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On the Interval Controllability of Fractional Systems with Control Delay

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Abstract

In this paper we discuss the interval controllability of fractional systems with control delay. We give the concept and some results of interval controllability. As a main result we give the necessary conditions and sufficient conditions for the interval controllability of fractional systems with control delay.

Keywords: interval controllability, fractional control systems, control delay

1. Introduction

The investigation of fractional differential systems have wide application in sciences and engineering such as physics, chemistry, mechanics. So in recent years, the research of fractional differential systems is extensive around the world, and there has been a significant development (Anatoly A. K., Hari M. S. & Juan, J. T., 2006; Podlubny, I., 1999; Lakshmikantham, V., 2008; Yong, Z., 2008; Shantanu, D., 2008; Jiang, W., 2010; Jiang, W., 2011). We also find that the phenomena of time delay exist in many systems, such as economics, biological, physiological and spaceflight systems, specially, time delay frequently exists in control. Up to now a lot of fabulous accomplishments have been given by many scholars (Hale, J., 1992; Zuxiu Z., 1994; Sebakhy, O. & Bayoumi, M. M., 1971; Wansheng T. & Guangquan, L., 1995; Jiang, W. & WenZhong, S., 2001; Jiang, W., 2006; Jiang, W., 2012; Hai, Z., Jinde, C. & Wei, J., 2013; Xian-Feng, Z., Jiang, W. & Liang-Gen, H., 2013; Xian-Feng, Z., Jiang, W. & Liang-Gen, H., 2013; Song Liu & et al., 2016; Xian-Feng, Z., Fuli, Y. & Wei, J., 2015). The fractional systems with control delay have two factors (fractional order differential, and delay) synchronously. So the results of this paper must be useful.

Definition 1 Riemann-Liouville's fractional integral of order $\alpha > 0$ for a function $f : R^+ \to R$ is defined as

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta) d\theta.$$

Here $\Gamma(\cdot)$ is a Gamma-Function.

Definition 2 Caputo's fractional derivative of order α $(0 \le m \le \alpha < m+1)$ for a function $f : R^+ \to R$ is defined as

$${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha+1)} \int_0^t \frac{f^{(m+1)}(\theta)}{(t-\theta)^{\alpha-m}} d\theta.$$

Here $\Gamma(\cdot)$ is a Gamma-Function.

In this paper we study the fractional control systems with control delay

$$\begin{cases} {}^cD^{\alpha} x(t) = Ax(t) + Bu(t) + Cu(t-h), & t \ge 0, \\ x(0) = x_0, & \\ u(t) = \psi(t), & -h \le t \le 0, \end{cases} \tag{1}$$

where $x(t) \in R^n$is state vector; $u(t) \in R^m$ is control vector, $A \in R^{n\times n}$, B, C$\in R^{n\times m}$ are any matrices;$h > 0$ is time control delay; and $\psi(t)$ the initial control function. $0 < \alpha \le 1$, ${}^cD^{\alpha} x(t)$ denotes α order-Caputo fractional derivative of $x(t)$.

In the investigation for controllability of fractional systems with control delay (1), we find that controllability of such systems is closely related to time interval. So we must to pay much attention to the study of the controllability for time interval.

For the time interval $I = [a, b]$, we give the concept of interval controllability. There $a \in R$ and $b \in R, b > a$ or $b = +\infty$.

Definition 3 The system (1) is said to be controllable in interval I if one can at any time $\bar{t} \in I$ reach any state from any admissible initial state and initial control.

For systems (1), the time delay h play a very important role in the controllability. In this paper we discuss the interval controllability of fractional systems with control delay (1). In section 2, we give some preliminaries. In section 3, we give some results about controllability the interval $[0, h]$ for fractional systems with control delay (1). In section 4, the necessary and sufficient conditions for controllability the interval (h, ∞) for fractional systems with control delay (1).

2. Preliminaries

In this section, we give some preliminaries.

Lemma 1 The general solution of system

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

can be written as:

$$x(t) = \Phi_{\alpha,1}(A, t)x(0) + \int_0^t \Phi_{\alpha,\alpha}(A, t - \tau)f(\tau)d\tau. \tag{3}$$

Here

$$\Phi_{\alpha,\beta}(A, t) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k+\beta-1}}{\Gamma(\alpha k + \beta)}.$$

Proof From Anatoly A.Kilbas, Hari M.Srivastava, Juan J.Trujillo(2006) and I.Podlubny (1999) (page 106), we know that the Laplace Transform of Caputo fractional derivative of function $f(t)$ is

$$L({}^cD^\alpha f(t))(s) = s^\alpha L(f(t))(s) - s^{\alpha-1} f(0).$$

We consider the Mittag-Leffler function in two parameters (Anatoly A.Kilbas, Hari M.Srivastava, Juan J.Trujillo(2006); I.Podlubny (1999))

