$ and $-k_{C^2,N_{j,2}} > k_{B^2,N_{j,2}} - k_{C^2,N_{j,2}}$, then $\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)} > k_{B^2,N_{j,2}} - k_{C^2,N_{j,2}}$). This implies that primes $P_{N_{j,2}}$ must be necessarily present in the sum $X^{n_1} + Y^{n_2}$. However, from Equation (35) one can note that primes $P_{N_{j,2}}$ are absent from Z^{n_3}, leading to $X^{n_1} + Y^{n_2} \neq Z^{n_3}$. *The conclusion is that an exception for Beal Conjecture in the situation correspondent to point E_2 is impossible to happen, in order that the conjecture remains valid.*

Now let ́s apply Equations (33) and (35) in Equation (3).

$$\begin{aligned} Z^{n_3} - X^{n_1} &= \left(\prod P_i^{k_{C^2,i}} \prod P_{N_{j,1}}^{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right) \\ &- \left(\prod P_i^0 \prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)}} \prod P_{N_{j,2}}^{\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)}} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_1=R_{Z,N_{j,3}}.n_1}\right) \\ &= \prod P_{N_{j,1}}^{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}} \Bigg[\left(\prod P_i^{k_{C^2,i}} \prod P_{N_{j,1}}^0 \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right) \\ &- \left(\prod P_i^0 \prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)}-k_{C^2,N_{j,1}}+k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^{\frac{-k_{C^2,N_{j,2}}.n_1}{(n_1-2)}} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_1=R_{Z,N_{j,3}}.n_1}\right)\Bigg] \end{aligned} \quad (37)$$

As one can note, as the powers of $P_{N_{j,1}}$ in Z^{n_3} and X^{n_1} are different from 0, then the term $\prod P_{N_{j,1}}^{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}}$ can be put at evidence. As for point E_1, $|k_{C^2,N_{j,1}}| < |k_{B^2,N_{j,1}}|$, then $k_{C^2,N_{j,1}} - k_{B^2,N_{j,1}} > 0$ (it is important to remind that $k_{C^2,N_{j,1}}$ and $k_{B^2,N_{j,1}}$are negative), and the term inside the bracket in Equation (37) is necessarily an integer number (because once $\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)} > -k_{B^2,N_{j,1}}$ and $-k_{B^2,N_{j,1}} > k_{C^2,N_{j,1}} - k_{B^2,N_{j,1}}$, then $\frac{-k_{B^2,N_{j,1}}.n_1}{(n_1-2)} > k_{C^2,N_{j,1}} - k_{B^2,N_{j,1}}$). This implies that primes $P_{N_{j,1}}$ must be necessarily present in the result of $Z^{n_3} - X^{n_1}$. However, from Equation (34) one can note that primes $P_{N_{j,1}}$ are absent from Y^{n_2}, what results in $Z^{n_3} - X^{n_1} \neq Y^{n_2}$. *The conclusion is that an exception for Beal Conjecture in the situation correspondent to point E_1 is impossible to happen, in order that the conjecture remains valid.*

Applying now Equations (34) and (35) in Equation (3), there comes that

$$Z^{n_3} - Y^{n_2} = \left(\prod P_i^{k_{C^2,i}} \prod P_{N_{j,1}}^{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right) - \left(\prod P_i^{k_{B^2,i}} \prod P_{N_{j,1}}^0 \prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}} \prod P_{N_{j,3}}^0\right) \quad (38)$$

As one can note, if the powers of P_i in Z^{n_3} and Y^{n_2} are different from 0, then the term $\prod P_i^{\min\{k_{C^2,i};k_{B^2,i}\}}$ can be put at evidence, resulting in Equation (39).

$$\begin{aligned} Z^{n_3} - Y^{n_2} &= \prod P_i^{\min\{k_{C^2,i};k_{B^2,i}\}} \Bigg[\left(\prod P_i^{k_{C^2,i}-\min\{k_{C^2,i};k_{B^2,i}\}} \prod P_{N_{j,1}}^{k_{C^2,N_{j,1}}-k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right) \\ &- \left(\prod P_i^{k_{B^2,i}-\min\{k_{C^2,i};k_{B^2,i}\}} \prod P_{N_{j,1}}^0 \prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{C^2,N_{j,2}}} \prod P_{N_{j,3}}^0\right)\Bigg] \end{aligned} \quad (39)$$

In this case, $Z^{n_3} - Y^{n_2}$ would have prime(s) P_i in its result, but from Equation (33) one can note that primes P_i are absent from X^{n_1}. This way, to keep the integrity of Equation 3, it is necessary that primes P_i do not exist simultaneously in Y and Z. Referring the primes P_i in Y as $P_{i,Y}$ and the primes P_i in Z as $P_{i,Z}$, it comes that an exception at point E_3 will have the form of Equations (40), (41) and (42).

$$X = \prod P_{i,Y}^{0} \prod P_{i,Z}^{0} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}}=R_{Z,N_{j,3}}} \tag{40}$$

$$Y = \prod P_{i,Y}^{\frac{k_{B^2,i,Y}}{n_2}} \prod P_{i,Z}^{0} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{0} \tag{41}$$

$$Z = \prod P_{i,Y}^{0} \prod P_{i,Z}^{\frac{k_{C^2,i,Z}}{n_3}} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{0} \tag{42}$$

2.3.2 Resume about the Analysis of Beal Conjecture for Situation 3

Situation 3 is divided in the situations described in Table 2. The Beal Conjecture was confirmed or not contradicted for all situations of Table 2, except for the ones described in lines 7, 8 and 9.

After a more detailed investigation on the exceptions at points E_1, E_2 and E_3 (see Figure 3 and Table 3), it was clear that these exceptions are impossible to happen at points E_1 and E_2, but not impossible to happen in situations described by point E_3.

2.4 Resume about the Analysis of Beal Conjecture for Situations 1, 2 and 3

As demonstrated in the previous sections, Beal Conjecture was confirmed / not contradicted in all situations related to Situations 1, 2 and 3, except for the situations described by point E_3 in Situation 3.

In this peculiar exception, it was proved by Equations (40), (41) and (42) that X, Y and Z must be pairwise coprime, that is, a prime that is present in one of them is necessarily absent from the others.

3. Extending the Approach

In fact, the initial approach adopted the first term as the reference for the development presented until now, what led to $A^n + B^2A^{n-2} = C^2A^{n-2}$, and consequently $X = A$, $Y = \sqrt[n_2]{B^2X^{n_1-2}}$, $Z = \sqrt[n_3]{C^2X^{n_1-2}}$.

One can also adopt the second element as the basis, resulting in $A^2B^{n-2} + B^n = C^2B^{n-2}$, and $X = \sqrt[n_1]{A^2B^{n_2-2}}$, $Y = B$, $Z = \sqrt[n_3]{C^2B^{n_2-2}}$. The same applies if one chooses the third element to be the reference, coming to $A^2C^{n-2} + B^2C^{n-2} = C^n$ and $X = \sqrt[n_1]{A^2C^{n_3-2}}$, $Y = \sqrt[n_2]{B^2C^{n_3-2}}$, $Z = C$.

