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The First Hochschild Cohomology of Square Algebras With it's Stability

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Abstract

In this paper, we study on a special case of generalized matrix algebra that we call it square algebra. According to that Hochschild cohomology play a significant role in Geometry for example in orbifolds, we study the first Hochschild cohomology of the square algebra the vanishing of its.

Keywords: First Hochschild cohomology, Hochschild cohomology, Square algebra

1. Introduction

Let $\mathcal{R}$ be a commutative ring (with unit), let A and B be $\mathcal{R}$-algebras and M be a left A-module and right B-module. A triangular algebra T over $\mathcal{R}$ is the following matrix

$$T = \begin{bmatrix} A & M \\ & B \end{bmatrix}.$$

Automorphisms, derivations, commuting mappings and Lie derivations on triangular algebras are studied by Cheung (Cheung, 2001) and (Cheung, 2003). Other useful and valuable literature concerning the structure of derivations and Lie derivations is (Ji & Qi, 2011). Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras which derivations of those considered in (Christensen, 1977), (Coelho, & Milies, 1993), (Donsig, Forrest & Marcoux, 1996).

A generalized matrix algebra is a generalization of triangular matrix algebra. In the triangular algebra $\mathcal{T}$, the element lies in the second row and second column is zero. In generalized matrix algebra, we put a right A-module and left B-module N in zero place. We denote the generalized matrix algebra by $\mathcal{G}$. Algebraic studying on derivations, generalized derivations and Lie derivations have been studied in (Du, & Wang, 2012), (Li & Wei, 2012), (Li, & Xiao, 2011).

Throughout this paper $\mathcal{R}$ is a commutative ring (with unit), A and B are R-algebras with units 1_A and 1_B, respectively, M is an $\mathcal{R}$-bimodule, left A-module and right B-module (A, B-module) and N is an $\mathcal{R}$-bimodule, right A-module and left B-module (B, A-module). Define bimodule homomorphisms $\Phi_{MN} : M \otimes_B N \longrightarrow A$ and $\Phi_{NM} : N \otimes_A M \longrightarrow B$ satisfying the following commutative diagrams:

$$\begin{tikzcd}
M \otimes_B N \otimes_A M \arrow[r, "\Phi_{MN} \otimes id_M"] \arrow[d, "id_M \otimes \Phi_{NM}"'] & A \otimes_A M \arrow[d] \\
M \otimes_B B \arrow[r] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \otimes_A M \otimes_B N \arrow[r, "\Phi_{NM} \otimes id_M"] \arrow[d, "id_M \otimes \Phi_N"'] & B \otimes_B N \arrow[d] \\
N \otimes_A A \arrow[r] & N.
\end{tikzcd}$$

For more details and applications see (Buchweitz, 2003). We define generalized matrix algebra

$$\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix} = \Big\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} : a \in A, b \in B, m \in M, n \in N \Big\},$$

with the usual 2×2 matrix-like addition and multiplication

$$\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + m_1 \otimes_B n_2 & a_1 m_2 + m_1 b_2 \\ n_1 a_2 + b_1 n_2 & n_1 \otimes_A m_2 + b_1 b_2 \end{bmatrix}$$

In this algebra, if $M \otimes_B N = 0 = N \otimes_A M$, then we denote it by S and we called that a square algebra.

Let $\mathcal{R}$ be a commutative ring (with unit), let A be an $\mathcal{R}$-algebra and M be an A-bimodule. For $n = 0, 1, 2, \ldots$, let $C^n(A, M)$ be the space of all n-linear (as a $\mathcal{R}$-module map) mappings from $A \times \cdots \times A$ into M and $C^0(A, M) = M$. Consider the sequence

$$0 \longrightarrow C^0(A, M) \xrightarrow{d^0} C^1(A, M) \xrightarrow{d^1} \cdots \Big(\tilde{C}(A, M)\Big)$$

in which

$$\begin{aligned} d^0 x(a) &= ax - xa \\ d^n f(a_1, a_2, \cdots, a_{n+1}) &= a_1 f(a_2, \cdots, a_{n+1}) \\ &\quad + (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1} \\ &\quad + \sum_{j=1}^{n} (-1)^j f(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1}) \end{aligned} \tag{1}$$

where $n \geq 1$, $x \in M$ and $a_1, \ldots, a_{n+1} \in A$. The above sequence is a complex for A and M. The n-th cohomology group of $\tilde{C}(A, E)$ is said to be n-th Hochschild cohomology group and denoted by $H^n(A, M)$, for more details see (Brodmann & Sharp, 1998), (Rotman, 2009). A derivation is a linear map $D : A \longrightarrow M$ such that $D(ab) = aD(b) + D(a)b$ $(a, b \in A)$ and for $x \in M$, we define the map $D_x : A \longrightarrow M$ by $D_x(a) = xa - ax$. The map D_x is a derivation and such derivations called inner derivations. Let $Der(A, M)$ denote all derivations and $Inn(A, M)$ denote all inner derivations.