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0).$$

We have

$$\begin{aligned} L(\Phi_{\alpha,\beta}(A, t))(s) &= \int_0^\infty e^{-st} t^{\beta-1} E_{\alpha,\beta}(At^\alpha)dt \\ &= \sum_{k=0}^{\infty} \tfrac{1}{\Gamma(\alpha k+\beta)} \int_0^\infty e^{-st} t^{\beta-1}(At^\alpha)^k dt \\ &= \sum_{k=0}^{\infty} \tfrac{A^k}{\Gamma(\alpha k+\beta)} \int_0^\infty e^{-st} t^{\alpha k+\beta-1} dt = \sum_{k=0}^{\infty} \tfrac{A^k}{\Gamma(\alpha k+\beta)} \int_0^\infty e^{-h} \tfrac{h^{\alpha k+\beta-1}}{s^{\alpha k+\beta-1}} \tfrac{1}{s} dh \\ &= \sum_{k=0}^{\infty} \tfrac{A^k s^{-(\alpha k+\beta)}}{\Gamma(\alpha k+\beta)} \int_0^\infty e^{-h} h^{\alpha k+\beta-1} dh = \sum_{k=0}^{\infty} \tfrac{A^k s^{-(\alpha k+\beta)}}{\Gamma(\alpha k+\beta)} \Gamma(\alpha k + \beta) \\ &= \sum_{k=0}^{\infty} (As^{-\alpha})^k s^{-\beta} = (I - As^{-\alpha})^{-1} s^{-\beta} \\ &= (s^\alpha I - A)^{-1} s^{\alpha-\beta}. \end{aligned}$$

Take Laplace Transform for systems (2), we have

$$s^\alpha L(x(t))(s) - s^{\alpha-1} x(0) = AL(x(t))(s) + L(f(t))(s).$$

That is

$$\begin{aligned} L(x(t))(s) &= (s^\alpha I - A)^{-1} s^{\alpha-1} x(0) + (s^\alpha I - A)^{-1} L(f(t))(s) \\ &= L(\Phi_{\alpha,1}(A, t))(s)x(0) + L(\Phi_{\alpha,\alpha}(A, t))(s)L(f(t))(s) \\ &= L(\Phi_{\alpha,1}(A, t))(s)x(0) + L(\Phi_{\alpha,\alpha}(A, t) * f(t))(s). \end{aligned}$$

There

$$\Phi_{\alpha,\alpha}(A, t) * f(t) = \int_0^t \Phi_{\alpha,\alpha}(A, t - \tau)f(\tau)d\tau$$

is the convolution of $\Phi_{\alpha,\alpha}(A, t)$ and $f(t)$ (Anatoly A.Kilbas, Hari M.Srivastava, Juan J.Trujillo(2006); I.Podlubny (1999)). Then we have

$$L(x(t))(s) = L(\Phi_{\alpha,1}(A, t))(s)x(0) + L(\textstyle\int_0^t \Phi_{\alpha,\alpha}(A, t - \tau)f(\tau)d\tau)(s).$$

That is

$$x(t) = \Phi_{\alpha,1}(A,t))x_0 + \int_0^t \Phi_{\alpha,\alpha}(A,t-\tau)f(\tau)d\tau.$$

From Lemma 1, we have

Theorem 1 The general solution of system (1) can be written as that when $t \geq h$

$$\begin{aligned} x(t) &= \Phi_{\alpha,1}(A,t)x_0 \\ &\quad + \int_0^{t-h}(\Phi_{\alpha,\alpha}(A,t-\tau)B + \Phi_{\alpha,\alpha}(A,t-\tau-h)C)u(\tau)d\tau \\ &\quad + \int_{t-h}^{t}\Phi_{\alpha,\alpha}(A,t-\tau)Bu(\tau)d\tau \\ &\quad + \int_{-h}^{0}\Phi_{\alpha,\alpha}(A,t-\tau-h)C\psi(\tau)d\tau. \end{aligned} \tag{4a}$$

when $0 \leq t \leq h$

$$\begin{aligned} x(t) &= \Phi_{\alpha,1}(A,t)x_0 \\ &\quad + \int_0^{t}(\Phi_{\alpha,\alpha}(A,t-\tau)Bu(\tau)d\tau \\ &\quad + \int_{-h}^{t-h}\Phi_{\alpha,\alpha}(A,t-\tau-h)C\psi(\tau)d\tau. \end{aligned} \tag{4b}$$

Let

$$\langle A|B,C\rangle = \alpha + A\alpha + A^2\alpha + \cdots + A^{n-1}\alpha + \underline{+}A\underline{+}A^2\underline{+}\cdots + A^{n-1}\underline{,}$$

where n is the order of A and $\alpha = ImageB, \underline{=}ImageC$. Then the space $\langle A|B,C\rangle$ is spanned by the columns of matrix

$$[B, AB, A^2B, \cdots, A^{n-1}B, C, AC, A^2C, \cdots, A^{n-1}C].$$

Lemma 2 For Beta function

$$B(z,w) = \int_0^1 s^{z-1}(1-s)^{w-1}ds, (Re(z) > 0, Re(w) > 0),$$

we have

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

The proof of this Lemma can be seen in I.Podlubny (1999)(page 7).

