On S-quasi-Dedekind Modules

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Abstract

Let \( R \) be a commutative ring and \( M \) an unital \( R \)-module. A proper submodule \( L \) of \( M \) is called primary submodule of \( M \), if \( rm \in L \), where \( r \in R, m \in M \), then \( m \in L \) or \( r^nM \subseteq L \) for some positive integer \( n \). A submodule \( K \) of \( M \) is called semi-small submodule of \( M \) if, \( K + L \neq M \) for each primary submodule \( L \) of \( M \). An \( R \)-module \( M \) is called S-quasi-Dedekind module if, for each \( f \in \text{End}_R(M), f \neq 0 \) implies \( \text{Ker } f \) semi-small in \( M \). In this paper we introduce the concept of S-quasi-Dedekind modules as a generalisation of small quasi-Dedekind modules, and gives some of their properties, characterizations and examples. Another hand we study the relationships of S-quasi-Dedekind modules with some classes of modules and their endomorphism rings.

Keywords: Primary submodules, semi-small submodules, quasi-Dedekind modules, S-quasi-Dedekind modules

1. Introduction

Throughout all rings are associative, commutative with identity and all modules are unitary \( R \)-module. A submodule \( K \) of \( M \) is small in \( M \) if, \( K + N \neq M \) for each submodule \( N \) of \( M \). A proper submodule \( L \) of \( M \) is called primary submodule of \( M \), if \( rm \in L \), where \( r \in R, m \in M \), then \( m \in L \) or \( r^nM \subseteq L \) for some positive integer \( n \). A submodule \( K \) of \( M \) is called semi-small submodule of \( M \) if \( K + L \neq M \) for each primary submodule \( L \) of \( M \). An \( R \)-module \( M \) is called quasi-Dedekind module if any nonzero endomorphism of \( M \) is a monomorphism. An \( R \)-module \( M \) is called small quasi-Dedekind module if, for each \( f \in \text{End}_R(M), f \neq 0 \) implies \( \text{Ker } f \) small in \( M \). An \( R \)-module \( M \) is called S-quasi-Dedekind module if, for each \( f \in \text{End}_R(M), f \neq 0 \) implies \( \text{Ker } f \) semi-small in \( M \). Mijbass introduce and study the concept of quasi-Dedekind module (Mijbass, A. S. (1997)). Ghawi study the concept of small quasi-Dedekind module (Ghawi, Th. Y. (2010)). In this paper we introduce and study the concept of S-quasi-Dedekind as a generalization of small quasi-Dedekind module.

In the first section, we introduce S-quasi-Dedekind modules and study some basic properties of this concept.

In the second section, we study the relations between S-quasi-Dedekind modules and other related modules.

In third section, we study the endomorphism ring of S-quasi-Dedekind module.

2. Some Properties of S-quasi-Dedekind Modules

In this section, we introduce the concept of S-quasi-Dedekind module as a generalization of quasi-Dedekind module and give some basic properties examples and characterization of this concept.

Definition 1

1. A proper submodule \( L \) of \( M \) is called primary submodule of \( M \), if \( rm \in L \), where \( r \in R, m \in M \), then \( m \in L \) or \( r^nM \subseteq L \) for some positive integer \( n \).

2. An ideal \( I \) in a ring \( R \) is called primary ideal in \( R \), if \( xy \in I \), where \( x, y \in R \), then either \( x^n \in I \) or \( y^k \in I \) for some positive integers \( n \) and \( k \).

Definition 2 Let \( M \) be an \( R \)-module and \( N \leq M \).

1. \( N \) is called small submodule of \( M \) (\( N \ll M \), for short) if \( N + L \neq M \) for each submodule \( L \) of \( M \).

2. \( N \) is called semi-small submodule of \( M \) (\( N \ll_2 M \), for short) if \( N + L \neq M \) for each primary submodule \( L \) of \( M \).

3. An ideal \( J \) in a ring \( R \) is called semi-small ideal in \( R \) if \( I + J \neq R \), for each primary ideal \( I \) of \( R \).
Remark 1

1. Each small submodule is semi-small submodule.
2. For each module $M$, we have $\{0\}$ is a semi-small submodule of $M$.
3. If $M$ is semi-simple module, then $\{0\}$ is the only semi-small submodule.

Definition 3 Let $M$ be an $R$-module.

1. $M$ is called small quasi-Dedekind if for all $f \in \text{End}_R(M)$, $f \neq 0$ implies $\text{Ker} f \ll M$.
2. $M$ is called S-quasi-Dedekind if for all $f \in \text{End}_R(M)$, $f \neq 0$ implies $\text{Ker} f \ll_s M$.

Example 1

1. $\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind.
2. Let $p$ is a prime integer and $\mathbb{Z}(p^\infty) = \{a^p + \mathbb{Z}/a^k \dv{a \text{ integers} \text{ and} k \text{ is positive}}\}$. The only submodules of $\mathbb{Z}(p^\infty)$ are $0 \leq a^p + \mathbb{Z} \leq a^{p+1} + \mathbb{Z} \leq ...$
   Hence the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is S-quasi-Dedekind.

Remark 2

1. It is clear that every quasi-Dedekind $R$-module is a S-quasi-Dedekind $R$-module. But the converse is not true in general, for example $\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$-module is S-quasi-Dedekind but it is not quasi-Dedekind.
2. Every small quasi-Dedekind $R$-module is a S-quasi-Dedekind $R$-module.
3. The direct sum of S-quasi-Dedekind modules is not necessary that a S-quasi-Dedekind module, for example each of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ as $\mathbb{Z}$-module is S-quasi-Dedekind. But $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is not a S-quasi-Dedekind $\mathbb{Z}$-module.
4. Every integral domain $R$ is a S-quasi-Dedekind $R$-module (Mijbass, A. S. (1997)). But the converse need not be in general; for example $\mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$-module is S-quasi-Dedekind module, but $\mathbb{Z}/4\mathbb{Z}$ is not an integral domain.

Proposition 1 Let $M$ be a semi-simple $R$-module. Then $M$ is S-quasi-Dedekind if and only if $M$ is quasi-Dedekind.

Proof. $\Rightarrow$) Let $f \in \text{End}_R(M)$, $f \neq 0$. Since $M$ is S-quasi-Dedekind, then $\text{Ker} f \ll_s M$. But $M$ is semi-simple, so $\text{Ker} f = \{0\}$. Thus $M$ is quasi-Dedekind.