As one can see, a solution of $A^2 + B^2 = C^2$ *in reals* (in which at least A or B or C is integer) may lead to three similar types of solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$ (see Table 4).

Table 4. Types of solutions for $X^{n_1} + Y^{n_2} = Z^{n_3}$ that can be obtained using the first, second and third elements as unaltered basis. Source: Di Gregorio (2013).

SOLUTIONS FOR $\boldsymbol{X^{n1} + Y^{n_2} = Z^{n_3}}$	***X***	***Y***	***Z***
I	A	$\sqrt[n_2]{B^2A^{n_1-2}}$	$\sqrt[n_3]{C^2A^{n_1-2}}$
II	$\sqrt[n_1]{A^2B^{n_2-2}}$	B	$\sqrt[n_3]{C^2B^{n_2-2}}$
III	$\sqrt[n_1]{A^2C^{n_3-2}}$	$\sqrt[n_2]{B^2C^{n_3-2}}$	C

Now, it is clear that:

- In solution type I, for $n_1 > 2$, Y and Z depend on X=A, even if $n_2 = 2$ and/or $n_3 = 2$.
- In solution type II, for $n_2 > 2$, X and Z depend on Y=B, even if $n_1 = 2$ and/or $n_3 = 2$.
- In solution type III, for $n_3 > 2$, X and Y depend on Z=C, even if $n_1 = 2$ and/or $n_2 = 2$.

The development performed for solution type I is analogous for the solutions type II and III and will be presented in sections 3.2 and 3.3.

3.1 Demonstration for Solution Type I

This demonstration was already performed in section 2.

3.2 Demonstration for Solution Type II

Demonstration for solution type II is essentially the same comparing to the one performed in section 2 for solution type I (so it will not be repeated), since one can exchange the order of the terms of the first member of Equation (3) and transform $Y^{n_2} + X^{n_1} = Z^{n_3}$ in the format $X^{n_1} + Y^{n_2} = Z^{n_3}$ by exchanging the variables $X \leftrightarrow Y$, $A \leftrightarrow B$ and $n_1 \leftrightarrow n_2$, resulting in a configuration and conclusions analogous to solution type I (see section 2.4).

3.3 Demonstration for Solution Type III

As there are some arguments slightly different from solution type I, solution type III will be explored now from the beginning, to avoid any doubts.

Let ́s start from an equation (1) *in reals*. Multiplying (1) by C^{n-2} there comes that:

$$A^2C^{n-2} + B^2C^{n-2} = C^n \tag{43}$$

Comparing Equation (43) to Equation (3), in reals, and by making $n = n_3$, one can have

$$Z = C \tag{44}$$

$$X = \sqrt[n_1]{A^2C^{n_3-2}} = \sqrt[n_1]{A^2Z^{n_3-2}} \tag{45}$$

$$Y = \sqrt[n_2]{B^2C^{n_3-2}} = \sqrt[n_2]{B^2Z^{n_3-2}} \tag{46}$$

By equalities (44), (45) and (46) one can obtain real solutions X, Y, Z for the Beal equation starting from real solutions A, B, C for Pythagoras ́equation.

It is also possible to obtain real solutions A, B, C for Pythagoras ́equation starting from real solutions X, Y, Z for Beal equation, using the following transforms

$$C = Z \tag{47}$$

$$A^2 = \frac{X^{n_1}}{Z^{n_3-2}} \tag{48}$$

$$B^2 = \frac{Y^{n_2}}{Z^{n_3-2}} \tag{49}$$

From Equations (44), (45) and (46) one can note that, in principle, unless that $n_3 = 2$ (situation in which the power of C is zero, resulting in the unit) or $C = 0$ (trivial solution), the variable $C = Z$ is always present in the transforms.

As we want to investigate integer solutions for Equation (3), let ́s assume X, Y, Z integers and $n_3 > 2$. This imply that A^2 and $B^2$are necessarily rational numbers (integers or non-integers), because they are written as quotients of integers as shown in Equations (48) and (49).

Writing X^{n_1}, Y^{n_2} and Z^{n_3-2} in the form of infinite products, these Equations become:

$$A^2 = \frac{P_1^{n_1k_{X,1}} \dots P_\infty^{n_1k_{X,\infty}}}{P_1^{(n_3-2)k_{Z,1}} \dots P_\infty^{(n_3-2)k_{Z,\infty}}} \tag{50}$$

$$B^2 = \frac{P_1^{n_2k_{Y,1}} \dots P_\infty^{n_2k_{Y,\infty}}}{P_1^{(n_3-2)k_{Z,1}} \dots P_\infty^{(n_3-2)k_{Z,\infty}}} \tag{51}$$

Or equivalently:

$$A^2 = P_1^{n_1k_{X,1}-(n_3-2)k_{Z,1}} \dots P_\infty^{n_1k_{X,\infty}-(n_3-2)k_{Z,\infty}} \tag{52}$$

$$B^2 = P_1^{n_2k_{Y,1}-(n_3-2)k_{Z,1}} \dots P_\infty^{n_2k_{Y,\infty}-(n_3-2)k_{Z,\infty}} \tag{53}$$

Since A^2 and $B^2$are necessarily rational numbers, three situations may occur for solution type III:

- Situation 1-III: A^2 and B^2 are integers;
- Situation 2-III: A^2 or B^2 is integer and the other is non-integer;
- Situation 3-III: A^2 and B^2 are non-integers.

3.3.1 Situation 1-III Analysis (A^2 and $B^2$are Integers)

Assuming that A^2 and B^2 are integers, all the powers of the prime factors P_i, $k_{A^2,i}, k_{B^2,i} \geq 0$, respectively, $\forall i$. Contrariwise, a prime factor could be raised to a negative power, going to the denominator and leading A^2 and B^2 to be non-integers, what conflicts with Situation 1-III fundamental hypothesis. This condition can be expressed in the general form by Equations (54) and (55), which are illustrated in Figure 4.

$$n_1k_{X,i} - (n_3-2)k_{Z,i} = k_{A^2,i} \therefore k_{X,i} = \frac{k_{A^2,i}}{n_1} + \frac{(n_3-2)}{n_1}k_{Z,i} \tag{54}$$

$$n_2k_{Y,i} - (n_3-2)k_{Z,i} = k_{B^2,i} \therefore k_{Y,i} = \frac{k_{B^2,i}}{n_2} + \frac{(n_3-2)}{n_2}k_{Z,i} \tag{55}$$

As one can see, Equations (54) and (55) have the same form of Equations (17) and (18), in order that the analysis is analogous to the one performed in Situation I for solution type I (see Figure 4 and Table 5).

