The First Hochschild Cohomology of Square Algebras With it’s Stability

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Received: April 18, 2017 Accepted: May 8, 2017 Online Published: July 15, 2017
doi:10.5539/jmr.v9n4p200 URL: https://doi.org/10.5539/jmr.v9n4p200

Abstract

In this paper, we study on a special case of generalized matrix algebra that we call it square algebra. According to that Hochschild cohomology play a significant role in Geometry for example in orbifolds, we study the first Hochschild cohomology of the square algebra the vanishing of its.

Keywords: First Hochschild cohomology, Hochschild cohomology, Square algebra

1. Introduction

Let $\mathcal{R}$ be a commutative ring (with unit), let $A$ and $B$ be $\mathcal{R}$-algebras and $M$ be a left $A$-module and right $B$-module. A triangular algebra $T$ over $\mathcal{R}$ is the following matrix

$$T = \begin{bmatrix} A & M \\ B & \end{bmatrix}.$$

Automorphisms, derivations, commuting mappings and Lie derivations on triangular algebras are studied by Cheung (Cheung, 2001) and (Cheung, 2003). Other useful and valuable literature concerning the structure of derivations and Lie derivations is (Ji & Qi, 2011). Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras which derivations of those considered in (Christensen, 1977), (Coelho, & Milies, 1993), (Donsig, Forrest & Marcoux, 1996).

A generalized matrix algebra is a generalization of triangular matrix algebra. In the triangular algebra $T$, the element lies in the second row and second column is zero. In generalized matrix algebra, we put a right $A$-module and left $B$-module $N$ in zero place. We denote the generalized matrix algebra by $\mathcal{G}$. Algebraic studying on derivations, generalized derivations and Lie derivations have been studied in (Du, & Wang, 2012), (Li & Wei, 2012), (Li, & Xiao, 2011).

Throughout this paper $\mathcal{R}$ is a commutative ring (with unit), $A$ and $B$ are $\mathcal{R}$-algebras with units $1_A$ and $1_B$, respectively, $M$ is an $\mathcal{R}$-bimodule, left $A$-module and right $B$-module ($A$, $B$-module) and $N$ is an $\mathcal{R}$-bimodule, right $A$-module and left $B$-module ($B$, $A$-module). Define bimodule homomorphisms $\Phi_{MN} : M \otimes_B N \rightarrow A$ and $\Phi_{NM} : N \otimes_A M \rightarrow B$ satisfying the following commutative diagrams:

$$\begin{align*}
\Phi_{MN} \otimes id_M : M \otimes_B N \otimes_A M & \rightarrow A \otimes_A M \\
A \otimes_A M & \rightarrow A \\
\Phi_{NM} \otimes id_M : M \otimes_B N \otimes_A M & \rightarrow A \otimes_A M \\
M & \rightarrow M
\end{align*}$$

$$\begin{align*}
id_M \otimes \Phi_{NM} : M \otimes_B N \otimes_A M & \rightarrow A \otimes_A M \\
A \otimes_A M & \rightarrow A \\
id_M \otimes \Phi_{NM} : M \otimes_B N \otimes_A M & \rightarrow A \otimes_A M \\
M & \rightarrow M
\end{align*}$$
Let $R$ be a commutative ring (with unit), let $D$ be the space of all $n$-linear (as a $R$-module) maps $f: A \times \cdots \times A \to M$ and $C^0(A, M) = M$. Consider the sequence

$$0 \to C^0(A, M) \xrightarrow{d^0} C^1(A, M) \xrightarrow{d^1} \cdots \xrightarrow{d^n} \tilde{C}(A, M)$$

in which

$$d^0(x(a)) = ax - xa$$

$$d^n(f(a_1, a_2, \cdots, a_{n+1})) = a_1 f(a_2, \cdots, a_{n+1}) + (-1)^{a+1} f(a_1, \cdots, a_n + 1) + \sum_{j=1}^{n} (-1)^j f(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1})$$

where $n \geq 1$, $x \in M$ and $a_1, \ldots, a_{n+1} \in A$. The above sequence is a complex for $A$ and $M$. The $n$-th cohomology group of $\tilde{C}(A, E)$ is said to be $n$-th Hochschild cohomology group and denoted by $H^n(A, M)$, for more details see (Brodmann & Sharp, 1998), (Rotman, 2009). A derivation is a linear map $D: A \to M$ such that $D(ab) = aD(b) + D(a)b$ ($a, b \in A$) and for $x \in M$, we define the map $D_x: A \to M$ by $D_x(a) = xa - ax$. The map $D_x$ is a derivation and such derivations called inner derivations. Let $\text{Der}(A, M)$ denote all derivations and $\text{Inn}(A, M)$ denote all inner derivations.