$\Leftarrow$ It is clear.

Proposition 2 Let $M$ be a finitely generated $R$-module. Then $M$ is S-quasi-Dedekind if and only if $M$ is small quasi-Dedekind.

Proof. $\Rightarrow$) Let $f \in \text{End}_R(M)$, $f \neq 0$. Suppose that $N$ is a proper submodule of $M$ such that $\text{Ker} f + N = M$. Since $M$ is a finitely generated $R$-module, then there exists a maximal submodule $L$ such that $N \subseteq L$. Thus $\text{Ker} f + L = M$. But $L$ is a primary submodule of $M$ and $\text{Ker} f \ll_s M$, so $L = M$, a contradiction. Thus $\text{Ker} f \ll M$ and $M$ is small quasi-Dedekind.

$\Leftarrow$ It is clear.

Corollary 1 Let $R$ be an Artinian principal ideal ring and let $M$ be a co-Hopfian $R$-module. Then $M$ is S-quasi-Dedekind if and only if $M$ is small quasi-Dedekind.

Proof. Since $R$ is an Artinian principal ideal ring and $M$ is a co-Hopfian $R$-module, then by (Barry & all, (1997)), $M$ is a finitely generated $R$-module, thus the result is obtained.

Corollary 2 Let $R$ be an Artinian principal ideal ring and let $M$ be a weakly co-Hopfian $R$-module. Then $M$ is S-quasi-Dedekind if and only if $M$ is small quasi-Dedekind.

Proof. Since $R$ is an Artinian principal ideal ring and $M$ is a weakly co-Hopfian $R$-module, then (Barry & all, (2010)), $M$ is a finitely generated $R$-module, thus the result is obtained.
Corollary 3 Let R be an Artinian principal ideal ring and let M be a Dedekind finite R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.

Proof. Since R is an Artinian principal ideal ring and M is a Dedekind finite R-module, then by (Barry & all, (2011)), M is a finitely generated R-module, thus the result is obtained.

Definition 4 An R-module is called a multiplication R-module if every submodule N of M is of the form IM, for some ideal I of R.

Proposition 3 Let M be a multiplication R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.

Proof. \( \Rightarrow \) Let \( f \in \text{End}_R(M) \) such that \( f \neq 0 \). Suppose that \( N \) is a proper submodule such that \( \text{ker} f + N = M \). Since M is a multiplication R-module, then by (El-Bast & all, (1988)), there exists a prime submodule L such that \( N \subseteq L \). Thus \( \text{ker} f + L = M \). But L is a primary submodule of M and \( \text{ker} f \ll M \), so \( L = M \). This is a contradiction. Thus \( \text{ker} f \ll M \) and M is small quasi-Dedekind.

\( \Leftarrow \) It is clear.

Corollary 4 Let M be a cyclic R-module. Then M is S-quasi-Dedekind if and only if M is small quasi-Dedekind.

Lemma 1 Let M be an R-module and let \( N \ll M \). If \( K \subseteq N \), then \( K \ll M \).

Proof. Let \( K + L = M \), for some primary submodule L of M. Since \( K \subseteq N \), then \( N + L = M \), and because \( N \ll M \), M = L, a contradiction.

The following theorem is a characterization of S-quasi-Dedekind modules.

Theorem 1 Let M be an R-module. Then M is S-quasi-Dedekind if and only if \( \text{Hom}(M/N, M) = \{0\} \), for all N \( \ll M \).

Proof. \( \Rightarrow \) Suppose that there exists \( N \ll M \) such that \( \text{Hom}(M/N, M) \neq \{0\} \), then there exists \( \phi : M/N \rightarrow M \), \( \phi \neq 0 \). Hence \( \phi \circ \pi \in \text{End}_A(M) \), where \( \pi \) is the canonical projection, and \( \phi \circ \pi = 0 \) which implies \( \text{ker} (\phi \circ \pi) \ll M \), but \( N \subseteq \text{ker} (\phi \circ \pi) \), so \( N \ll M \) by Lemma 1 which implies a contradiction.

\( \Leftarrow \) Suppose that there exists \( f \in \text{End}_A(M) \), \( f \neq 0 \) such that \( \text{ker} f \ll M \), define \( g : M \rightarrow M \) by \( g(m + \text{ker} f) = f(m) \), for all \( m \in M \). So g is well-defined and \( g \neq 0 \). Hence \( \text{Hom}(M/\text{ker} f, M) \neq \{0\} \) which is a contradiction.

Proposition 4 Let M be an R-module and let \( \overline{R} = R/J \), where J is an ideal of R such that \( J \subseteq \text{ann}_R(M) \). Then M is a S-quasi-Dedekind R-module if and only if M is a S-quasi-Dedekind R-module.

Proof. \( \Rightarrow \) We have \( \text{Hom}_R(M/K, M) = \text{Hom}_R(M/K, M) \), for all \( K \subseteq M \) by (Kasch, F. (1982)). Thus, if M is a S-quasi-Dedekind R-module then \( \text{Hom}_R(M/K, M) = \{0\} \) for all \( K \ll M \), so \( \text{Hom}_R(M/K, M) = \{0\} \) for all \( K \ll M \). Thus M is a S-quasi-Dedekind R-module.

\( \Leftarrow \) The proof of the converse is similiary.

Definition 5 Let M be an R-module and let \( N \ll M \). N is called quasi-invertible if \( \text{Hom}(M/N, M) = \{0\} \).

Lemma 2 (Mijbass (1997), Proposition 1.14) Let M be an R-module and let \( N \subseteq M \). Then \( \text{ann}_R(M) = \text{ann}_R(N) \).

Proposition 5 Let M be a S-quasi-Dedekind R-module. Then \( \text{ann}_R(M) = \text{ann}_R(N) \) for all \( N \ll M \).

Proof. Since M is a S-quasi-Dedekind R-module, so by theorem 1 \( \text{Hom}(M/N, M) = \{0\} \) for all \( N \ll M \) which implies \( N \) is a quasi-invertible submodule for all \( N \ll M \). Thus by lemma 2 \( \text{ann}_R(M) = \text{ann}_R(N) \) for all \( N \ll M \).

Lemma 3 (Abdullah & all, (2011), Proposition 1.16)

Let M and \( M' \) be R-modules and let \( f : M \rightarrow M' \) be an R-epimorphism.

If \( K \ll M \) such that \( \text{ker} f \subseteq K \), then \( f(K) \ll M' \).