Table 5. Behavior of the powers of a prime P_i illustrated in Figure 4.

Powers of P_i		**C1**	**C2**	**C3**	**C4**
		$k_{B^2,i} = 0$	$k_{B^2,i} > 0$	$k_{A^2,i} = 0$	$k_{A^2,i} > 0$
L1	$k_{Z,i} = 0$	$k_{Y,i} = 0$ ($\nexists P_i in\ Z;\ \nexists P_i in\ Y$)	$k_{Y,i} > 0$ ($\nexists P_i in\ Z;\ \exists P_i in\ Y$)	$k_{X,i} = 0$ ($\nexists P_i in\ Z;\ \nexists P_i in\ X$)	$k_{X,i} > 0$ ($\nexists P_i in\ Z;\ \exists P_i in\ X$)
L2	$k_{Z,i} > 0$	$k_{Y,i} > 0$ ($\exists P_i in\ Z;\ \exists P_i in\ Y$)	$k_{Y,i} > 0$ ($\exists P_i in\ Z;\ \exists P_i in\ Y$)	$k_{X,i} > 0$ ($\exists P_i in\ Z;\ \exists P_i in\ X$)	$k_{X,i} > 0$ ($\exists P_i in\ Z;\ \exists P_i in\ X$)

Considering a hypothetical situation where $k_{X,i}, k_{Y,i}, k_{Z,i}$ are integers simultaneously (as always happens in integer solutions for Equation 3), if a prime factor P_i exists in Z (and Z must have at least one prime factor since it is an integer by initial hypothesis), then P_i necessarily is present in Y (see Table 5 cells (2,1) and (2,2)) and in X (see Table 5 cells (2,3) and (2,4)). Note that the first number in brackets represents the line and the second one represents the column). *This proves Beal Conjecture for Situation 1-III.*

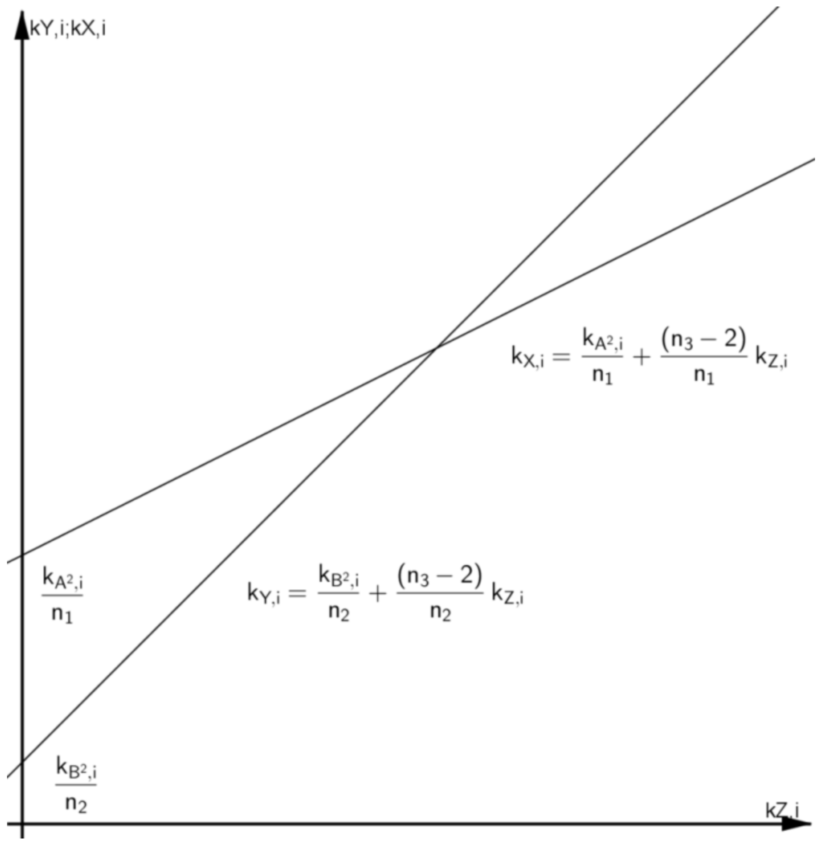

Figure 4. Graphic arbitrary representation of equations (54) and (55), which rule the non-negative powers of primes P_i in $A^2$and B^2, respectively.

3.3.2 Situation 2-III Analysis (A^2 or $B^2$is Integer and the Other is a Non-integer Rational Number)

Writing A^2(non-integer) in the form $I_{A^2} + \varepsilon_{A^2}$, in which I_{A^2} represents the whole part of A^2 and $0 < \varepsilon_{A^2} < 1$ is the decimal part, and as $A^2 + B^2 = C^2$, it follows that:

$$I_{A^2} + \varepsilon_{A^2} + B^2 = C^2 \tag{56}$$

$$B^2 = C^2 - I_{A^2} - \varepsilon_{A^2} \tag{57}$$

Once $C^2 > I_{A^2}$ and $\varepsilon_{A^2} = 1 - \varepsilon_{B^2}$, there comes that

$$B^2 = (C^2 - I_{A^2} - 1) + \varepsilon_{B^2} \tag{58}$$

Since C^2 and I_{A^2} are integers, then $(C^2 - I_{A^2} - 1) = I_{B^2}$, in which I_{B^2} is the whole part of B^2.

This results that $B^2 = I_{B^2} + \varepsilon_{B^2}$, that is, if A^2 is a non-integer rational number, then B^2 also is, because $0 < \varepsilon_{B^2} < 1$. *Therefore, Situation 2-III is impossible to happen, leaving only Situation 3-III to be analyzed, in order that Beal Conjecture was not contradicted by Situation 2-III.*

3.3.3 Situation 3-III Analysis (A^2 and B^2 are Non-Integer Rational Numbers)

Suppose the prime factors with negative powers in B^2 are $P_{N_1}, \cdots, P_{N_f}$ (there should be one or more primes P_{N_j} in B^2 since it is a non-integer rational number by hypothesis of Situation 3-III). It is important to highlight that C is integer according to the initial hypothesis, resulting $k_{C^2,i} \geq 0, \forall i$.