Thus, we have

$$H^1(A, M) = \frac{\text{Der}(A, M)}{\text{Inn}(A, M)}.$$

In this paper, we describe $H^1(S, S)$ and vanishing of $H^1(S, X)$, where $X$ is a two sided $S$-module (bimodule) is investigated.

2. Structure of $H^1(S, S)$

We begin with the following simple properties of derivations on $S$ as follows:

**Proposition 1** Let $D: S \to S$ be a derivation, then there are derivations $d_A: A \to A$, $d_B: B \to B$, $R$-linear maps $\tau: M \to M$ and $\sigma: N \to N$ and elements $m_D \in M$ and $n_D \in N$ such that

(i) $D \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_D \\ n_D & 0 \end{bmatrix} = -D \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$.

(ii) $D \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau(m) \\ 0 & 0 \end{bmatrix}$ and $D \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sigma(n) & 0 \end{bmatrix}$. 
Let $D$ be a derivation. By the following relations and simple calculation we obtain (i)-(vii):

$$D \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} d_A(a) & \tau(m) \\ \sigma(n) & d_B(b) \end{bmatrix}$$

Moreover, $D$ is a derivation on $S$.

**Proof.** Let $D$ be a derivation. By the following relations and simple calculation we obtain (i)-(vii):

$$D \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = D \begin{bmatrix} a & 0 \\ 0 & m \end{bmatrix}, \quad D \begin{bmatrix} 0 & mb \\ 0 & 0 \end{bmatrix} = D \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}.$$ 

Conversely, consider,

$$D \begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} = D \begin{bmatrix} a_1 a_2 & a_1 m_1 + m_1 b_1 \\ n_1 a_1 + b_1 n_1 & b_1 b_2 \end{bmatrix} = \begin{bmatrix} d_A(a_1 a_2) & d_A(a_1 m_1 + m_1 b_1) \\ \sigma(n_1 a_1 + b_1 n_1) & d_B(b_1 b_2) \end{bmatrix}.$$ 

Thus $D$ is a derivation on $S$.

Let $a_0 \in A$ and $b_0 \in B$, then Rosenblum $\mathcal{R}$-linear map $\tau_{M}^{a_0,b_0} : M \rightarrow M$ is defined by

$$\tau_{M}^{a_0,b_0} = a_0 \cdot -m - m \cdot b_0$$

for each $m \in M$.

Now, let $Z(A)$ be the center of $A$ and $Z(B)$ be the center of $B$, $x \in Z(A)$ and $y \in Z(B)$. Then the Rosenblum $\mathcal{R}$-linear map $\tau_{M}^{x,y}$ is called a central Rosenblum $\mathcal{R}$-linear map. We denote the set of all central Rosenblum $\mathcal{R}$-linear maps by $ZR_{A,B}(M)$. Also, we have

$$ZR_{A,B}(M) \subseteq \text{Hom}_{A,B}(M).$$
An $\mathcal{R}$-map $\tau_M : M \to M$ is called a generalized Rosenblum $\mathcal{R}$-linear map if there exist derivations $d_A$ and $d_B$ on $A$ and $B$, respectively, such that $\tau_M$ satisfies

$$
\tau(abm) = d_A(ab)m + a\tau(m)b + amd_B(b)
$$

for each $a \in A$, $b \in B$ and $m \in M$. Similarly, an $\mathcal{R}$-map $\tau_N : N \to N$ is called a generalized Rosenblum $\mathcal{R}$-linear map if there exist derivations $d_A$ and $d_B$ on $A$ and $B$, respectively, such that $\tau_N$ satisfies

$$
\tau(bna) = d_B(bna) + b\tau_N(n)a + bnd_A(a)
$$

for each $a \in A$, $b \in B$ and $m \in M$.

**Lemma 2** Let $\varphi \in \text{Hom}_{A,B}(M)$ and $\sigma \in \text{Hom}_{B,A}(N)$. Then the map $d_{\varphi,\sigma} : S \to S$ given by

$$
d_{\varphi,\sigma} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} \varphi(m) \\ \sigma(n) \end{bmatrix},
$$

is a derivation. Moreover, $d_{\varphi,\sigma}$ is an inner derivation if and only if $\varphi = \tau_M^{x,y}$ and $\sigma = \tau_N^{x,y}$, where $\tau_M^{x,y} \in ZR_{A,B}(M)$ and $\tau_N^{x,y} \in ZR_{B,A}(N)$.