Thus, we have

$$H^1(A, M) = \frac{Der(A, M)}{Inn(A, M)}.$$

In this paper, we describe $H^1(S, S)$ and vanishing of $H^1(S, X)$, where X is a two sided S-module (bimodule) is investigated.

2. Structure of $H^1(S, S)$

We begin with the following simple properties of derivations on S as follows:

Proposition 1 *Let $D : S \longrightarrow S$ be a derivation, then there are derivations $d_A : A \longrightarrow A$, $d_B : B \longrightarrow B$, $\mathcal{R}$-linear maps $\tau : M \longrightarrow M$ and $\sigma : N \longrightarrow N$ and elements $m_D \in M$ and $n_D \in N$ such that*

(i) $D\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_D \\ n_D & 0 \end{bmatrix} = -D\left(\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\right)$,

(ii) $D\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \tau(m) \\ 0 & 0 \end{bmatrix}$ and $D\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \sigma(n) & 0 \end{bmatrix}$,

(iii) $D\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} d_A(a) & am_D \\ n_Da & 0 \end{bmatrix}$ and $D\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & -m_Db \\ -bn_D & d_B(b) \end{bmatrix}$,

(iv) $\tau(am) = d_A(a)m + a\tau(m)$,

(v) $\tau(mb) = \tau(m)b + md_B(b)$,

(vi) $\sigma(na) = nd_A(a) + \sigma(n)a$,

(vii) $\sigma(bn) = b\sigma(n) + d_B(b)n$,

for all $a \in A, b \in B, m \in M, n \in N$.

Conversely, if d_A *and* d_B *are derivations on A and B, respectively, and if* $\tau : M \longrightarrow M$ *and* $\sigma : N \longrightarrow N$ *are any* $\mathcal{R}$*-linear maps satisfy (i), (ii), (iii) and (iv) then the map*

$$D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} d_A(a) & \tau(m) \\ \sigma(n) & d_B(b) \end{bmatrix}$$

defines a derivation on S.

Proof. Let D be a derivation. By the following relations and simple calculation we obtain (i)-(vii):

$$D\left(\begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right),\ D\left(\begin{bmatrix} 0 & mb \\ 0 & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right),$$

and

$$D\left(\begin{bmatrix} 0 & 0 \\ bn & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right),\ D\left(\begin{bmatrix} 0 & 0 \\ na & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Conversely, consider,

$$\begin{aligned} D\left(\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix}\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix}\right) &= D\left(\begin{bmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ n_1a_1 + b_1n_2 & b_1b_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} d_A(a_1a_2) & \tau(a_1m_2 + m_1b_2) \\ \sigma(n_1a_1 + b_1n_2) & d_B(b_1b_2). \end{bmatrix} \end{aligned}$$

Moreover,

$$\begin{aligned} \begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} D\left(\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix}\right) + D\left(\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix}\right)\begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} &= \begin{bmatrix} a_1d_A(a_2) & a_1\tau(m_2) + m_1d_B(b_2) \\ n_1d_A(a_2) + b_1\sigma(n_2) & b_1d_B(b_2) \end{bmatrix} \\ &\quad + \begin{bmatrix} d_A(a_1)a_2 & d_A(a_1)m_2 + \tau(m_1)b_2 \\ \sigma(n_1)a_2 + d_B(b_1)n_2 & d_B(b_1)b_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1d_A(a_2) + d_A(a_1)a_2 & \begin{array}{r} a_1\tau(m_2) + m_1d_B(b_2) \\ +d_A(a_1)m_2 + \tau(m_1)b_2 \end{array} \\ \begin{array}{l} n_1d_A(a_2) + b_1\sigma(n_2) \\ +\sigma(n_1)a_2 + d_B(b_1)n_2 \end{array} & b_1d_B(b_2) + d_B(b_1)b_2 \end{bmatrix} \\ &= \begin{bmatrix} d_A(a_1a_2) & \tau(a_1m_2) + \tau(m_1b_2) \\ \sigma(n_1a_2) + \sigma(b_1n_2) & d_B(b_1b_2) \end{bmatrix} \end{aligned}$$

Thus D is a derivation on S.