Lemma 3

$$^{c}D^{\alpha}\Phi_{\alpha,1}(A,t) = A\Phi_{\alpha,1}(A,t). \tag{5}$$

Proof From Lemma 2, we have

$$\begin{aligned} ^{c}D^{\alpha}\Phi_{\alpha,1}(A,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Phi'_{\alpha,1}(A,\theta)}{(t-\theta)^{\alpha}}d\theta \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{\infty}\int_0^t \frac{A^k\theta^{\alpha k-1}(\alpha k)}{(t-\theta)^{\alpha}\Gamma(\alpha k+1)}d\theta \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\int_0^t \frac{A^{k+1}\theta^{\alpha(k+1)-1}(\alpha(k+1))}{(t-\theta)^{\alpha}\Gamma(\alpha(k+1)+1)}d\theta \\ &= \sum_{k=0}^{\infty}\int_0^1 \frac{A^{k+1}s^{\alpha(k+1)-1}t^{\alpha(k+1)-1}}{\Gamma(1-\alpha)t^{\alpha}(1-s)^{\alpha}\Gamma(\alpha(k+1))}tds \\ &= \sum_{k=0}^{\infty}\frac{A^{k+1}t^{\alpha k}}{\Gamma(1-\alpha)\Gamma(\alpha(k+1))}\int_0^1 s^{\alpha(k+1)-1}(1-s)^{(-\alpha+1)-1}ds \\ &= \sum_{k=0}^{\infty}\frac{A^{k+1}t^{\alpha k}B(\alpha(k+1),1-\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha(k+1))} \\ &= A\sum_{k=0}^{\infty}\frac{A^k t^{\alpha k}}{\Gamma(\alpha k+1)} \\ &= A\Phi_{\alpha,1}(A,t). \end{aligned}$$

For any $z \in R^n$ and $t > h$, define $W(t) : R^n \to R^n$ by

$$\begin{aligned} W(t)z &= \int_0^{t-h}[(\Phi_{\alpha,\alpha}(A,t-\tau)B + \Phi_{\alpha,\alpha}(A,t-\tau-h)C) \\ &\quad \times(\Phi_{\alpha,\alpha}(A,t-\tau)B + \Phi_{\alpha,\alpha}(A,t-\tau-h)C)^T]zd\tau \\ &\quad + \int_{t-h}^{t}[\Phi_{\alpha,\alpha}(A,t-\tau)BB^T(\Phi_{\alpha,\alpha}(A,t-\tau))^T]zd\tau. \end{aligned}$$

3. The Controllability of (1) in Interval $[0, h]$

In this section, we discuss the controllability of (1) in interval $[0, h]$.

Let $I = [0, \infty), I_1 = [0, h], I_2 = (h, \infty)$.

From Definition 3, we know that system (1) is controllable in interval I_1 if and only if for any state $\bar{x} \in R^n$, any time $\bar{t} \in I_1$, and any admissible initial state x_0 and initial control $u(t) = \psi(t)(-h \le t \le 0)$, there is a control $\bar{u}(t)$ such that the solution $x(t)$ of system (1) from x_0 can reach $\bar{x}$ at time $\bar{t}$, that is $x(\bar{t}) = \bar{x}$.

If $0 \le \bar{t} \le h$, systems (1) become

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + Bu(t) + C\psi(t-h), & 0 \le t \le \bar{t}, \\ x(0) = x_0, & \\ u(t) = \psi(t), & -h \le t \le 0. \end{cases} \tag{6}$$

From Theorem 1, the general solution of system (1) can be written as that when $0 \le t \le h$,

$$\begin{aligned} x(t) &= \Phi_{\alpha,1}(A, t)x(0) \\ &\quad + \int_0^t \Phi_{\alpha,\alpha}(A, t-\tau)Bu(\tau)d\tau \\ &\quad + \int_{-h}^{t-h} \Phi_{\alpha,\alpha}(A, t-\tau-h)C\psi(\tau)d\tau. \end{aligned} \tag{7}$$

The control term $Cu(t-h)$ can't play a part in the controllability of (1).