**Proof.** The first statement follows immediately from assume that $\varphi = \tau_M^{x,y}$ and $\sigma = \tau_N^{x,y}$ where $x \in Z(A)$ and $y \in Z(B)$. Then

$$
d_{\varphi,\sigma} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} xa & xm \\ yn & yb \end{bmatrix} = \begin{bmatrix} xa - ax & xm - my \\ yn - nx & yb - by \end{bmatrix} = \begin{bmatrix} 0 & \varphi(m) \\ \sigma(n) & 0 \end{bmatrix}.
$$

Hence $d_{\varphi,\sigma}$ is inner. Conversely, assume that $d_{\varphi,\sigma}$ is inner. Then there exists $\begin{bmatrix} x & z \\ w & y \end{bmatrix} \in S$ such that $d_{\varphi,\sigma} = d_{\varphi,\sigma} \begin{bmatrix} x & z \\ w & y \end{bmatrix}$. Then

$$
d_{\varphi,\sigma} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} xa - ax & zm + zb - az - my \\ wa + yn - nx - bw & yb - by \end{bmatrix}.
$$

If $d_{\varphi,\sigma} \begin{bmatrix} a & m \\ n & b \end{bmatrix} = d_{\varphi,\sigma}$, then $xa - ax = 0$ for each $a \in A$ and $yb - by = 0$ for each $b \in B$. In particular, $x \in Z(A)$ and $y \in Z(B)$.

Moreover, we have

$$
\varphi(m) = zm + zb - az - my
$$

and

$$
\sigma(n) = wa + yn - nx - bw.
$$

Since $\varphi \in \text{Hom}_{A,B}(M)$ and $\sigma \in \text{Hom}_{B,A}(N)$, it follows that $zb - az = 0$ and $wa - bw = 0$. Hence $\varphi(m) = zm + my = \tau_M^{x,y}(m)$ and $\sigma(n) = yn - nx = \tau_N^{x,y}(n)$. In particular, $\varphi \in ZR_{A,B}(M)$ and $\sigma \in ZR_{B,A}(N)$.

We can now state the main result of this section for describing $H^1(S,S)$.

**Theorem 3** If $H^1(A,A) = 0 = H^1(B,B)$, then

$$
H^1(S,S) \cong \frac{\text{Hom}_{A,B}(M) \times \text{Hom}_{B,A}(N)}{ZR_{A,B}(M) \times ZR_{B,A}(N)}
$$

(2)
Proof. Define $\phi : \text{Hom}_{A,B}(M) \times \text{Hom}_{B,A}(N) \to H^1(S,S)$ by

$$\phi(\varphi, \sigma) = \tilde{d}_{\varphi, \sigma},$$

where $\tilde{d}_{\varphi, \sigma}$ represents the equivalence class of $d_{\varphi, \sigma}$ in $H^1(S,S)$. Clearly, $\phi$ is $\mathcal{R}$-linear.

We shall show that $\phi$ is surjective. Let $d : S \to S$ be a derivation. Then there are derivations $d_A$, $d_B$, and $\mathcal{R}$-linear maps $\tau : M \to M$, $\sigma : N \to N$ and elements $m_d \in M$, $n_d \in N$ that satisfy in the conditions (i)-(vii) of Proposition 1. Since $H^1(A,A) = H_1(B,B) = 0$, we can find $x \in A$ and $y \in B$ such that $d_A = d_x$ and $d_B = d_y$. Define $d_0 : S \to S$ by

$$d_0 \left[ \begin{array}{c} a \\ n \\ b \end{array} \right] = \left[ \begin{array}{c} d_x(a) \\ \tau_N^x(n) + (n_d a - b n_d) \\ d_y(b) \end{array} \right].$$

Then $d_0$ is an inner derivation on $S$ induced by $\left[ \begin{array}{c} x \\ -n_d \\ y \end{array} \right]$. Furthermore, if $d_1 = d - d_0$, then $d_1$ is a derivation and

$$d_1 \left[ \begin{array}{c} a \\ m \\ n \\ b \end{array} \right] = \left[ \begin{array}{c} d_x(a) \\ \tau(m) + (am_d - m_d b) \\ d_y(b) \\ \tau_N^y(m) + (n_d a - b n_d) \\ 0 \end{array} \right].$$

where $\tau = \tau_M^x$ and $\sigma_1 = \sigma - \tau_N^y$. It follows from Proposition 1, that $\tau_1 \in \text{Hom}_{A,B}(M)$ and $\sigma_1 \in \text{Hom}_{B,A}(N)$. Finally, $\tilde{d} = \tilde{d}_1 = \varphi(\tau_1, \sigma_1)$, and so $\varphi$ is surjective. This implies that

$$H^1(S,S) \cong \frac{\text{Hom}_{A,B}(M) \times \text{Hom}_{B,A}(N)}{\ker \varphi}.$$  \hspace{1cm} (3)