Proposition 5 Let \( M_1, M_2 \) be R-modules such that \( M_1 \cong M_2 \). Then \( M_1 \) is a S-quasi-Dedekind R-module if and only if \( M_2 \) is a S-quasi-Dedekind R-module.

Proof. \( \Rightarrow \) Let \( f \in \text{End}_R(M_2) \), \( f \neq 0 \). To prove \( \text{ker} f \ll M_2 \). Since \( M_1 \cong M_2 \), there exists an isomorphism \( g : M_1 \rightarrow M_2 \) and \( g^{-1} : M_2 \rightarrow M_1 \). We have. Hence \( h = g^{-1} \circ f \circ g \in \text{End}_R(M_1) \), \( h \neq 0 \). So \( \text{ker} h \ll M_1 \), then \( g(\text{ker} h) \ll M_2 \) by lemma 3. But we can show that \( g(\text{ker} h) = \text{ker} f \) as follows; let \( y = g(x) \), \( x \in \text{ker} h \). Hence \( h(x) = 0 \); that is \( g^{-1} \circ f \circ g(x) = 0 \), then \( g^{-1} \circ f(y) = 0 \), so \( g^{-1}(f(y)) = 0 \) and hence \( f(y) = 0 \), since \( g^{-1} \) is a monomorphism, so that \( y \in \text{ker} f \). Now let \( y \in \text{ker} f \), then \( f(y) = 0 \), but \( y \in M_2 \), so there exists an \( x \in M_1 \) such that \( y = g(x) \), since g is surjective. Thus \( f(g(x)) = 0 \) and so \( f(\text{ker} f) = 0 \); that is \( h(x) = 0 \). Hence \( x \in \text{ker} h \). This implies \( y = g(x) \in g(\text{ker} h) \), thus \( \text{ker} f = \text{ker} h \), hence \( \text{ker} f \ll M_2 \).

\( \Leftarrow \) The proof the converse is similiary.
Lemma 4 Let $M, M'$ be injective $R$-modules that can be embedded in each other. Then $M \cong M'$.

Proof. Since $M'$ is injective, we may assume that $M = M' \oplus X$ and that there exists a monomorphism $f : M \rightarrow M'$. Note first that if $x_0 + f(x_1) + \ldots + f(x_n) = 0$, where $x_i \in X$, then all $x_i = 0$. In fact, $x_0 \in \text{Im} f \subseteq M'$ implies $x_0 = 0$, and so $x_1 + f(x_2) + \ldots + f^{n-1}(x_n) = 0$, since $f$ is a monomorphism. By induction, we see that all $x_i = 0$. Therefore, we have $M'' = X \oplus f(X) \oplus f^2(X) \oplus \ldots \subseteq M$. Let $E = E(f(M'')) \subseteq M'$, and write $M' = E \oplus Y$. Since $M'' = X \oplus f(M'')$, $E(M'') = E(X \oplus f(M'')) = E(X) \oplus E(f(M'')) = X \oplus E$. On the other hand $E(M'') = E(f(M'')) = E$, so $X \oplus E \cong E$. From this, we deduce that $M = X \oplus M' = X \oplus E \oplus Y \cong E \oplus Y = M'$.

Proposition 5 Let $M, M'$ be $R$-modules that can be embedded in each other. Then $E(M)$ is a $S$-quasi-Dedekind $R$-module if and only if $E(M')$ is a $S$-quasi-Dedekind $R$-module, where $E(M)$ is an injective hull of $M$.

Proof. Fix an embedding $f : M \rightarrow M'$. Then $f(M) \subseteq M' \subseteq E(M')$, so $E(M')$ contains a copy of $E(f(M)) \cong E(M)$. By symmetric $E(M)$ also contains a copy of $E(M')$. Since $E(M), E(M')$ are injective, then by lemma 4, $E(M) \cong E(M')$. Hence by Proposition 5, the result is obtained.

Definition 6 Let $S$ be submodule of an $R$-module $M$. A submodule $C$ of $M$ is said to be a complement to $S$ in $M$ if $C$ is maximal with respect to the property that $C \cap S = \{0\}$.

Remark 3

1. By Zorn’s lemma, any submodule $S$ of an $R$-module has a complement; in fact, any submodule $C_0$ with $C_0 \cap S = \{0\}$ can be enlarged into a complement to $S$ in $M$.

2. If $C$ is a complement to $S$, then we have $C \oplus S \leq_e M$.

Proposition 7 Let $M$ be any $R$-module and let $g : M \rightarrow E(M)$. If $g$ is an injective endomorphism of $M$, then the following assertions are verified.

1. $E(M)$ is a $S$-quasi-Dedekind $R$-module.

2. If $N \leq_e M$, then $E(N)$ is a $S$-quasi-Dedekind $R$-module.

3. For any $N \leq M$, there exists $K \leq M$ such that $E(N) \oplus E(K)$ is a $S$-quasi-Dedekind $R$-module.

4. If $M$ and $M'$ are $R$-modules that can be embedded in each other for any injective $R$-module $M'$, then $M'$ is a $S$-quasi-Dedekind $R$-module.

Proof.

1. Let $f \in \text{End}_{\text{fg}}(E(M))$ such that $f \neq 0$ and $g = f_{|_M}$. Since $f_{|_M}$ injective, we have $M \cap \text{Ker} f = \{0\}$. Therefore $M \leq_e E(M)$ implies that $\text{Ker} f = \{0\}$, so $\text{Ker} f \ll_e E(M)$. Thus $E(M)$ is a $S$-quasi-Dedekind $R$-module.

2. $E(M)$, if $N \leq_e M$, then $E(N)$ and $E(M)$ is injective, so the inclusion $N \rightarrow E(M)$ is an injective enveloppe of $M$. Thus $E(M) = E(N)$, and so the result is obtained.

3. By Zorn’s lemma, there exists a maximal submodule $K$ of $M$ with respect $N \cap K = \{0\}$. Then $N \cap K \leq_e M$ and so by the proof of (2) $E(M) \cong E(N \oplus K) \cong E(N) \oplus E(K)$. Thus $E(N) \oplus E(K)$ is a $S$-quasi-Dedekind $R$-module.

4. Since $M'$ is an $R$-module injective, then $E(M') = M'$. By the proposition 6, $E(M) \cong E(M') = M'$, so $M'$ is a $S$-quasi-Dedekind $R$-module.