Let $a_0 \in A$ and $b_0 \in B$, then Rosenblum $\mathcal{R}$-linear map $\tau_M^{a_0,b_0} : M \longrightarrow M$ is defined by

$$\tau_M^{a_0,b_0}(m) = a_0 \cdot m - m \cdot b_0 \ \ for\ each\ \ m \in M.$$

Now, let $Z(A)$ be the center of A and $Z(B)$ be the center of B, $x \in Z(A)$ and $y \in Z(B)$. Then the Rosenblum $\mathcal{R}$-linear map $\tau_M^{x,y}$ is called a central Rosenblum $\mathcal{R}$-linear map. We denote the set of all central Rosenblum $\mathcal{R}$-linear maps by $ZR_{A,B}(M)$. Also, we have

$$ZR_{A,B}(M) \subseteq Hom_{A,B}(M).$$

An $\mathcal{R}$-map $\tau_M : M \longrightarrow M$ is called a generalized Rosenblum $\mathcal{R}$-linear map if there exist derivations d_A and d_B on A and B, respectively, such that τ_M satisfies

$$\tau(amb) = d_A(a)mb + a\tau(m)b + amd_B(b)$$

for each $a \in A$, $b \in B$ and $m \in M$. Similarly, an $\mathcal{R}$-map $\tau_N : N \longrightarrow N$ is called a generalized Rosenblum $\mathcal{R}$-linear map if there exist derivations d_A and d_B on A and B, respectively, such that τ_N satisfies

$$\tau(bna) = d_B(b)na + b\tau_N(n)a + bnd_A(a)$$

for each $a \in A$, $b \in B$ and $m \in M$.

Lemma 2 *Let $\varphi \in Hom_{A,B}(M)$ and $\sigma \in Hom_{B,A}(N)$. Then the map $d_{\varphi,\sigma} : S \longrightarrow S$ given by*

$$d_{\varphi,\sigma}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} 0 & \varphi(m) \\ \sigma(n) & 0 \end{bmatrix},$$

is a derivation. Moreover, $d_{\varphi,\sigma}$ is an inner derivation if and only if $\varphi = \tau_M^{x,y}$ and $\sigma = \tau_N^{y,x}$, where $\tau_M^{x,y} \in ZR_{A,B}(M)$ and $\tau_N^{y,x} \in ZR_{B,A}(N)$.

Proof. The first statement follows immediately from assume that $\varphi = \tau_M^{x,y}$ and $\sigma = \tau_N^{y,x}$ where $x \in Z(A)$ and $y \in Z(B)$. Then

$$\begin{aligned} d_{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) &= \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} - \begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \\ &= \begin{bmatrix} xa & xm \\ yn & yb \end{bmatrix} - \begin{bmatrix} ax & my \\ nx & by \end{bmatrix} \\ &= \begin{bmatrix} xa - ax & xm - my \\ yn - nx & yn - by \end{bmatrix} = \begin{bmatrix} 0 & xm - my \\ yn - nx & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \varphi(m) \\ \sigma(n) & 0 \end{bmatrix}. \end{aligned}$$

Hence $d_{\varphi,\sigma}$ is inner. Conversely, assume that $d_{\varphi,\sigma}$ is inner. Then there exists $\begin{bmatrix} x & z \\ w & y \end{bmatrix} \in S$ such that $d_{\varphi,\sigma} = d_{\begin{bmatrix} x & z \\ w & y \end{bmatrix}}$. Then,

$$\begin{aligned} d_{\begin{bmatrix} x & z \\ w & y \end{bmatrix}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) &= \begin{bmatrix} x & z \\ w & y \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} - \begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} x & z \\ w & y \end{bmatrix} \\ &= \begin{bmatrix} xa - ax & xm + zb - az - my \\ wa + yn - nx - bw & yb - by \end{bmatrix}. \end{aligned}$$

If $d_{\begin{bmatrix} x & z \\ w & y \end{bmatrix}} = d_{\varphi,\sigma}$, then $xa - ax = 0$ for each $a \in A$ and $yb - by = 0$ for each $b \in B$. In particular, $x \in Z(A)$ and $y \in Z(B)$. Moreover, we have

$$\varphi(m) = xm + zb - az - my$$

and

$$\sigma(n) = wa + yn - nx - bw.$$

Since $\varphi \in Hom_{A,B}(M)$ and $\sigma \in Hom_{B,A}(N)$, it follows that $zb - az = 0$ and $wa - bw = 0$. Hence $\varphi(m) = xm - my = \tau_M^{x,y}(m)$ and $\sigma(n) = yn - nx = \tau_N^{y,x}(n)$. In particular, $\varphi \in ZR_{A,B}(M)$ and $\sigma \in ZR_{B,A}(N)$.

We can now state the main result of this section for describing $H^1(S, S)$.