However, $(\varphi, \sigma) \in \ker \varphi$ if and only if $d_{\varphi, \sigma}$ is inner. By Lemma 2, $\ker \varphi = ZR_{A,B}(M) \times ZR_{B,A}(N)$. Thus, by this fact and (3), (2) holds.

**Corollary 4** Let $A$ and $B$ be a commutative ring. By hypothesis of the above Theorem, we have $H^1(S,S) \cong \text{Hom}_{A,B}(M) \times \text{Hom}_{B,A}(N)$.

### 3. Vanishing of the First Cohomology Group

Let $X$ be a unitary $S$-bimodule, denote $X_{AA} = 1_A X 1_A$, $X_{BB} = 1_B X 1_B$, $X_{AB} = 1_A X 1_B$, and $X_{BA} = 1_B X 1_A$. For example, when $X = S$, we have $X_{AA} = A$, $X_{BB} = B$, $X_{AB} = M$ and $X_{BA} = N$. In this section, the relations between the first cohomology of $S$ with coefficients in $X$ and those of $A$ and $B$ with coefficients in $X_{AA}$ and $X_{BB}$, respectively, whenever $X_{AB} = 0$, are investigated.

We started by illustrating the structure of derivations from a square algebra into its bimodules.

Let $\delta : S \to X$ be a derivation. Then $\delta : A \to X_{AA}$ defined by $\delta_A(a) = 1_A \delta \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right]$ and $\delta_B : B \to X_{BB}$ defined by $\delta_B(b) = 1_B \delta \left[ \begin{array}{c} 0 \\ 0 \\ b \end{array} \right]$ are derivations. Moreover, the $\mathcal{R}$-linear maps $\tau : M \to X_{AB}$, defined by $\tau(m) = 1_A \delta \left[ \begin{array}{c} 0 \\ 0 \\ m \end{array} \right]$ and $\sigma : N \to X_{BA}$ defined by $\sigma(n) = 1_B \delta \left[ \begin{array}{c} 0 \\ 0 \\ n \end{array} \right]$ satisfy

(i) $\tau(am) = \sigma(n m) + \delta_A(a)m$,
(ii) $\tau(mb) = \sigma(m b) + m \delta_B(b)$,
(iii) $\sigma(n a) = \sigma(n a) + n \delta_A(a)$,
(iv) $\sigma(b n) = b \sigma(n) + b \delta_B(b)$. $n$

Conversely, if $\delta_1$ and $\delta_2$ are derivation from $A$ and $B$ into $X_{AA}$ and $X_{BB}$, respectively, and $\tau : M \to X_{AB}$ and $\sigma : N \to X_{BA}$ are any $\mathcal{R}$-linear maps satisfying in (i), (ii), (iii) and (iv), then the map $D \left[ \begin{array}{c} a \\ m \\ n \\ b \end{array} \right] = \delta_1(a) + \delta_2(b) + \tau(n) + \sigma(m)$ defines a derivation from $S$ into $X$. If $X_{AB} = 0 = X_{BA}$, then we may assume that $\tau$ and $\sigma$ are zero. Note that, in this case, $\delta_A(a)m = m \delta_B(b) = 0 = \delta_B(b)n = n \delta_A(a)$, for every $a \in A$, $b \in B$, $m \in M$ and $n \in N$. 