Writing B^2 and C^2 in the notation of infinite products, then $A^2 + B^2 = C^2$ becomes

$$A^2 + \left(P_1^{k_{B^2,1}} \dots P_{N_1}^{k_{B^2,N_1}} \dots P_{N_f}^{k_{B^2,N_f}} \dots P_\infty^{k_{B^2,\infty}}\right) = \left(P_1^{k_{C^2,1}} \dots P_{N_1}^{k_{C^2,N_1}} \dots P_{N_f}^{k_{C^2,N_f}} \dots P_\infty^{k_{C^2,\infty}}\right) \therefore$$

$$\left(P_1^{k_{C^2,1}} \dots P_{N_1}^{k_{C^2,N_1}} \dots P_{N_f}^{k_{C^2,N_f}} \dots P_\infty^{k_{C^2,\infty}}\right) - \left(P_1^{k_{B^2,1}} \dots P_{N_1}^{k_{B^2,N_1}} \dots P_{N_f}^{k_{B^2,N_f}} \dots P_\infty^{k_{B^2,\infty}}\right) = A^2 \tag{59}$$

The powers of $P_{N_1}, \cdots, P_{N_f}$ in B^2 are necessarily negative by hypothesis ($k_{B^2,N_1}, \cdots, k_{B^2,N_f} < 0$) and can be put at evidence, resulting

$$\left[\left(P_1^{k_{C^2,1}} \dots P_{N_1}^{k_{C^2,N_1} - k_{B^2,N_1}} \dots P_{N_f}^{k_{C^2,N_f} - k_{B^2,N_f}} \dots P_\infty^{k_{C^2,\infty}}\right) - \left(P_1^{k_{B^2,1}} \dots P_{n_1}^0 \dots P_{n_f}^0 \dots P_\infty^{k_{B^2,\infty}}\right)\right] P_{N_1}^{k_{B^2,N_1}} \dots P_{N_f}^{k_{B^2,N_f}} = A^2 \tag{60}$$

Now, for all primes inside the brackets in Equation (60), their respective powers are not negative, resulting that the content inside the brackets is a positive integer number (here named T), what brings to Equation (61).

$$[T] P_{N_1}^{k_{B^2,N_1}} \dots P_{N_f}^{k_{B^2,N_f}} = A^2 \tag{61}$$

As one can see, the only terms with negative powers in the left member of Equation (61) are $P_{N_1}, \cdots, P_{N_f}$, resulting that these primes are necessarily responsible for the negative powers in A^2, since it is a non-integer rational number by hypothesis of Situation 3-III. It is important to highlight that the powers of $P_{N_1}, \cdots, P_{N_f}$ in A^2 are not necessarily $k_{B^2,N_1}, \cdots, k_{B^2,N_f}$, because if T have primes $P_{N_1}, \cdots, P_{N_f}$ in its factorization their powers would be altered. Thus, A^2 can be written in the form of Equation (62).

$$\frac{\left(P_1^{k_{T,1} = k_{A^2,1}} \dots P_{N_1}^{k_{T,N_1}} \dots P_{N_f}^{k_{T,N_f}} \dots P_\infty^{k_{T,\infty} = k_{A^2,\infty}}\right)}{P_{N_1}^{-k_{B^2,N_1}} \dots P_{N_f}^{-k_{B^2,N_f}}} = A^2 \tag{62}$$

Since the primes with negative powers in B^2 and A^2 are the same, the equations that rule the behavior of their powers can be represented in the same cartesian plane, as illustrated in Figure 5. In this situation, from Equations (52) and (53) one can obtain Equations (63) and (64) for Situation 3-III, as analogous to Equations (25) and (26) for Situation 3-I, respectively.

$$n_1 k_{X,N_j} - (n_3 - 2) k_{Z,N_j} = k_{A^2,N_j} \therefore k_{X,N_j} = \frac{k_{A^2,N_j}}{n_1} + \frac{(n_3 - 2)}{n_1} k_{Z,N_j} \tag{63}$$

$$n_2 k_{Y,N_j} - (n_3 - 2) k_{Z,N_j} = k_{B^2,N_j} \therefore k_{Y,N_j} = \frac{k_{B^2,N_j}}{n_2} + \frac{(n_3 - 2)}{n_2} k_{Z,N_j} \tag{64}$$

Equations (63) and (64) can be represented by the arbitrary straight lines in Figure 5. It is important to highlight that the graphic shows (63) and (64) as continuous functions, but in fact the target of the study focuses on $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ integers, implying in a discrete behavior that would be difficult to represent graphically. However, this simplification does not affect the conclusions, because $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ integers are a particular case of $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ reals.

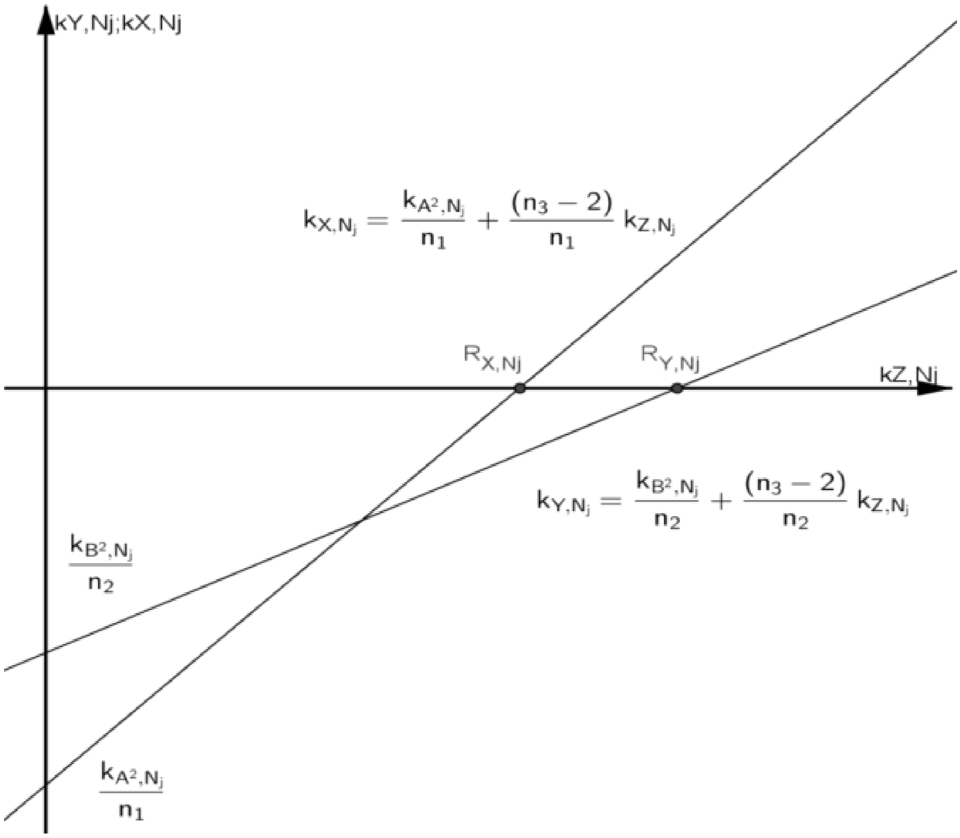

Figure 5. Graphic arbitrary representation of equations (63) and (64), which rule the negative powers of primes P_{N_j} in $A^2$and B^2, respectively.

From Figure 5 one can note that both functions k_{Y,N_j} and k_{X,N_j} are crescent, since $n_3 > 2$, $n_2 > 0$ and $n_1 > 0$ and their angular coefficients have the same form of the linear functions presented in Figure 4. However, in Situation 3-III $k_{B^2,N_j}, k_{A^2,N_j} < 0$, leading to negative linear coefficients $\frac{k_{B^2,N_j}}{n_2}$ and $\frac{k_{A^2,N_j}}{n_1}$.