Consider fractional control systems without control delay

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + Bu(t), & t \ge 0, \\ x(0) = x_0, & \\ u(0) = \psi(0). & \end{cases} \tag{8}$$

Theorem 2 System (1) is controllable in interval I_1 if and only if system (8) is controllable in interval I_1.

Proof From Section 2, we have that the solution of (8) can be written as

$$\begin{aligned} x(t) &= \Phi_{\alpha,1}(A, t)x(0) \\ &\quad + \int_0^t \Phi_{\alpha,\alpha}(A, t-\tau)Bu(\tau)d\tau. \end{aligned} \tag{9}$$

We firstly prove sufficiency. We say system (8) is controllable in interval I_1.

For any state $\bar{x} \in R^n$, any time $\bar{t} \in I_1$, and any admissible initial state x_0 and initial control $u(t) = \psi(t)(-h \le t \le 0)$, let $\hat{x} = \bar{x} - \int_{-h}^{\bar{t}-h} \Phi_{\alpha,\alpha}(A, \bar{t}-\tau-h)C\psi(\tau)d\tau$. For system (8) is controllable in interval I_1, from (9), there is control $u^*(t)$ with $u^*(0) = \psi(0)$ such that

$$\begin{aligned} \hat{x} &= \Phi_{\alpha,1}(A, \bar{t})x(0) \\ &\quad + \int_0^{\bar{t}} \Phi_{\alpha,\alpha}(A, \bar{t}-\tau)Bu^*(\tau)d\tau. \end{aligned}$$

That is

$$\begin{aligned} \bar{x} &= \Phi_{\alpha,1}(A, \bar{t})x(0) \\ &\quad + \int_0^{\bar{t}} \Phi_{\alpha,\alpha}(A, \bar{t}-\tau)Bu^*(\tau)d\tau + \int_{-h}^{\bar{t}-h} \Phi_{\alpha,\alpha}(A, \bar{t}-\tau-h)C\psi(\tau)d\tau. \end{aligned}$$

Let

$$\bar{u}(t) = \begin{cases} u^*(t) & 0 \le t \le \bar{t}, \\ \psi(t) & -h \le t \le 0. \end{cases}$$

By the definition, we know that system (1) is controllable in interval I_1.

The proof of the necessity of Theorem 2 can be easily gotten from the proof of the sufficiency in inverse way.

From Shantanu Das(2008), we have that system (8) is controllable in interval I_1 if and only if the rank of matrix $[B, AB, A^2B, \cdots, A^{n-1}B]$ is n. So from Theorem 2, we have

Theorem 3 System (1) is controllable in interval I_1 if and only if

$$rank[B, AB, A^2B, \cdots, A^{n-1}B] = n.$$

From Shantanu Das(2008), we know that system (8) is controllable in interval $I_1 = [0, h]$ if and only if it is controllable in interval $I = [0, \infty)$. But for system (1), this conclusion is not true. We will discuss the controllability of (1) in interval (h, ∞) and give an example to illustrate that in next section.

4. The Controllability of (1) in Interval (h, ∞)

In this section we discuss the controllability of (1) in interval (h, ∞). Firstly we give a conclusion about the relationship of $ImW(t_1)$ and $\langle A|B, C\rangle$.

Theorem 4 If $\Phi_{\alpha,1}(A, h)$ is nonsingular, and system (1) is controllable at any time $t_1 > h$, then $ImW(t_1) = \langle A|B, C\rangle$.

Proof To show $ImW(t_1) = \langle A|B, C\rangle$ is equivalence to showing that

$$KerW(t_1) = \bigcap_{i=0}^{n-1} KerB^T(A^T)^i \bigcap_{j=0}^{n-1} KerC^T(A^T)^j. \tag{10}$$

If $\bar{x} \in KerW(t_1)$ and $\bar{x} \neq 0$,then

$$\begin{aligned} 0 &= \bar{x}^T W(t_1)\bar{x} \\ &= \int_0^{t_1-h} \|(\Phi_{\alpha,\alpha}(A, t_1-\tau)B + \Phi_{\alpha,\alpha}(A, t_1-\tau-h)C)^T \bar{x}\|^2 d\tau \\ &\quad + \int_{t_1-h}^{t_1} \|B^T \Phi_{\alpha,\alpha}(A, t_1-\tau)^T \bar{x}\|^2 d\tau, \end{aligned}$$

that is

$$\begin{cases} 0 = (\Phi_{\alpha,\alpha}(A, t_1-\tau)B + \Phi_{\alpha,\alpha}(A, t_1-\tau-h)C)^T \bar{x}, & 0 \le \tau \le t_1 - h, \\ 0 = B^T \Phi_{\alpha,\alpha}(A, t_1-\tau)^T \bar{x}, & t_1 - h < \tau \le t_1, \end{cases} \tag{11}$$