Lemma 5 (Lam, T. Y. (1999), P. 213)

Let $R$ be a quasi-Frobenius ring. Then any right $R$-module $M$ can be embedded in a free module.

Proposition 8 Let $R$ be a quasi-Frobenius ring and let $M$ be a finitely generated $R$-module. Then $E(M)$ is a $S$-quasi-Dedekind $R$-module if and only $E(M)$ is a small $S$-quasi-Dedekind $R$-module.

Proof. By lemma 5, we have $M \subseteq F$ for some free module $F$. Since $M$ is finitely generated, we have $M \subseteq F_0 \subseteq F$ for some free module $F_0$ of finite rank. Thus by (Lam, T. Y. (1999), P.412), $F_0$ is an injective $R$-module, so can be found inside $F_0$. Thus $E(M)$ is a direct summand of $F_0$ and so is also finitely generated. Thus by proposition 2, the result is obtained.
Lemma 6 (Lam, T. Y. (1999), P.412-413)
For any ring, the following are equivalent:
1-R is quasi-Frobenius.
2-A right R-module is projective if and only if it is injective.

Proposition 9 Let R be a quasi-Frobenius ring and let M a projective R-module. If M is a S-quasi-Dedekind R-module, then E(M) is a S-quasi-Dedekind R-module.

Proof. By lemma 6, M is injective and so E(M) is a S-quasi-Dedekind R-module.

Proposition 10 Let M be a quasi-injective R-module, T = End_R(M) and m ∈ M. If mR is a simple R-module, then T.m is a S-quasi-Dedekind S-module.

Proof. Let t ∈ T such that tm ≠ 0. Consider the R-epimorphism: \( \phi : mR \rightarrow tmR \) given by left multiplication by t. Since mR is simple, \( \phi \) is an isomorphism. Let \( \psi = \phi^{-1} \) and extend \( \psi \) to an endomorphism \( g \in T \). Now \( gtm = \psi(tm) = \phi^{-1}(tm) = m \), so \( m \in T.m \). Thus \( T.m \) is a simple T-module. We have \( \forall f \in End_T(T.m), f ≠ 0, Ker f ≪_T T.m \). Hence \( T.m \) is a S-quasi-Dedekind S-module.

Proposition 11 Let M be a S-quasi-Dedekind and quasi-injective R-module, let N ≤ M such that for all U ≤ N, U ≪ M implies U ≪ N. Then N is a S-quasi-Dedekind R-module.

Proof. Let \( f \in End_R(N) \), \( f ≠ 0 \). To prove that \( Ker f ≪_S N \). Since M is a quasi-injectif R-module, there exists \( g \in End_R(M) \) such that \( g \circ i = i \circ f \), where \( i \) is the inclusion mapping. Then \( g(N) = f(N) ≠ 0 \). So \( Ker f ≪_S M \). But \( Ker f ≪_S Kerg \), hence \( Ker f ≪_S M \). On the other hand \( Ker f ≤ N \), so by hypothesis \( Ker f ≪_S N \). Thus N is a S-quasi-Dedekind R-module.

Proposition 12 Every direct summand of a finitely generated S-quasi-Dedekind module is a S-quasi-Dedekind module.

Proof. Let \( M = N ⊕ K \) such that M is a S-quasi-Dedekind R-module. Let \( f : K → K, f ≠ 0 \). We have \( h = i \circ f \circ p ∈ End_R(M), h ≠ 0 \), where \( p \) is the natural projection and \( i \) is the inclusion mapping. Hence \( Ker h ≪ M \), so \( Ker f ≪ M \) since M is finitely generated. But \( Ker f ≪_K Kerh \), so \( Ker f ≪ M \). On the other hand \( Ker f ≤ K \) implies \( Ker f ≪ K \) by (Ali, A. H. (2010.), Prop. 1.12). Thus K is a S-quasi-Dedekind R-module.

Remark 4 If M is a S-quasi-Dedekind R-module, \( N ≤ M \). Then it is not necessary that \( M/N \) is a S-quasi-Dedekind R-module; for example the \( \mathbb{Z} - \) module \( M = \mathbb{Z} \) is S-quasi-Dedekind. Let \( N = 12\mathbb{Z} ≤ \mathbb{Z} \), then \( M/N = \mathbb{Z}/12\mathbb{Z} \) is not a S-quasi-Dedekind \( \mathbb{Z} \)-module.

Remark 5 The homomorphic image of an S-quasi-Dedekind module is not necessary S-quasi-Dedekind; for example \( \mathbb{Z} \) as \( \mathbb{Z} \)- module S-quasi-Dedekind. But \( \pi : \mathbb{Z} → \mathbb{Z}/12\mathbb{Z}, \) where \( \pi \) is the natural projection. However \( \mathbb{Z}/12\mathbb{Z} \) as \( \mathbb{Z} \)-module is not S-quasi-Dedekind.

Lemma 7 (Abdullah & all, (2011), Prop. 1.18)
Let N and K are submodules of an R-module M such that \( N ≤ K \) and \( N ≤ L \) for each primary submodule L of M, if \( N ≪_S M \), then \( K/N ≪_S M/N \) if and only if \( K ≪_S M \).

Proposition 13 Let M be a S-quasi-Dedekind R-module such that \( M/U \) is projectif for all \( U ≪_S M \). Let \( N ≪_S M \) such that \( N ≤ L \), for each primary submodule L of M. Then \( M/N \) is a S-quasi-Dedekind R-module for all \( N ≤ M \).

Proof. Let \( K/N ≪_S M/N \), so by lemma 7, \( K ≪_S M \).

Suppose that \( Hom(M/N)/(K/N), M/N) ≠ [0] \), but \( Hom(M/N)/(K/N), M/N) ≅ Hom(M/K, M/N) \), so there exists \( f : M/K → M/N, f ≠ 0 \). Since M/K is projective, then there exists \( g : M/K → M \) such that \( \pi \circ g = f \), where \( \pi \) is the canonical projection.

Hence \( \pi \circ g(M/K) = f(M/K) ≠ 0 \), so \( g ≠ 0 \). But \( g ∈ Hom(M/K, M) \), \( K ≪_S M \). Thus \( Hom(M/K, M) ≠ [0] \), \( K ≪_S M \); that is M is not S-quasi-Dedekind, which is a contradiction. Thus \( M/N \) is a S-quasi-Dedekind R-module.