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Now, we have the following:

**Theorem 5** If $X_{AB} = 0 = X_{BA}$, where $X$ is a unitary $S$-module. Then

$$H^1(S, X) = H^1(A, X_{AA}) \oplus H^1(B, X_{BB})$$

**Proof.** Suppose that $X_{AB} = 0 = X_{BA}$ and consider the $\mathcal{R}$-linear map $\rho : \text{Der}(S, X) \longrightarrow H^1(A, X_{AA}) \oplus H^1(B, X_{BB})$ defined by

$$\delta \rightarrow (\delta_A + \text{Inn}(A, X_{AA}), \delta_B + \text{Inn}(B, X_{BB})).$$

If $\delta_1 \in \text{Der}(A, X_{AA})$ and $\delta_2 \in \text{Der}(B, X_{BB})$, then

$$D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \delta_1(a) + \delta_2(b)$$

is a derivation from $S$ into $X$ and

$$\rho(D) = (\delta_A + N^1(A, X_{AA}), \delta_B + N^1(B, X_{BB})) = (\delta_1 + N^1(A, X_{AA}), \delta_2 + N^1(B, X_{BB})).$$

The last equation is deduced from the fact that $\delta_A(a) = 1_A(\delta_1(a) + \delta_2(0))1_A = \delta_1(a)$ and $\delta_B(b) = 1_B(\delta_1(0) + \delta_2(b))$. Thus $\rho$ is surjective.

If $\delta \in \ker \rho$, then $\delta_A \in \text{Inn}(A, X_{AA})$ and $\delta_B \in \text{Inn}(B, X_{BB})$. Then $\delta_A(a) = ax - xa$ for some $x \in X_{AA}$ and $\delta_B(b) = by - yb$ for some $y \in X_{BB}$. Then

$$D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \delta_A(a) + \delta_B(b) = (ax - xa) + (by - yb)$$

$$= \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} x - x \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) + \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 \\ 1_B & 0 \end{bmatrix} y - y \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} x - x \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) + \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 0 \\ 1_B & 0 \end{bmatrix} y - y \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Thus $D = \delta_{x+y}$. It is straightforward to show that

$$\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = 1_A \delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) 1_A - 1_B \delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right) 1_A \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly,

$$\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = 1_B \delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) 1_B - \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} 1_A \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right) 1_A.$$

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Also,
\[
\delta \left( \begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix} \right) = 1_B \delta \left( \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \right) 1_A \begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix} - \begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix} 1_B \delta \left( \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \right) 1_A
\]
and
\[
\delta \left( \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} \right) = 1_A \delta \left( \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \right) 1_B \begin{bmatrix} a & 0 \\ n & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ n & b \end{bmatrix} 1_A \delta \left( \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \right) 1_B.
\]
These follow that
\[
(\delta - D) \left( \begin{bmatrix} a \\ n \\ b \end{bmatrix} \right) = -\delta \left( \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} \right) 1_A \left( \begin{bmatrix} a \\ m \\ n \\ b \end{bmatrix} \right).
\]

Therefore, we have \(\delta - D \in \text{Inn}(S, X)\), and so \(\delta \in \text{Inn}(S, X)\).

Conversely, let \(\delta \in \text{Inn}(S, X)\). Then there exists \(x \in X\) such that
\[
\delta \left( \begin{bmatrix} a \\ n \\ b \end{bmatrix} \right) = \begin{bmatrix} a \\ m \\ n \\ b \end{bmatrix} - x \begin{bmatrix} a \\ m \\ n \\ b \end{bmatrix}.
\]
So that
\[
\delta_A(a) = 1_A \delta \left( \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \right) 1_A = 1_A \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} - x \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} 1_A
\]
\[
= \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} 1_A = 1_A - 1_A x 1_A \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \delta_{1a \times 1a}(a).
\]
Similarly, \(\delta_B(b) = \delta_{1a \times 1a}(b)\). Hence \(\delta_A\) and \(\delta_B\) are inner and so \(\delta \in \ker \rho\). Thus \(\text{Inn}(S, X) = \ker \rho\). We conclude that
\[
H^1(S, X) = \frac{\text{Der}(S, X)}{\ker \rho} = \frac{\text{Der}(S, X)}{\ker \rho} = H^1(A, X_{AA}) \oplus H^1(B, X_{BB}).
\]

**Corollary 6** \(H^1(S, M) = 0 = H^1(S, N)\).

**Proof.** With \(X = M (X = N)\) we have
\[
H^1(S, M) = H^1(A, 0) \oplus H^1(B, 0) \quad \left( H^1(S, N) = H^1(A, 0) \oplus H^1(B, 0) \right)
\]
and this is zero.

**Corollary 7** \(H^1(S, A) = 0\) where
\[
S = \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} \mid a \in A \right\}.
\]

**Example 8** If \(S = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_n \\ \mathbb{Z}_n & \mathbb{Z} \end{bmatrix} \) for \(n > 1\), then \(H^1(S, \mathbb{Z}_n) = 0\).