Studying Figure 5 in details, the behavior of the powers $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ can be resumed in Table 6.

Table 6. Behavior of the powers of a prime $\mathrm{P_{N_j}}$ illustrated in Figure 5.

<table>
<tr><th colspan="2" rowspan="2">Powers of $\mathbf{P_{N_j}}$</th><th>C1</th><th>C2</th></tr>
<tr><th>$\boldsymbol{k_{Y,N_j}}$</th><th>$\boldsymbol{k_{X,N_j}}$</th></tr>
<tr><td>L1</td><td>$k_{Z,N_j} = 0$</td><td>$k_{Y,N_j} < 0$
$(\nexists P_{N_j} in\ Z;\ \exists P_{N_j} in\ Y\ denomin.;$
$Y\ non\ integer)$</td><td>$k_{X,N_j} < 0$
$(\nexists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ X\ denominator;$
$X\ non\ integer)$</td></tr>
<tr><td>L2</td><td>$0 < k_{Z,N_j} < min\{R_{Y,N_j}; R_{X,N_j}\}$</td><td>$k_{Y,N_j} < 0$
$(\exists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ Y\ denominator;$
$Y\ non\ integer)$</td><td>$k_{X,N_j} < 0$
$(\exists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ X\ denominator;$
$X\ non\ integer)$</td></tr>
<tr><td>L3</td><td rowspan="2">$k_{Z,N_j} = min\{R_{Y,N_j}; R_{X,N_j}\}$</td><td>$If\ R_{Y,N_j} < R_{X,N_j}$
$k_{Y,N_j} = 0$
$(\exists P_{N_j} in\ Z; \nexists P_{N_j} in\ Y)$</td><td>$If\ R_{Y,N_j} < R_{X,N_j}$
$k_{X,N_j} < 0$
$(\exists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ X\ denominator;$
$X\ non\ integer)$</td></tr>
<tr><td>L4</td><td>$If\ R_{Y,N_j} > R_{X,N_j}$
$k_{Y,N_j} < 0$
$(\exists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ Y\ denominator;$
$Y\ non\ integer)$</td><td>$If\ R_{Y,N_j} > R_{X,N_j}$
$k_{X,N_j} = 0$
$(\exists P_{N_j} in\ Z;\ \nexists P_{N_j} in\ X)$</td></tr>
<tr><td>L5</td><td>$\min\{R_{Y,N_j}; R_{X,N_j}\} < k_{Z,N_j}$
$< max\{R_{Y,N_j}; R_{X,N_j}\}$</td><td>$If\ R_{Y,N_j} < R_{X,N_j}$
$k_{Y,N_j} > 0$
$(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ Y)$</td><td>$If\ R_{Y,N_j} < R_{X,N_j}$
$k_{X,N_j} < 0$
$(\exists P_{N_j} in\ Z;$
$\exists P_{N_j} in\ X\ denominator;$
$X\ non\ integer)$</td></tr>
</table>

L6		$If\ R_{Y,N_j}>R_{X,N_j}$ $k_{Y,N_j}<0$ $(\exists P_{N_j} in\ Z;$ $\exists P_{N_j} in\ Y\ denominator;$ $Y\ non\ integer)$	$If\ R_{Y,N_j}>R_{X,N_j}$ $k_{X,N_j}>0$ $(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ X)$
L7	$k_{Z,N_j}=max\{R_{Y,N_j};R_{X,N_j}\}$	$If\ R_{Y,N_j}<R_{X,N_j}$ $k_{Y,N_j}>0$ $(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ Y)$	$If\ R_{Y,N_j}<R_{X,N_j}$ $k_{X,N_j}=0$ $(\exists P_{N_j} in\ Z;\ \nexists P_{N_j} in\ X)$
L8		$If\ R_{Y,N_j}>R_{X,N_j}$ $k_{Y,N_j}=0$ $(\exists P_{N_j} in\ Z;\ \nexists P_{N_j} in\ Y)$	$If\ R_{Y,N_j}>R_{X,N_j}$ $k_{X,N_j}>0$ $(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ X)$
L9	$k_{Z,N_j}=R_{Y,N_j}=R_{X,N_j}$	$If\ R_{Y,N_j}=R_{X,N_j}$ $k_{Y,N_j}=0\ and\ k_{X,N_j}=0$ $(\exists P_{N_j} in\ Z;\ \nexists P_{N_j} in\ Y; \nexists P_{N_j} in\ X)$	
L10	$k_{Z,N_j}>max\{R_{Y,N_j};R_{X,N_j}\}$	$k_{Y,N_j}>0$ $(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ Y)$	$k_{X,N_j}>0$ $(\exists P_{N_j} in\ Z;\ \exists P_{N_j} in\ X)$

Considering a hypothetical situation where $k_{X,N_j}, k_{Y,N_j}, k_{Z,N_j}$ are integers simultaneously (as always happens in integer solutions for Equation 3), from Table 6 one can conclude:

a) Lines 1 to 6 have *Y* and / or *X* as non-integer rational numbers. *As Beal Conjecture is only about integer solutions for Equation (3), this conclusion does not contradict Beal Conjecture*.

b) Line 10 shows that if a prime factor P_{N_j} exists in *Z*, then P_{N_j} necessarily is present in *Y* and in *X*. This is in accordance to Beal Conjecture.

c) Lines 7, 8 and 9 reveal that, for primes P_{N_j}, there is a possibility that they exist in *Z* and, do not exist in *Y* and/or do not exist in *X*, for *X*, *Y* and *Z* integers. *This conclusion, in principle, could allow somehow a contradiction to Beal Conjecture and we will dedicate the following section to investigate it*.

It is important to highlight that an exception to Beal Conjecture may only occur in case each power k_{Z,N_j} behave as one of the situations described in Table 6 cell (7;2), cell (8;1) or line 9, what seems to be very rare, but not impossible. This also imply in Z having no primes P_i different from $P_{N_1}, \cdots, P_{N_f}$, because if this doesnʼt happen, the primes P_i would be ruled by Situation 1-III, in which Beal Conjecture was already proved to be valid.