For (1) is controllable, from theorem 1, for any t_1 and any $\hat{x}$, for $x(t_1) = \int_{-h}^{0} \Phi_{\alpha,\alpha}(A, t_1-\tau-h)C)\psi(\tau)d\tau$ there is $\bar{u}$ such that

$$\begin{cases} 0 &= \Phi_{\alpha,1}(A, t_1)\hat{x} \\ &\quad + \int_0^{t_1-h}(\Phi_{\alpha,\alpha}(A, t_1-\tau)B + \Phi_{\alpha,\alpha}(A, t_1-\tau-h)C)\bar{u}(\tau)d\tau \\ &\quad + \int_{t_1-h}^{t_1} \Phi_{\alpha,\alpha}(A, t_1-\tau)B\bar{u}(\tau)d\tau. \end{cases}$$

Left time $\bar{x}^T$, from (11), we have

$$0 = \bar{x}^T \Phi_{\alpha,1}(A, t_1)\hat{x}. \tag{12}$$

Let $t_1 \to h$ and $\hat{x} = \Phi_{\alpha,1}^{-1}(A, h)BB^T \bar{x}$, we have

$$0 = \bar{x}^T BB^T \bar{x}.$$

That is

$$B^T \bar{x} = 0.$$

Take Caputo's fractional derivative (12) for t_1 , from Lemma 3 we have

$$0 = \bar{x}^T A\Phi_{\alpha,1}(A, t_1)\hat{x}.$$

Let $t_1 \to h$ and $\hat{x} = \Phi_{\alpha,1}^{-1}(A, h)BB^T A^T \bar{x}$, we have

$$0 = \bar{x}^T ABB^T A^T \bar{x}.$$

That is

$$B^T A^T \bar{x} = 0.$$

Repeatedly take k times Caputo's fractional derivative for the (12), and let $t_1 \to h$,$\hat{x} = \Phi_{\alpha,1}^{-1}(A, h)BB^T(A^T)^k \bar{x}$ we have

$$0 = \bar{x}^T A^k BB^T(A^T)^k \bar{x},$$

That is

$$B^T(A^T)^k \bar{x} = 0, \qquad for \quad k = 0, 1, 2, \cdots, n-1. \tag{13}$$

For (12), let $t_1 \to h$ and $\hat{x} = \Phi_{\alpha,1}^{-1}(A, h)CC^T \bar{x}$, we have

$$0 = \bar{x}^T CC^T \bar{x}.$$

That is

$$C^T \bar{x} = 0.$$

Take Caputo's fractional derivative (12) for t_1 , from Lemma 3 we have

$$0 = \bar{x}^T \Phi_{\alpha,1}(A, t_1) A\hat{x}.$$

Let $t_1 \to h$ and $\hat{x} = \Phi^{-1}_{\alpha,1}(A, h)CC^T A^T \bar{x}$, we have

$$0 = \bar{x}^T ACC^T A^T \bar{x}.$$

That is

$$C^T A^T \bar{x} = 0.$$

Repeatedly take k times Caputo's fractional derivative for the (12), and let $t_1 \to h$,$\hat{x} = \Phi^{-1}_{\alpha,1}(A, h)CC^T (A^T)^k \bar{x}$ we have

$$0 = \bar{x}^T A^k CC^T (A^T)^k \bar{x},$$

That is

$$C^T (A^T)^k \bar{x} = 0, \qquad for \quad k = 0, 1, 2, \cdots, n-1. \tag{14}$$

From (13) and (14) we have $\bar{x} \in \bigcap_{i=0}^{n-1} KerB^T (A^T)^i \bigcap_{j=0}^{n-1} KerC^T (A^T)^j$. That is

$$KerW(t_1) \subset \bigcap_{i=0}^{n-1} KerB^T (A^T)^i \bigcap_{j=0}^{n-1} KerC^T (A^T)^j. \tag{15}$$

Conversely, let $\bar{x} \in \bigcap_{i=0}^{n-1} KerB^T (A^T)^i \bigcap_{j=0}^{n-1} KerC^T (A^T)^j$, (13) and (14) is true.