Proposition 14 Let M be a quasi-projective R-module and let \( N ≪_S M \) such that \( g^{-1}(N) ≪_S M, \) for each \( g \in \text{End}_R(M). \)

If \( N ≤ L, \) for each primary submodule L of M, then \( M/N \) is a S-quasi-Dedekind R-module.

Proof. Let \( f ∈ \text{End}_R(M/N) \) such that \( f ≠ 0 \). Since M is quasi-projective, there exists \( g ∈ \text{End}_R(M) \) such that \( \pi \circ g = f \circ \pi \) where \( \pi \) is the canonical projection.

Let \( Ker f = L/N = \{ x + N : f(x + N) = N \} = \{ x + N : \pi \circ g(x) = N \} = \{ x + N : g(x) + N = N \} = \{ x + N : g(x) ∈ N \} = \{ x + N : x ∈ g^{-1}(N) \} = g^{-1}(N)/N. \) Thus \( Ker f = g^{-1}(N)/N. \)

But \( g^{-1}(N) ≪_S M, \) so by lemma 7, \( g^{-1}(N)/N ≪_S M/N. \) That is \( Ker f ≪_S M/N. \)
3. S-quasi-Dedekind Modules and Other Related Modules

In this section, we study the relations between S-quasi-Dedekind modules and other related modules.

**Definition 7**

1. An R-module $M$ is called indecomposable if $M \neq \{0\}$ and it is not a direct sum of two nonzero submodules.

2. A left principal indecomposable module of a ring $R$ is a left submodule of $R$, that is a direct summand of $R$ and is an indecomposable module.

**Proposition 15** Let $R$ be an Artinian ring which is quasi-Frobenius. Then every principal indecomposable $R$-module has a $S$-quasi-Dedekind socle.

**Proof.** For any primitive idempotent $e$, consider the principal indecomposable $R$-module $eR$. Since $eR$ is projective, then by lemma 6, it is also injective. Let $M$ be simple submodule of $eR$. Clearly $eR = E(M)$, so $M \leq eR$. In particular $\text{Soc}(eR) = M$ is $S$-quasi-Dedekind.

**Proposition 16** Let $R$ be quasi-Frobenius ring and two principal indecomposable $R$-modules $M, M'$ such that $M \cong M'$. Then there exists two $S$-quasi-Dedekind $R$-modules $M_1, M_2$ such that $M_1 \cong M_2$.

**Proof.** Let $M_1 = \text{Soc}(M)$ and $M_2 = \text{Soc}(M')$. Then by (Lam, T. Y. (1999), P.423), $M_1, M_2$ are simple $R$-modules. If $M \cong M'$, then $M_1 \cong M_2$ and $M_1, M_2$ are $S$-quasi-Dedekind $R$-modules.

**Proposition 17** Let $M$ be an $R$-module such that every nonzero factor module of $M$ is indecomposable. Then $M$ is a $S$-quasi-Dedekind module $R$-module.

**Proof.** Let $L$ be a proper submodule of $M$. Suppose that $M = L + K$, where $K \leq M$. We have $M/L \cap K \cong M/L \oplus M/K$. But $M/L \cap K$ is indecomposable so $M/L \neq \{0\}$ and $M/K = \{0\}$. Hence $M = K$. Thus $L \ll M$ and so $M$ is a $S$-quasi-Dedekind module $R$-module.

**Proposition 18** Let $M$ be an indecomposable $R$-module with finite length such that $\forall f \in \text{End}_R(M), f$ is not nilpotent. Then $M$ is a $S$-quasi-Dedekind module $R$-module.

**Proof.** Let $f \in \text{End}_R(M)$ such that $f \neq 0$. Since $f$ is not nilpotent, then by (Anderson, F.W.& all (1973), P.138) $\text{Ker} f = \{0\}$. Thus $M$ is a $S$-quasi-Dedekind module $R$-module.

**Definition 8** An $R$-module $M$ is said to have the direct summand intersection property (briefly SIP) if the intersection of any two direct summands is again a direct summand.

**Lemma 8** Let $M$ be an indecomposable $R$-module and $N$ be any $R$-module. If $M \oplus N$ has the SIP, then every nonzero $R$-homomorphism from $M$ to $N$ is a monomorphism.

**Proof.** Assume $\text{Hom}(M, N) \neq \{0\}$ and let $f$ be a nonzero $R$-homomorphism from $M$ to $N$. Since $M \oplus N$ has the SIP, then $\text{Ker} f$ is a direct summand of $M$. But $M$ is indecomposable so $\text{Ker} f = \{0\}$ and $f$ is a monomorphism.

**Proposition 19** Let $M$ be an indecomposable $R$-module and let $N$ be any $R$-module such that $\text{Hom}(M, N) \neq \{0\}$. If $M \oplus N$ has the SIP, then $M$ is $S$-quasi-Dedekind. In particular, if $M \oplus M$ has the SIP, then $M$ is $S$-quasi-Dedekind.

**Proof.** By lemma 8, there is a monomorphism $f$ from $M$ to $N$. Let $g \in \text{End}_R(M)$ such that $g \neq 0$. We claim that $\text{Ker} g \ll M$. Assume that $\text{Ker} g \ll M$, then $\text{Ker} g = \{0\}$. Since $f$ is a monomorphism, then $\text{Ker} f \circ g = \text{Ker} f \neq \{0\}$. This is a contradiction. Thus $\text{Ker} f \ll M$. Hence $M$ is $S$-quasi-Dedekind.

**Definition 9** Let $M$ be an $R$-module.

1. $M$ is called local if it has exactly one maximal submodule that contains all proper submodules of $M$.

2. $M$ is called hollow if $M \neq \{0\}$ and every proper submodule of $M$ is small in $M$.

**Remark 6**

1. Every proper submodule of a local module $M$ is semi-small in $M$.

2. Every Hollow $R$-module is $S$-quasi-Dedekind. But the converse is not true in general; for example $\mathbb{Z}$ as $\mathbb{Z}$-module is $S$-quasi-Dedekind, but it is not Hollow.
**Remark 7**

According to (Naoum, A.G. (1990), theorem 3.2), there exists 0 \(\notin R\) such that \(\ann_R(M) = \{0\}\). Let \(M\) be a finitely generated faithful multiplication \(R\)-module and let \(N = IM\) be a proper submodule of \(M\). Then \(I_{\ann_R(M)}\) is a hollow \(R\)-module, so by (Naoum, A.G. (1990), theorem 2.2) Let \(M\) be a finitely generated faithful multiplication \(R\)-module and let \(N = IM\) be a proper submodule of \(M\). Then \(I \ll_s R\) if and only if \(N \ll_s M\).