As in Situation 3-III B^2 is a non-integer rational number, it can be written in the form of Equation (65). In fact, in the case of an exception for Beal Conjecture, the primes of Z must be exactly $P_{N_1}, \cdots, P_{N_f}$, because if one of them (supposedly P_{N_θ}) is absent from Z, then Y will never be integer, once the resultant power of P_{N_θ} inside the radical symbol "$\sqrt{}$" is

$\left(0-(-k_{B^2,N_\theta})\right)<0$, leading to a non-integer radicand, as shown in Equation (66).

$$B^2=\frac{P_1^{k_{B^2,1}}\dots P_{N_1}^0\dots P_{N_f}^0\dots P_\infty^{k_{B^2,\infty}}}{P_{N_1}^{-k_{B^2,N_1}}\dots P_{N_f}^{-k_{B^2,N_f}}} \tag{65}$$

$$Y=\sqrt[n_2]{B^2Z^{n_3-2}}=\sqrt[n_2]{\frac{P_1^{k_{B^2,1}}\dots P_{N_1}^0\dots P_{N_\theta}^0\dots P_{N_f}^0\dots P_\infty^{k_{B^2,\infty}}}{P_{N_1}^{-k_{B^2,N_1}}\dots P_{N_\theta}^{-k_{B^2,N_\theta}}\dots P_{N_f}^{-k_{B^2,N_f}}}P_{N_1}^{(n_3-2)k_{Z,N_1}}\dots P_{N_\theta}^0\dots P_{N_f}^{(n_3-2)k_{Z,N_f}}} \tag{66}$$

This way, C=Z can be written as

$$C=Z=P_{N_1}^{k_{Z,N_1}}\dots P_{N_f}^{k_{Z,N_f}} \tag{67}$$

From Equation (66) one can note that, in an exception for Beal Conjecture in Situation 3-III, it is also necessary that the non-negative powers of primes P_i in B^2 are multiple of n_2 ($k_{B^2,i}=\alpha_{B^2,i}n_2$; where $\alpha_{B^2,i}$ is a non-negative coefficient), in order that they can go out of Y ́s radical. A similar behavior happens with the non-negative powers of primes P_i in A^2 , that must be multiple of n_1 ($k_{A^2,i}=\alpha_{A^2,i}n_1$; where $\alpha_{A^2,i}$ is a non-negative coefficient), in order that they can go out of X ́s radical.

3.3.3.1 Analysis for the Exceptions (Table 6 lines 7, 8 and 9)

Taking Equations (64) and (63) and making $k_{Y,N_j}=0$ and $k_{X,N_j}=0$, one can obtain R_{Y,N_j} and R_{X,N_j}, respectively (Equations (68) and (69)).

$$R_{Y,N_j}=\frac{-k_{B^2,N_j}}{(n_3-2)} \tag{68}$$

$$R_{X,N_j}=\frac{-k_{A^2,N_j}}{(n_3-2)} \tag{69}$$

Isolating the term k_{Z,N_j} in Equations (63) and (64) one can also obtain Equation (70), which can be graphically represented in Figure 6.

$$n_1k_{X,N_j}-n_2k_{Y,N_j}=k_{A^2,N_j}-k_{B^2,N_j} \tag{70}$$

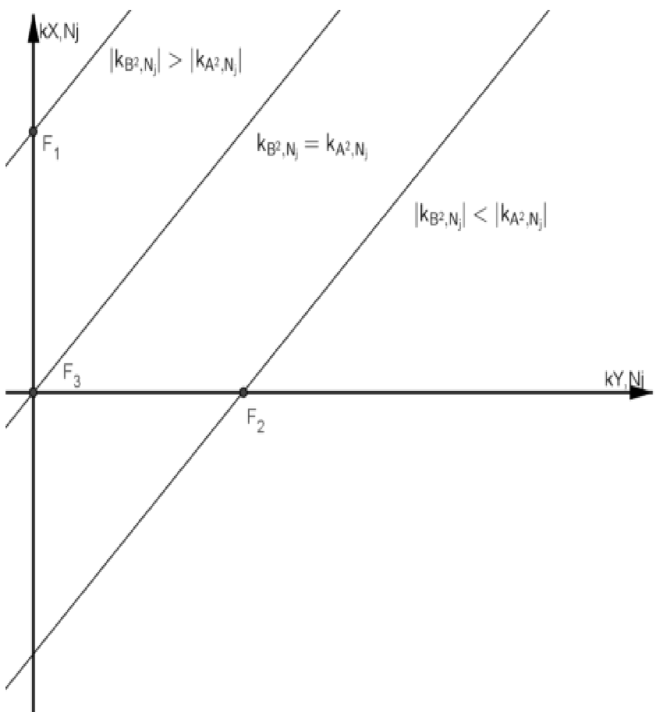

Figure 6. Graphic arbitrary representation of equation (70) according to the possibilities of the negative powers of primes P_{N_j} in $B^2$and A^2.

In Figure 6 we selected three points (F_1- correspondent to cell (8;1), F_2- correspondent to cell (7;2) and F_3- correspondent to line 9, in Table 6), which represent the exceptions to Beal Conjecture that are being investigated. At these points, primes P_{N_j} with positive powers exist in Z and, do not exist in Y and/or do not exist in X. Table 7 can enlighten these situations.

It is important to highlight that, for each prime P_{N_j}, only one of the situations F_1, F_2 or F_3 can happen, but each P_{N_j} must necessarily fit one of this three options to compose an exception to Beal Conjecture.

In order to distinguish them let ́s adopt the following convention for the primes P_{N_j}:

- primes in situation correspondent to point F_1 will be referred as $P_{N_{j,1}}$;
- primes in situation correspondent to point F_2 will be referred as $P_{N_{j,2}}$;
- primes in situation correspondent to point F_3 will be referred as $P_{N_{j,3}}$.

Table 7. Points of exceptions to Beal Conjecture for Situation 3-III.

POINT	$\boldsymbol{k_{Y,N_j}}$	$\boldsymbol{k_{X,N_j}}$	$\boldsymbol{k_{Z,N_j}}$	**CONDITION**
F_1	0	$\frac{k_{A^2,N_{j,1}}-k_{B^2,N_{j,1}}}{n_1}$	$R_{Y,N_j}=\frac{-k_{B^2,N_{j,1}}}{(n_3-2)}$	$\lvert k_{B^2,N_{j,1}}\rvert > \lvert k_{A^2,N_{j,1}}\rvert$
F_2	$\frac{k_{B^2,N_{j,2}}-k_{A^2,N_{j,2}}}{n_2}$	0	$R_{X,N_{j,2}}=\frac{-k_{A^2,N_{j,2}}}{(n_3-2)}$	$\lvert k_{B^2,N_{j,2}}\rvert < \lvert k_{A^2,N_{j,2}}\rvert$
F_3	0	0	$R_{Y,N_{j,3}}=R_{X,N_{j,3}}$	$k_{B^2,N_{j,3}}=k_{A^2,N_{j,3}}$

From Table 7 one can write X, Y and Z as the following generic expressions:

$$Z=\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}}{(n_3-2)}}\prod P_{N_{j,2}}^{\frac{-k_{A^2,N_{j,2}}}{(n_3-2)}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}=R_{X,N_{j,3}}} \tag{71}$$

$$Y=\prod P_i^{\frac{k_{B^2,i}}{n_2}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^{\frac{k_{B^2,N_{j,2}}-k_{A^2,N_{j,2}}}{n_2}}\prod P_{N_{j,3}}^0 \tag{72}$$

$$X=\prod P_i^{\frac{k_{A^2,i}}{n_1}}\prod P_{N_{j,1}}^{\frac{k_{A^2,N_{j,1}}-k_{B^2,N_{j,1}}}{n_1}}\prod P_{N_{j,2}}^0\prod P_{N_{j,3}}^0 \tag{73}$$