From Cayley Hamilton Theorem (Zheng Zuxiu (1994)), we have

$$\Phi_{\alpha,\alpha}(A, t_1) = \sum_{k=0}^{\infty} \frac{A^k {t_1}^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \sum_{k=0}^{n-1} \gamma_k(t_1) A^k \tag{16}$$

Then from (13),(14) and (16), for $t_1 - h < \tau \leq t_1$,

$$B^T \Phi_{\alpha,\alpha}(A, t_1 - \tau)^T A^T \bar{x} = \sum_{k=0}^{n-1} \gamma_k(t_1 - \tau) B^T (A^T)^k \bar{x} = 0.$$

and for $0 \leq \tau \leq t_1 - h$,

$$\begin{aligned}&(\Phi_{\alpha,\alpha}(A, t_1 - \tau)B + \Phi_{\alpha,\alpha}(A, t_1 - \tau - h)C)^T \bar{x} \\ &= \sum_{k=0}^{n-1} \gamma_k(t_1 - \tau) B^T (A^T)^k \bar{x} + \sum_{k=0}^{n-1} \gamma_k(t_1 - \tau - h) C^T (A^T)^k \bar{x} = 0\end{aligned}$$

therefore $\bar{x} \in KerW(t_1)$. That is

$$KerW(t_1) \supset \bigcap_{i=0}^{n-1} KerB^T (A^T)^i \bigcap_{j=0}^{n-1} KerC^T (A^T)^j. \tag{17}$$

From (15) and (17), we have (10) is true and the proof of Theorem 4 is over.

Theorem 5 If $\Phi_{\alpha,1}(A, h)$ is nonsingular, the system (1) is controllable in (h, ∞), that is for any time $t_1 > h$ system (1) is controllable at t_1 if and only if $W(t_1)$ is nonsingular.

Proof If for any time $t_1 \geq h$, $W(t_1)$ is nonsingular, for any $\bar{x} \in R^n$, let

$$z = \bar{x} - \Phi_{\alpha,1}(A, t_1)x(0) - \int_{-h}^{0} \Phi_{\alpha,\alpha}(A, t_1 - \tau - h)C\psi(\tau)d\tau. \tag{18}$$

From (4a), let

$$\bar{u}(\tau) = \begin{cases} (B^T \Phi_{\alpha,\alpha}(A, t_1 - \tau)^T \\ \qquad +C^T \Phi_{\alpha,\alpha}(A, t_1 - \tau - h)^T) W(t_1)^{-1} z, & 0 \leq \tau \leq t_1 - h, \\ B^T \Phi_{\alpha,\alpha}(A, t_1 - \tau)^T W(t_1)^{-1} z, & t_1 - h < \tau \leq t_1, \end{cases}$$

we have

$$\begin{aligned} x(t_1) &= \Phi_{\alpha,1}(A,t_1)x(0) + \int_{-h}^{0} \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C\psi(\tau)d\tau \\ &\quad + \int_0^{t_1-h}(\Phi_{\alpha,\alpha}(A,t_1-\tau)B + \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C)\bar{u}(\tau)d\tau \\ &\quad + \int_{t_1-h}^{t_1} \Phi_{\alpha,\alpha}(A,t_1-\tau)B\bar{u}(\tau)d\tau \\ &= \Phi_{\alpha,1}(A,t_1)x(0) + \int_{-h}^{0} \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C\psi(\tau)d\tau \\ &\quad + \int_0^{t_1-h}(\Phi_{\alpha,\alpha}(A,t_1-\tau)B + \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C) \\ &\qquad (B^T\Phi_{\alpha,\alpha}(A,t_1-\tau)^T + C^T\Phi_{\alpha,\alpha}(A,t_1-\tau-h)^T)W(t_1)^{-1}zd\tau \\ &\quad + \int_{t_1-h}^{t_1} \Phi_{\alpha,\alpha}(A,t_1-\tau)BB^T\Phi_{\alpha,\alpha}(A,t_1-\tau)^T W(t_1)^{-1}zd\tau. \end{aligned}$$

From the definition of $W(t_1)$ and (18), we have

$$\begin{aligned} x(t_1) &= \Phi_{\alpha,1}(A,t_1)x(0) + \int_{-h}^{0} \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C\psi(\tau)d\tau \\ &\quad + W(t_1)W(t_1)^{-1}z \\ &= \Phi_{\alpha,1}(A,t_1)x(0) + \int_{-h}^{0} \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C\psi(\tau)d\tau + z \\ &= \bar{x}. \end{aligned}$$

From Definition 3, we have that the system (1) is controllable in (h,∞).