### 4. Stability of the First Hochschild Cohomology

Let \(A\) and \(R\) be Banach algebras such that \(A\) is a Banach \(R\)-algebra with compatible actions, that is
\[
\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha)
\]
for all \(a, b \in A, \alpha \in R\). Let \(X\) be a Banach \(A\)-bimodule and a Banach \(R\)-bimodule with compatible actions, that is
\[
\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha)
\]
\[
x \cdot (a \cdot \alpha) = (x \cdot a) \cdot \alpha, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha)
\]
\[
a \cdot (a \cdot x) = (a \cdot a) \cdot x, \quad x \cdot (a \cdot a) = (x \cdot a) \cdot a
\]
for all \( a \in A, \alpha \in \mathcal{R}, x \in X \). Then we say that \( X \) is a Banach \( A \)-module. If moreover
\[
\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathcal{R}, x \in X)
\]
then \( X \) is called a commutative \( A \)-\( \mathcal{R} \)-module.

Let \( A \) and \( B \) be Banach \( \mathcal{R} \)-algebras with units \( 1_A \) and \( 1_B \), respectively, \( M \) is a Banach \( \mathcal{R} \)-bimodule, left Banach \( A \)-module and right Banach \( B \)-module \((A, B\text{-module})\) and \( N \) is a Banach \( \mathcal{R} \)-bimodule, right Banach \( A \)-module and left \( B \)-module \((B, A\text{-module})\). Then \( S = \{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} | a \in A, b \in B, m \in M, n \in N \} \) is a Banach \( \mathcal{R} \)-algebra equipped with the defined operations in section 1 and the following norm
\[
\| \begin{bmatrix} a & m \\ n & b \end{bmatrix} \| = \|a\|_A + \|b\|_B + \|m\|_M + \|n\|_N.
\]

Let \( X \) be a unitary \( S \)-bimodule and \( X_{AA}, X_{BB}, X_{AB} \) and \( X_{BA} \) be similar to section 3. Assume that \( X_{AB} = 0 = X_{BA} \). Let \( \alpha \in \mathcal{R} \) and let \( f_1, f_2, f_3 : S \rightarrow X \) be mappings. Define
\[
D_\alpha[f_1, f_2, f_3](s_1, s_2) = f_1(\alpha s_1 + s_2) - \alpha f_2(s_1) - f_3(s_2),
\]
and
\[
\delta[f_1, f_2, f_3](s_1, s_2) = s_1 f(s_2) - f(s_1 s_2) + f(s_1) s_2,
\]
for all \( s_1, s_2 \in S \). Similar to section 3, we obtain the mappings \( f_\alpha : A \rightarrow X_{AA} \) and \( f_\beta : B \rightarrow X_{BB} \) for \( i = 1, 2, 3 \) that are defined as
\[
f_\alpha(a) = e_A f_1(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}) e_A \quad \text{and} \quad f_\beta(b) = e_B f_3(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) e_B.
\]
for all \( a \in A \) and \( b \in B \).

**Theorem 9** Let \( \lambda, \gamma \in \mathbb{R}^+ \) and \( f_1, f_2, f_3 : S \rightarrow X \) be mappings that satisfy
\[
\|D_\alpha[f_1, f_2, f_3](s_1, s_2)\| \leq \lambda, \quad (8)
\]
\[
\|\delta[f_1, f_2, f_3](s_1, s_2)\| \leq \gamma. \quad (9)
\]

If for any \( s_i = 0, i = 1, 2 \), we have \( f_i(s_i) = 0 \), then there exists a unique inner derivation \( D \) such that
\[
\|f_1(s) - D(s)\| \leq 6\lambda, \quad (10)
\]
\[
\|f_2(s) - D(s)\| \leq 12\lambda, \quad (11)
\]
\[
\|f_3(s) - D(s)\| \leq 12\lambda, \quad (12)
\]
for all \( s \in S \).

**Proof.** Let \( \alpha = 1_\mathcal{R} \) (unit of \( \mathcal{R} \)) and \( s_2 = 0 \), then
\[
\|f_1(s_1) - f_2(s_1)\| \leq \lambda, \quad (13)
\]
for all \( s_1 \in S \). Similarly,
\[
\|f_1(s_1) - f_3(s_1)\| \leq \lambda, \quad (14)
\]
for all \( s_2 \in S \). By repeating the above stated relations we obtain the desire.

**Acknowledgements**

Collate acknowledgements in a separate section at the end of the article before the references. List here those individuals who provided help during the research (e.g., providing language help, writing assistance or proof reading the article, etc.).

**References**


https://doi.org/10.1017/CBO9780511629204

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