Applying Equations (71) and (72) in Equation (3), there comes that:

$$\begin{aligned}
&Z^{n_3}-Y^{n_2}\\
&=\left(\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)}}\prod P_{N_{j,2}}^{\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_3=R_{X,N_{j,3}}.n_3}\right)\\
&-\left(\prod P_i^{k_{B^2,i}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{A^2,N_{j,2}}}\prod P_{N_{j,3}}^0\right)\\
&=\prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{A^2,N_{j,2}}}\left[\left(\prod P_i^0\prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)}}\prod P_{N_{j,2}}^{\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)}-k_{B^2,N_{j,2}}+k_{A^2,N_{j,2}}}\prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_3=R_{X,N_{j,3}}.n_3}\right)\right.\\
&\left.-\left(\prod P_i^{k_{B^2,i}}\prod P_{N_{j,1}}^0\prod P_{N_{j,2}}^0\prod P_{N_{j,3}}^0\right)\right]
\end{aligned} \tag{74}$$

As one can note, as the powers of $P_{N_{j,2}}$ in Z^{n_3} and Y^{n_2} are different from 0, then the term $\prod P_{N_{j,2}}^{k_{B^2,N_{j,2}}-k_{A^2,N_{j,2}}}$ can be put at evidence. As for point F_2, $|k_{B^2,N_{j,2}}| < |k_{A^2,N_{j,2}}|$, then $k_{B^2,N_{j,2}} - k_{A^2,N_{j,2}} > 0$ (it is important to remind that $k_{A^2,N_{j,2}}$ and $k_{B^2,N_{j,2}}$are negative), and the term inside the bracket in Equation (74) is necessarily an integer number (because once $\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)} > -k_{A^2,N_{j,2}}$ and $-k_{A^2,N_{j,2}} > k_{B^2,N_{j,2}} - k_{A^2,N_{j,2}}$, then $\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)} > k_{B^2,N_{j,2}} - k_{A^2,N_{j,2}}$). This implies that primes $P_{N_{j,2}}$ must be necessarily present in the result of $Z^{n_3} - Y^{n_2}$. However, from Equation (73) one can note that primes $P_{N_{j,2}}$ are absent from X^{n_1}, leading to $Z^{n_3} - Y^{n_2} \neq X^{n_1}$. *The conclusion is that an exception for Beal Conjecture in the situation correspondent to point F_2 is impossible to happen, in order that the conjecture remains valid.*

Now let ́s apply Equations (71) and (73) in Equation (3).

$$
\begin{aligned}
&Z^{n_3} - X^{n_1}\\
&= \left(\prod P_i^0 \prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)}} \prod P_{N_{j,2}}^{\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)}} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_3 = R_{X,N_{j,3}}.n_3}\right)\\
&- \left(\prod P_i^{k_{A^2,i}} \prod P_{N_{j,1}}^{k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right)\\
&= \prod P_{N_{j,1}}^{k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}} \left[\left(\prod P_i^0 \prod P_{N_{j,1}}^{\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)} - k_{A^2,N_{j,1}} + k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^{\frac{-k_{A^2,N_{j,2}}.n_3}{(n_3-2)}} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}}.n_3 = R_{X,N_{j,3}}.n_3}\right)\right.\\
&\left. - \left(\prod P_i^{k_{A^2,i}} \prod P_{N_{j,1}}^0 \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right)\right]
\end{aligned}
\tag{75}
$$

As one can note, as the powers of $P_{N_{j,1}}$ in Z^{n_3} and X^{n_1} are different from 0, then the term $\prod P_{N_{j,1}}^{k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}}$ can be put at evidence. As for point F_1, $|k_{A^2,N_{j,1}}| < |k_{B^2,N_{j,1}}|$, then $k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}} > 0$ (it is important to remind that $k_{A^2,N_{j,1}}$ and $k_{B^2,N_{j,1}}$are negative), and the term inside the bracket in Equation (75) is necessarily an integer number (because once $\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)} > -k_{B^2,N_{j,1}}$ and $-k_{B^2,N_{j,1}} > k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}$, then $\frac{-k_{B^2,N_{j,1}}.n_3}{(n_3-2)} > k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}$). This implies that primes $P_{N_{j,1}}$ must be necessarily present in the result of $Z^{n_3} - X^{n_1}$. However, from Equation (72) one can note that primes $P_{N_{j,1}}$ are absent from Y^{n_2}, what results in $Z^{n_3} - X^{n_1} \neq Y^{n_2}$. *The conclusion is that an exception for Beal Conjecture in the situation correspondent to point F_1 is impossible to happen, in order that the conjecture remains valid.*

Applying now Equations (72) and (73) in Equation (3), there comes that

$$
X^{n_1} + Y^{n_2} = \left(\prod P_i^{k_{A^2,i}} \prod P_{N_{j,1}}^{k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^0 \prod P_{N_{j,3}}^0\right) + \left(\prod P_i^{k_{B^2,i}} \prod P_{N_{j,1}}^0 \prod P_{N_{j,2}}^{k_{B^2,N_{j,2}} - k_{A^2,N_{j,2}}} \prod P_{N_{j,3}}^0\right) \tag{76}
$$

As one can note, if the powers of P_i in X^{n_1} and Y^{n_2} are different from 0, then the term $\prod P_i^{\min\{k_{A^2,i}; k_{B^2,i}\}}$ can be put at evidence, resulting in Equation (77).

$$X^{n_1} + Y^{n_2} = \prod P_i^{\min\{k_{A^2,i};k_{B^2,i}\}} \left[\left(\prod P_i^{k_{A^2,i} - \min\{k_{A^2,i};k_{B^2,i}\}} \prod P_{N_{j,1}}^{k_{A^2,N_{j,1}} - k_{B^2,N_{j,1}}} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{0} \right) + \left(\prod P_i^{k_{B^2,i} - \min\{k_{A^2,i};k_{B^2,i}\}} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{k_{B^2,N_{j,2}} - k_{A^2,N_{j,2}}} \prod P_{N_{j,3}}^{0} \right) \right] \tag{77}$$

In this case, $X^{n_1} + Y^{n_2}$ would have prime(s) P_i in its result, but from Equation (71) one can note that primes P_i are absent from Z^{n_3}. This way, to keep the integrity of Equation 3, it is necessary that primes P_i do not exist simultaneously in Y and X. Referring the primes P_i in Y as $P_{i,Y}$ and the primes P_i in X as $P_{i,X}$, it comes that an exception at point F_3 will have the form of Equations (78), (79) and (80).

$$Z = \prod P_{i,Y}^{0} \prod P_{i,X}^{0} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{R_{Y,N_{j,3}} = R_{X,N_{j,3}}} \tag{78}$$

$$Y = \prod P_{i,Y}^{\frac{k_{B^2,i,Y}}{n_2}} \prod P_{i,X}^{0} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{0} \tag{79}$$

$$X = \prod P_{i,Y}^{0} \prod P_{i,X}^{\frac{k_{A^2,i,X}}{n_1}} \prod P_{N_{j,1}}^{0} \prod P_{N_{j,2}}^{0} \prod P_{N_{j,3}}^{0} \tag{80}$$

3.3.3.2 Resume about the Analysis of Beal Conjecture for Situation 3-III

Situation 3-III is divided in the situations described in Table 6. The Beal Conjecture was confirmed or not contradicted for all situations of Table 6, except for the ones described in lines 7, 8 and 9.