Now we prove that if system (1) is controllable in (h,∞) then for any time $t_1 > h$ such that $W(t_1)$ is nonsingular. If this is not true, then there is $\bar{x} \neq 0$ such that

$$\begin{aligned} 0 &= \bar{x}^T W(t_1)\bar{x} \\ &= \int_0^{t_1-h} \|(\Phi_{\alpha,\alpha}(A,t_1-\tau)B + \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C)^T\bar{x}\|^2 d\tau \\ &\quad + \int_{t_1-h}^{t_1} \|B^T\Phi_{\alpha,\alpha}(A,t_1-\tau)^T\bar{x}\|^2 d\tau, \end{aligned}$$

that is

$$\begin{cases} 0 = (\Phi_{\alpha,\alpha}(A,t_1-\tau)B + \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C)^T\bar{x}, & 0 \le \tau \le t_1-h, \\ 0 = B^T\Phi_{\alpha,\alpha}(A,t_1-\tau)^T\bar{x}, & t_1-h < \tau \le t_1, \end{cases} \tag{19}$$

For (1) is controllable, from theorem 1, for $x(t_1) = \int_{-h}^{0} \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C)\psi(\tau)d\tau$ there is $\bar{u}$ such that

$$\begin{cases} 0 &= \Phi_{\alpha,1}(A,t_1)\bar{x} \\ &\quad + \int_0^{t_1-h}(\Phi_{\alpha,\alpha}(A,t_1-\tau)B + \Phi_{\alpha,\alpha}(A,t_1-\tau-h)C)\bar{u}(\tau)d\tau \\ &\quad + \int_{t_1-h}^{t_1} \Phi_{\alpha,\alpha}(A,t_1-\tau)B\bar{u}(\tau)d\tau. \end{cases}$$

Time $\bar{x}^T$, from (19) we have

$$0 = \bar{x}^T\Phi_{\alpha,1}(A,t_1)\bar{x}.$$

We have that $\Phi_{\alpha,1}(A,t_1)$ is singular for any time $t_1 \ge h$. That is a contradiction with $\Phi_{\alpha,1}(A,h)$ is nonsingular. The proof of Theorem 5 is over.

From the proof of Theorem 5 we can see that $\Phi_{\alpha,1}(A,h)$ is nonsingular can be changed as that if there is $\bar{t} \in [h,\infty)$, $\Phi_{\alpha,1}(A,\bar{t})$ is nonsingular. Theorem 5 can be generalized as

Theorem 6 For system (1) we have

1^0. If for any time $t_1 \in (h,\infty)$, $W(t_1)$ is nonsingular, then system (1) is controllable in (h,∞).

2^0. If there is $\bar{t} \in (h,\infty)$, $\Phi_{\alpha,1}(A,\bar{t})$ is nonsingular and for any time $t_1 \in (h,\infty)$ system (1) is controllable at t_1, then $W(t_1)$ is nonsingular.

Now we discuss the algebraic property for the controllability of system (1).

Theorem 7 If $\Phi_{\alpha,1}(A,h)$ is nonsingular and the system (1) is controllable in (h,∞), then

$$rank[B, AB, A^2B, \cdots, A^{n-1}B, C, AC, A^2C, \cdots, A^{n-1}C] = n.$$

Proof From Theorem 4 , for any time $t_1 \ge h$, $ImW(t_1) = \langle A|B,C\rangle$.

From Theorem 5, $ImW(t_1) = R^n$. So, we have

$$rank[B, AB, A^2B, \cdots, A^{n-1}B, C, AC, A^2C, \cdots, A^{n-1}C] = n.$$

Remark If $\Phi_{\alpha,1}(A, h)$ is nonsingular, and

$$rank[B, AB, A^2B, \cdots, A^{n-1}B, C, AC, A^2C, \cdots, A^{n-1}C] \neq n,$$

from Theorem 7 we can judge that the system (1) is uncontrollable in (h, ∞).

Now we give two examples to illustrate the use of Theorem 6 and Theorem 7.

Example 1 Consider the fractional control systems with control delay

$$\begin{cases} {}^cD^{\frac{3}{4}}x_1(t) = x_2(t) + u_1(t), & t \geq 0, \\ {}^cD^{\frac{3}{4}}x_2(t) = u_2(t-h), & t \geq 0, \\ x(0) = x_0, & \\ u(t) = \psi(t), & -h \leq t \leq 0. \end{cases} \tag{20}$$

Compare (20) with (1), we have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha = \frac{3}{4}.$$

Then

$$\begin{aligned} \Phi_{\alpha,\beta}(A, t) &= \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} = \frac{t^{\beta-1}}{\Gamma(\beta)} I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &= \begin{pmatrix} \frac{t^{\beta-1}}{\Gamma(\beta)} & \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ 0 & \frac{t^{\beta-1}}{\Gamma(\beta)} \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} \Phi_{\frac{3}{4},1}(A, t) &= \begin{pmatrix} 1 & \frac{t^{\frac{3}{4}}}{\Gamma(\frac{3}{4}+1)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{4t^{\frac{3}{4}}}{3\Gamma(\frac{3}{4})} \\ 0 & 1 \end{pmatrix}, \\ \Phi_{\frac{3}{4},\frac{3}{4}}(A, t) &= \begin{pmatrix} \frac{t^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{2t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \\ 0 & \frac{t^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} \end{pmatrix} = \begin{pmatrix} \frac{t^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} \\ 0 & \frac{t^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} \end{pmatrix}. \end{aligned}$$