**Lemma 9**

Let \(M\) be a faithful multiplication \(R\)-module, then \(\ann_M(r) = \ann_R(r).M\).

**Proposition 20**

Every local module \(M\) is a \(S\)-quasi-Dedekind module.

**Proposition 21**

Let \(M\) be a hollow \(R\)-module. Then \(M/N\) is a \(S\)-quasi-Dedekind \(R\)-module, for all proper submodule \(N\) of \(M\).

**Proposition 22**

Let \(M\) be an \(R\)-module such that for some proper submodule \(N\) of \(M\), \(M/N\) is Hollow and \(N \ll M\). Then \(M\) is a \(S\)-quasi-Dedekind \(R\)-module.

**Definition 10**

An \(R\)-module \(M\) is called faithful if \(\ann_R(M) = \{0\}\).

**Definition 11**

An \(R\)-module \(M\) is said to have finite uniform dimension if it does not contain a direct sum of an infinite number of non-zero submodules.

**Definition 12**

An \(R\)-module \(M\) is scalar if, for all \(f \in \End_R(M)\) then there exists \(r \in R\) such that \(f(x) = rx\) for all \(x \in M\).

**Remark 7**

Let \(M\) be an \(R\)-module. Then

1. If \(M\) has finite uniform dimension, then \(M\) is weakly co-hopfian.
2. If \(M\) is scalar, then by (Mohamed-Ali, E. A. (2006), lemma 6.2), \(\End_R(M) \cong R/\ann_R(M)\).

**Proposition 23**

Let \(M\) be a semisimple \(R\)-module with finite uniform dimension. Then \(M\) is a finite direct sum of \(S\)-quasi-Dedekind \(R\)-modules.

Proof. Since \(M\) is a semisimple \(R\)-module with finite uniform dimension, \(M\) is finitely generated. Thus \(M\) is a finite direct sum of simples \(R\)-modules, and so \(M\) is a finite direct sum of \(S\)-quasi-Dedekind \(R\)-modules.

**Lemma 9**

Let \(M\) be a faithful multiplication \(R\)-module, then \(\ann_M(r) = \ann_R(r).M\).

**Proof.** We have \(\ann_M(r) \subseteq M\). Since \(M\) is multiplication \(R\)-module, so \(\ann_M(r) = (\ann_M(r) : M)M\). We claim that \(\ann_M(r) = (\ann_M(r) : M)\). To prove our assertion: Let \(a \in \ann_R(r)\), then \(ar = 0\) and \(arM = \{0\}\); that is \(aM \subseteq \ann_M(r)\), so \(a \in (\ann_M(r) : M)\). Thus \(\ann_M(r) \subseteq (\ann_M(r) : M)\). Now, if \(a \in (\ann_M(r) : M)\), then \(aM \subseteq \ann_M(r)\), so \(raM = \{0\}\), this implies \(ra \in \ann_R(M) = \{0\}\). Thus \(a \in (\ann_R(r) : M)\) and hence \(\ann_M(r) = \ann_R(r).M\).

**Lemma 10**

(Abdullah & all, (2011), theorem 2.2) Let \(M\) be a finitely generated faithful multiplication \(R\)-module and let \(N = IM\) be a proper submodule of \(M\). Then \(I \ll_s R\) if and only if \(N \ll_s M\).

**Lemma 11**

Let \(M\) be a local \(R\)-module. Then \(M\) is a hollow and cyclic \(R\)-module.

**Proposition 24**

Let \(M\) be a semisimple \(R\)-module with finite uniform dimension. Then \(M\) is a finite direct sum of simples \(R\)-modules, and so \(M\) is a finite direct sum of \(S\)-quasi-Dedekind \(R\)-modules.

**Theorem 2**

Let \(M\) be a finitely generated faithful multiplication \(R\)-module. Then \(M\) is a \(S\)-quasi-Dedekind \(R\)-module if and only if \(M\) is a \(S\)-quasi-Dedekind \(R\)-module.

**Proof.** \(\Rightarrow\) Let \(f : R \rightarrow R\) be a nonzero \(R\)-homomorphism. Then for each \(a \in R\), \(f(a) = ar\) for some \(0 \neq r \in R\). Define \(g : M \rightarrow M\) by \(g(m) = rm\) for all \(m \in M\). It follows that \(g \neq 0\), since if \(g = 0\), then \(rM = \{0\}\) and so \(r \in \ann_R(M) = \{0\}\), which is a contradiction.

Since \(M\) is \(S\)-quasi-Dedekind, then \(\ker f \ll_s M\). But \(\ker f = \{m \in M : g(m) = rm = 0\} = \ann_M(r)M\) and by lemma 9 \(\ann_M(r) = \ann_R(r)M\), hence by lemma 10 \(\ann_M(r) = \ann_R(r)\) and so \(\ann_R(r) \ll_s R\).

However it is easy to see that \(\ker f = \ann_R(r)\). Hence \(\ker f \ll_s R\) and hence \(R\) is a \(S\)-quasi-Dedekind \(R\)-module.

\(\Leftarrow\) Let \(f : M \rightarrow M\) such that \(f \neq 0\). To prove \(\ker f \ll_s M\). Since \(M\) is a finitely generated multiplication \(R\)-module so by (Naoum, A.G. (1990), theorem 3.2), there exists \(0 \neq r \in R\) such that \(f(m) = rm\) for \(m \in M\) and \(\ker f = \{m \in M : f(m) = rm = 0\} = \ann_M(r)\).

Now define \(g : R \rightarrow R\) by \(g(a) = ra\) for all \(a \in R\), hence \(g \neq 0\), since if \(g = 0\), then \(rR = \{0\}\) and so \(r = 0\) which is a contradiction. Thus \(\ker f \ll_s R\), since \(R\) is \(S\)-quasi-Dedekind. But \(\ker f = \{a \in R : g(a) = ra = 0\} = \ann_R(r)\) and so
ann_R(r) \ll_r R. On the other hand by lemma 9 \text{ann}_M(r) = \text{ann}_R(r) \cdot M, so by lemma 10 \text{ann}_M(r) \ll_r M. Thus \text{Ker} f \ll_r M and M is a S-quasi-Dedekind R-module.