After a more detailed investigation on the exceptions at points F_1, F_2 and F_3 (see Figure 6 and Table 7), it was clear that these exceptions are impossible to happen at points F_1 and F_2, but not impossible to happen in situations described by point F_3.

3.3.4 Resume about the Analysis of Beal Conjecture for Situations 1-III, 2-III and 3-III

As demonstrated in the previous sections, the Beal Conjecture was confirmed / not contradicted in all situations related to Situations 1-III, 2-III and 3-III, except for the situations described by point F_3 in Situation 3-III.

In this peculiar type of exception, it was proved by Equations (78), (79) and (80) that X, Y and Z must be pairwise coprime, that is, a prime that is present in one of them is necessarily absent from the others.

4. Conclusion

As explored in the previous sections, it was proved that covering almost all possibilities for Situations 1, 2 and 3 applied to solutions type I, II and III, if there are integer solutions for Equation (3), then necessarily X, Y, Z have at least one prime factor in common. From Table 4 it is possible to note that in these situations there can be prime factors in common to X, Y, Z since:

- In solution type I, $n_1 > 2$, even if $n_2 = 2$ and/or $n_3 = 2$;
- In solution type II, $n_2 > 2$, even if $n_1 = 2$ and/or $n_3 = 2$;
- In solution type III, $n_3 > 2$, even if $n_1 = 2$ and/or $n_2 = 2$.

However, we discovered the general analytical form of counterexamples for these statements in the situations described by Situation 3 in which the equations that rule the negative powers P_{N_j} in B^2 and C^2 (solution type I), A^2 and C^2 (solution type II) or A^2 and B^2 (solution type III), respectively, are concurrent to points located in the power´s graphic horizontal axis (here called "Exception Points").

Thus, the exceptions to the conjecture may occur only when the reference basis X (in solution type I), Y (in solution type II) or Z (in solution type III), respectively, are composed only by primes P_{N_j} with powers correspondent to the exception points. Besides that, it is necessary that X, Y and Z are pairwise coprime in all cases and there are other conditions that must be obeyed. This peculiar configuration seems to be quite rare, but not impossible to happen. In Table 8 we resumed the general conditions that must be simultaneously fulfilled (in solutions type I, II or III) to allow exceptions to Beal Conjecture.

Table 8. Conditions that must be simultaneously fulfilled to allow exceptions to Beal Conjecture.

CRITERIA	Solution type I	Solution type II	Solution type III
Reference basis	$X=A$	$Y=B$	$Z=C$
Expression for the other elements	$Y = \sqrt[n_2]{B^2A^{n_1-2}}$ $Z = \sqrt[n_3]{C^2A^{n_1-2}}$	$X = \sqrt[n_1]{A^2B^{n_2-2}}$ $Z = \sqrt[n_3]{C^2B^{n_2-2}}$	$X = \sqrt[n_1]{A^2C^{n_3-2}}$ $Y = \sqrt[n_2]{B^2C^{n_3-2}}$
Rational terms	B^2 and C^2	A^2 and C^2	A^2 and B^2
Form of the powers of the primes P_{N_j} at the reference basis	$k_{X,N_j} = \frac{-k_{B^2,N_{j,3}}}{(n_1-2)} = \frac{-k_{C^2,N_{j,3}}}{(n_1-2)}$	$k_{Y,N_j} = \frac{-k_{A^2,N_{j,3}}}{(n_2-2)} = \frac{-k_{C^2,N_{j,3}}}{(n_2-2)}$	$k_{Z,N_j} = \frac{-k_{B^2,N_{j,3}}}{(n_3-2)} = \frac{-k_{A^2,N_{j,3}}}{(n_3-2)}$
Form of the powers of the primes P_i at the other elements	$k_{Y,i,Y} = \frac{k_{B^2,i,Y}}{n_2}$ $k_{Z,i,Z} = \frac{k_{C^2,i,Z}}{n_3}$	$k_{X,i,X} = \frac{k_{A^2,i,X}}{n_1}$ $k_{Z,i,Z} = \frac{k_{C^2,i,Z}}{n_3}$	$k_{Y,i,Y} = \frac{k_{B^2,i,Y}}{n_2}$ $k_{X,i,X} = \frac{k_{A^2,i,X}}{n_1}$
Conditions about the powers of the primes in A^2, B^2 and C^2	$-k_{B^2,N_{j,3}} = -k_{C^2,N_{j,3}}$ are multiple of (n_1-2) $k_{B^2,i,Y}$ are multiple of n_2 $k_{C^2,i,Z}$ are multiple of n_3	$-k_{A^2,N_{j,3}} = -k_{C^2,N_{j,3}}$ are multiple of (n_2-2) $k_{A^2,i,X}$ are multiple of n_1 $k_{C^2,i,Z}$ are multiple of n_3	$-k_{A^2,N_{j,3}} = -k_{B^2,N_{j,3}}$ are multiple of (n_3-2) $k_{B^2,i,Y}$ are multiple of n_2 $k_{A^2,i,X}$ are multiple of n_1
Conditions about the relation between the main variables	$A^2+B^2 = C^2$ $X^{n_1}+Y^{n_2} = Z^{n_3}$ ***X*, *Y* and *Z* are pairwise coprime.**		
Consequences of the previous requirements	***X*, *Y* or *Z* is even and the others are odd.** **The product $W = X^{n_1}Y^{n_2}Z^{n_3}$ has primes P_v with powers** $k_{W,v} \le Greater\{n_1k_{X,v}; n_2k_{Y,v}; n_3k_{Z,v}\},\ \forall v.$		

The Beal Conjecture states, in a more conservative way that, for $n_1, n_2, n_3 > 2$, the integer solutions for $X^{n1} + Y^{n2} = Z^{n3}$ have a common prime factor, meeting all the conditions presented above, for situations different than the one described by the Exception Points previously demonstrated. ***Therefore, the Beal Conjecture is proved to be correct for situations different than the ones described in Table 8, which rule the exceptions of the conjecture.*** ☐

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