$$\Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau)B + \Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau-h)C = \begin{pmatrix} \frac{(t-\tau)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} & \frac{2(t-\tau-h)^{\frac{1}{2}}}{\sqrt{\pi}} \\ 0 & \frac{(t-\tau-h)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} \end{pmatrix},$$

$$\begin{aligned} &[\Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau)B + \Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau-h)C][\Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau)B + \Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau-h)C]^T \\ &= \begin{pmatrix} \frac{(t-\tau)^{-\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2} + \frac{4(t-\tau-h)}{\pi} & \frac{2(t-\tau-h)^{\frac{1}{4}}}{\sqrt{\pi}\Gamma(\frac{3}{4})} \\ \frac{2(t-\tau-h)^{\frac{1}{4}}}{\sqrt{\pi}\Gamma(\frac{3}{4})} & \frac{(t-\tau-h)^{-\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} &[\Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau)B][\Phi_{\frac{3}{4},\frac{3}{4}}(A, t-\tau)B]^T = \begin{pmatrix} \frac{(t-\tau)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{(t-\tau)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} & 0 \\ 0 & 0 \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{(t-\tau)^{-\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

$$
\begin{aligned}
W(t) &= \int_0^{t-h}[(\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau)B+\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau-h)C)\\
&\quad\times(\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau)B+\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau-h)C)^T]d\tau\\
&\quad+\int_{t-h}^{t}[\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau)BB^T(\Phi_{\frac{3}{4},\frac{3}{4}}(A,t-\tau))^T]d\tau\\
&= \begin{pmatrix} \frac{2(t^{\frac{1}{2}}-h^{\frac{1}{2}})}{(\Gamma(\frac{3}{4}))^2}+\frac{2}{\pi}(t-h)^2 & \frac{8(t-h)^{\frac{5}{4}}}{5\sqrt{\pi}\Gamma(\frac{3}{4})}\\ \frac{8(t-h)^{\frac{5}{4}}}{5\sqrt{\pi}\Gamma(\frac{3}{4})} & \frac{2(t-h)^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2}\end{pmatrix}+\begin{pmatrix}\frac{2h^{\frac{1}{2}})}{(\Gamma(\frac{3}{4})^2} & 0\\ 0 & 0\end{pmatrix}\\
&= \begin{pmatrix} \frac{2t^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2}+\frac{2}{\pi}(t-h)^2 & \frac{8(t-h)^{\frac{5}{4}}}{5\sqrt{\pi}\Gamma(\frac{3}{4})}\\ \frac{8(t-h)^{\frac{5}{4}}}{5\sqrt{\pi}\Gamma(\frac{3}{4})} & \frac{2(t-h)^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2}\end{pmatrix}.
\end{aligned}
$$

For when $t > h$,

$$
\begin{aligned}
|W(t)| &= \left(\frac{2t^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2}+\frac{2}{\pi}(t-h)^2\right)\frac{2(t-h)^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^2}-\left(\frac{8(t-h)^{\frac{5}{4}}}{5\sqrt{\pi}\Gamma(\frac{3}{4})}\right)^2\\
&= \frac{4t^{\frac{1}{2}}(t-h)^{\frac{1}{2}}}{(\Gamma(\frac{3}{4}))^4}+\frac{36(t-h)^{\frac{5}{2}}}{25\pi(\Gamma(\frac{3}{4}))^2}>0,
\end{aligned}
$$

we know that $W(t)$ is nonsingular. From Theorem 6, we know that system (20) is controllable in (h,∞).

Example 2 For the fractional control systems with control delay

$$
\begin{cases}
{}^cD^{\frac{1}{2}}x_1(t)=x_2(t)+u_1(t)+u_2(t-h), & t\ge 0,\\
{}^cD^{\frac{1}{2}}x_2(t)=0, & t\ge 0,\\
x(0)=x_0,\\
u(t)=\psi(t), & -h\le t\le 0.
\end{cases} \tag{21}
$$

Compare (20) with (1), we have

$$
A=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad B=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad C=\begin{pmatrix}0&1\\0&0\end{pmatrix}.
$$

Then

$$
\Phi_{\alpha,1}(A,h)=\begin{pmatrix}-1&\frac{h^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\\0&1\end{pmatrix},
$$

$$
[B,AB,C,AC]=\begin{pmatrix}1&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}.
$$

$|\Phi_{\alpha,1}(A,h)|=-1\neq 0$, and

$$
rank[B,AB,C,AC]=1\neq 2,
$$

from Theorem 7 we have that the system (21) is uncontrollable in (h,∞).

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