**Corollary 5** Let M be an R-module. If M is a local faithful R-module. Then R is a S-quasi-Dedekind R-module.

*Proof.* Suppose that M is a local R-module, then by lemma 11, M is a hollow and cyclic R-module. But M is a faithful R-module, thus by theorem 2, R is a S-quasi-Dedekind.

**Corollary 6** Let R be an Artinian principal ideal ring and let M be an R-module module with finite uniform dimension. If M is a faithful multiplication R-module, then R is a S-quasi-Dedekind R-module.

*Proof.* Since M is an R-module module with finite uniform dimension, then M is a weakly co-Hopfian R-module, so M is a finitely generated R-module. But M is a faithful multiplication R-module, thus by theorem 2, R is a S-quasi-Dedekind.

**Definition 13** An R-module M is called monoform if for each nonzero submodule N of M and for each f \in Hom(N, M), f \neq 0 implies \text{Ker} f = \{0\}.

**Proposition 24** Every monoform R-module is a S-quasi-Dedekind R-module.

**Remark 8** The converse of proposition 24 is not true in general; for example \mathbb{Z}/4\mathbb{Z} as \mathbb{Z}-module is S-quasi-Dedekind, but it is not monoform.

**Definition 14** An R-module M is called anti-Hopfian if M is not simple and every nonzero factor module of M is isomorphic to M.

**Definition 15** Let M be an R-module. M is called generalized Hopfian (gH, for short), if for each f \in End_R(M), f surjective implies \text{Ker} f \ll_r M.

**Proposition 25** Let M be an anti-Hopfian R-module. If M is a gH R-module, then M is a S-quasi-Dedekind R-module.

*Proof.* Let f \in End_R(M) such that f \neq 0. Since M is anti-Hopfian R-module, so by (Hirano & all (1986)), f is surjective. But M is gH R-module implies \text{Ker} f \ll_r M. Thus \text{Ker} f \ll_r M and so M is a S-quasi-Dedekind R-module.

**Proposition 26** Let M be an anti-Hopfian quasi-projective R-module. If M is Dedekind finite module, then M is a S-quasi-Dedekind R-module.

*Proof.* Since M is Dedekind finite quasi-projective, then by (Ghorbani & all (2002) P.327), M is a gH R-module. Moreover M is an anti-Hopfian and gH R-module, thus by proposition 25, M is a S-quasi-Dedekind R-module.

**Definition 16** An R-module M is called special generalized Hopfian (sgH, for short), if whenever f is a left regular element of End_R(M); that is if f is not a left zero divisor, then \text{Ker} f \ll_r M.

**Theorem 3** Let M be a scalar R-module such that \text{ann}_R(M) is prime. If M is a sgH R-module, then M is a S-quasi-Dedekind R-module.

*Proof.* Since M is a scalar R-module, thus by remark 7 End_R(M) \cong R/\text{ann}_R(M). Thus End_R(M) is an integral domain. Hence for each f \in End_R(M), f \neq 0, f is nonzero divisor and since M is sgH, so we get \text{Ker} f \ll_r M. Thus \text{Ker} f \ll_r M and so M is a S-quasi-Dedekind R-module.

**Proposition 27** Let M be an anti-Hopfian R-module. If M is a sgH R-module, then M is a S-quasi-Dedekind R-module.

*Proof.* Since M is anti-Hopfian, then by ((Hirano & all (1986)), Theorem 14 P.129) End_R(M) is an integral domain, so that for each f \in End_R(M), f \neq 0 implies f is nonzero divisor. Hence \text{Ker} f \ll_r M, since M is sgH. Thus \text{Ker} f \ll_r M and so M is a S-quasi-Dedekind R-module.

**Definition 18** Let M be an R-module, put \mathbb{Z}(M) = \{m \in M : \text{ann}_M(m) \leq_e R\}. M is called nonsingular if \mathbb{Z}(M) = \{0\}, and singular if \mathbb{Z}(M) = M.

**Lemma 12** Let f : M \longrightarrow M’ of homomorphism of right R-modules. If N \leq_e M’, f^{-1}(N) \leq_e M.

*Proof.* Consider any e \in M’f^{-1}(N). Then f(e) \neq 0, so there exists r \in R such that f(e)r \in N\setminus\{0\}. Then clearly er \in f^{-1}(N)\setminus\{0\}. Thus f^{-1}(N) \leq_e M.

**Remark 9** Given N \leq_e M’ and any element y \in M’, let f : R_R \longrightarrow M’ be defined by f(r) = yr. Then the lemma 12 implies f^{-1}(N) = y^{-1}N = \{r \in R : yr \in N\} \leq_e R_R.

**Proposition 28** Let M be a nonsingular uniform R-module. Then M is a S-quasi-Dedekind R-module.

*Proof.* Let f \in End_R(M) such that f \neq 0. Then \text{Ker} f = \{0\}. If \text{Ker} f \neq \{0\}, then \text{Ker} f \leq_e M. For any y \in M, y^{-1}\text{Ker} f \leq_e R_R by remark 9. Now f(y)y^{-1}\text{Ker} f \leq f(y) \{y^{-1}\text{Ker} f\} \leq f(\text{Ker} f) = \{0\}, so f(y) \in \mathbb{Z}(M) = \{0\}, that is f = 0, a contradiction. Then
Corollary 7 Let \( M \) be a nonsingular uniform \( R \)-module. If \( M \) is injective, then \( E(M) \) is a S-quasi-Dedekind \( R \)-module.

Proof. Since \( M \) is injective, then \( E(M) = M \). By proposition 28, \( E(M) \) is a S-quasi-Dedekind \( R \)-module.

Remark 10 If \( M \) is a nonsingular module, then by (Lam, T. Y. (1999), P.277) \( \overline{E}(M) = E(M) \), where \( \overline{E}(M) \) is the rational hull of \( M \).

Corollary 8 Let \( M \) be a nonsingular uniform \( R \)-module. If \( M \) is injective, then \( \overline{E}(M) \) is a S-quasi-Dedekind \( R \)-module.

Proof. We have \( \overline{E}(M) = E(M) = M \). Thus \( \overline{E}(M) \) is a S-quasi-Dedekind \( R \)-module.

Proposition 33 Let \( M \) be a simple \( R \)-module. Then \( \text{End}_R(M) \) is a S-quasi-Dedekind \( R \)-module.

Proof. First show that \( \text{ann}_R(m) \subseteq \text{ann}_R(f(m)) \) for any \( m \in M \).

1. Follows from the fact \( \text{ann}_R(m) \subseteq \text{ann}_R(f(m)) \) for any \( m \in M \).

2. Follows directly from the definition.

Proposition 31 Let \( M \) be an \( R \)-module and set \( 0 \neq N \subseteq M \) such that \( N \) and \( M/N \) are both nonsingular. If \( M \) is uniform, then \( M \) and \( N \) are both S-quasi-Dedekind \( R \)-modules.

Proof. First show that \( N \) is a S-quasi-Dedekind \( R \)-module. By lemma 13, we have \( \mathbb{Z}(M) \cap N = \mathbb{Z}(N) = \{0\} \). Therefore the projection map from \( M \) to \( M/N \) induces an injective homomorphism \( \pi : \mathbb{Z}(M) \rightarrow M/N \). Thus by lemma 13, we have \( \pi(\mathbb{Z}(M)) \subseteq \mathbb{Z}(M/N) = \{0\} \), so \( \pi = 0 \). This implies that \( \mathbb{Z}(M) = \{0\} \). Then \( M \) is a nonsingular uniform \( R \)-module, and so by proposition 28, \( M \) is a S-quasi-Dedekind \( R \)-module. It is clear that \( N \) is a nonsingular uniform \( R \)-module. Then \( N \) is a S-quasi-Dedekind \( R \)-module.

4. Some Properties of the Endomorphism Ring of S-quasi-Dedekind Module

Proposition 33 Let \( M \) be a simple \( R \)-module. Then \( \text{End}_R(M) \) is a S-quasi-Dedekind \( R \)-module.

Proof. By Schur’s lemma \( \text{End}_R(M) \) is a division ring. Thus \( \text{End}_R(M) \) is a S-quasi-Dedekind \( R \)-module.

Proposition 34 Let \( M \) be an anti-Hopfian \( R \)-module. Then \( \text{End}_R(M) \) is a S-quasi-Dedekind \( R \)-module. Proof. Since \( M \) is anti-Hopfian, then by (Hirano, Y. & all (1986), Theorem 14, P.129), \( \text{End}_R(M) \) is an integral domain. Thus \( \text{End}_R(M) \) is a S-quasi-Dedekind \( R \)-module.
Proposition 35  Let $M$ be a nonsingular uniform $R$-module. Then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring.

Proof.  Let $f \neq 0 \neq g \in \text{End}_R(M)$, then by the proposition 28, $f,g$ are injectives and so $fg \neq 0$. Thus $\text{End}_R(M)$ is an integral domain. Hence $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring.

Proposition 36  Let $M$ be a scalar $R$-module with $\text{ann}_R(M)$ is a prime ideal of $R$, then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring.

Proof.  Since $M$ is a scalar $R$-module, then by remark 7, $\text{End}_R(M) \cong R/\text{ann}_R(M)$, so $\text{End}_R(M)$ is an integral domain. Hence $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring.

Corollary 9  If $M$ is scalar and prime $R$-module, then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring.

Proposition 37  Let $M$ be a scalar faithful $R$-module. $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring if and only if $R$ is a $S$-quasi-Dedekind ring.

Proof.  Suppose that $M$ is scalar $R$-module, so by remark 7, $\text{End}_R(M) \cong R/\text{ann}_R(M)$. But $M$ is faithful, thus $R/\text{ann}_R(M)$ is a $S$-quasi-Dedekind ring. Hence we have on the result.

Proposition 38  Let $R$ be an Artinian principal ideal ring and let $M$ be a weakly co-Hopfian multiplication faithful $R$-module. Then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring if and only if $R$ is a $S$-quasi-Dedekind ring.

Proof.  Suppose that $M$ is a weakly co-Hopfian $R$-module, so $M$ is a finitely generated $R$-module. Thus by (Naoum, A.G. (1990), theorem 3.2), $M$ is a $S$-quasi-Dedekind ring; that is $M$ is scalar faithful $R$-module. Thus by proposition 37, the result is obtained.

Proposition 39  Let $R$ be an Artinian principal ideal ring and let $M$ be a co-Hopfian multiplication faithful $R$-module. Then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring if and only if $R$ is a $S$-quasi-Dedekind ring.

Proof.  Suppose that $M$ is co-Hopfian $R$-module, so $M$ is a finitely generated $R$-module. Thus $M$ is scalar $R$-module; that is $M$ is scalar faithful $R$-module. Thus by proposition 37, the result is obtained.

Proposition 40  Let $R$ be an Artinian principal ideal ring and let $M$ be a Dedekind finite multiplication faithful $R$-module. Then $\text{End}_R(M)$ is a $S$-quasi-Dedekind ring if and only if $R$ is a $S$-quasi-Dedekind ring.

Proof.  Suppose that $M$ is a Dedekind finite $R$-module, so $M$ is a finitely generated $R$-module. Thus $M$ is scalar $R$-module; that is $M$ is scalar faithful $R$-module. Thus by proposition 37, the result is obtained.

Definition 18  Let $M$ be an $R$-module. $M$ is said quasi-prime if $\text{ann}_R(N)$ is a prime ideal of $R$.

Proposition 41  Let $M$ be a quasi-injective scalar and quasi-prime $R$-module. Then $\text{End}_R(N)$ is a $S$-quasi-Dedekind ring for all $0 \neq N \leq M$.

Proof.  Assume that $0 \neq N \leq M$. Since $M$ is a quasi-injective scalar $R$-module, then by (Shibab, B.N. (2004), Prop. 1.1.16), $N$ is a scalar $R$-module. Thus by remark 7, $\text{End}_R(N) \cong R/\text{ann}_R(N)$. But $M$ is a quasi-prime $R$-module, so $\text{ann}_R(N)$ is a prime ideal of $R$; that is $\text{End}_R(N)$ is an integral domain. Hence $\text{End}_R(N)$ is a $S$-quasi-Dedekind ring.

Corollary 10  Let $M$ be an injective scalar and quasi-prime $R$-module. Then $\text{End}_R(N)$ is a $S$-quasi-Dedekind ring for all $0 \neq N \leq M$.

Corollary 11  Let $M$ be a quasi-injective scalar $R$-module and let $0 \neq N \leq M$ be a faithful $R$-module. Then $\text{End}_R(N)$ is a $S$-quasi-Dedekind ring for all $0 \neq N \leq M$.

Proof.  It follows by (Shibab, B.N. (2004), Prop. 1.1.16) and proposition 37.

References


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