Theorem 3 If $H^1(A, A) = 0 = H^1(B, B)$, then

$$H^1(S, S) \cong \frac{Hom_{A,B}(M) \times Hom_{B,A}(N)}{ZR_{A,B}(M) \times ZR_{B,A}(N)}. \tag{2}$$

Proof. Define $\phi : Hom_{A,B}(M) \times Hom_{B,A}(N) \longrightarrow H^1(S, S)$ by

$$\phi(\varphi, \sigma) = \overline{d}_{\varphi,\sigma},$$

where $\overline{d}_{\varphi,\sigma}$ represents the equivalence class of $d_{\varphi,\sigma}$ in $H^1(S, S)$. Clearly, ϕ is $\mathcal{R}$-linear.

We shall show that ϕ is surjective. Let $d : S \longrightarrow S$ be a derivation. Then there are derivations d_A, d_B, and $\mathcal{R}$-linear maps $\tau : M \longrightarrow M$, $\sigma : N \longrightarrow N$ and elements $m_d \in M$, $n_d \in N$ that satisfy in the conditions (i)-(vii) of Proposition 1. Since $H^1(A, A) = H_1(B, B) = 0$, we can find $x \in A$ and $y \in B$ such that $d_A = d_x$ and $d_B = d_y$. Define $d_0 : S \longrightarrow S$ by

$$d_0\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} d_x(a) & \tau_M^{x,y}(m) + (am_d - m_d b) \\ \tau_N^{y,x}(n) + (n_d a - bn_d) & d_y(b) \end{bmatrix}.$$

Then d_0 is an inner derivation on S induced by $\begin{bmatrix} x & -m_d \\ -n_d & y \end{bmatrix}$. Furthermore, if $d_1 = d - d_0$, then d_1 is a derivation and

$$\begin{aligned} d_1\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) &= \begin{bmatrix} d_x(a) & \tau(m) + (am_d - m_d b) \\ \sigma(n) + (n_d a - bn_d) & d_y(b) \end{bmatrix} \\ &- \begin{bmatrix} d_x(a) & \tau_M^{x,y}(m) + (am_d - m_d b) \\ \tau_N^{y,x}(n) + (n_d a - bn_d) & d_y(b) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \tau(m) - \tau_M^{x,y}(m) \\ \sigma(n) - \tau_N^{y,x}(n) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau_1(m) \\ \sigma_1(n) & 0 \end{bmatrix} \end{aligned}$$

where $\tau_1 = \tau - \tau_M^{x,y}$ and $\sigma_1 = \sigma - \tau_N^{x,y}$. It follows from Proposition 1, that $\tau_1 \in Hom_{A,B}(M)$ and $\sigma_1 \in Hom_{B,A}(N)$. Finally, set $\bar{d} = \bar{d}_1 = \varphi(\tau_1, \sigma_1)$, and so φ is surjective. This implies that

$$H^1(S, S) \cong \frac{Hom_{A,B}(M) \times Hom_{B,A}(N)}{\ker\varphi}. \tag{3}$$

However, $(\varphi, \sigma) \in \ker\varphi$ if and only if $d_{\phi,\sigma}$ is inner. By Lemma 2, $\ker\varphi = ZR_{A,B}(M) \times ZR_{B,A}(N)$. Thus, by this fact and (3), (2) holds.

Corollary 4 Let A and B be a commutative ring. By hypothesis of the above Theorem, we have $H^1(S, S) \cong Hom_{A,B}(M) \times Hom_{B,A}(N)$.

3. Vanishing of the First Cohomology Group

Let X be a unitary S-bimodule, denote $X_{AA} = 1_A X\, 1_A$, $X_{BB} = 1_B X\, 1_B$, $X_{AB} = 1_A X\, 1_B$ and $X_{BA} = 1_B X\, 1_A$. For example, when $X = S$, we have $X_{AA} = A$, $X_{BB} = B$, $X_{AB} = M$ and $X_{BA} = N$. In this section, the relations between the first cohomology of S with coefficients in X and those of A and B with coefficients in X_{AA} and X_{BB}, respectively, whenever $X_{AB} = 0$, are investigated.

We started by illustrating the structure of derivations from a square algebra into its bimodules.

Let $\delta : S \longrightarrow X$ be a derivation. Then $\delta_A : A \longrightarrow X_{AA}$ defined by $\delta_A(a) = 1_A \delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) 1_A$ and $\delta_B : B \longrightarrow X_{BB}$ defined by $\delta_B(b) = 1_B \delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) 1_B$ are derivations. Moreover, the $\mathcal{R}$-linear maps $\tau : M \longrightarrow X_{AB}$, defined by $\tau(m) = 1_A \delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) 1_B$ and $\sigma : N \longrightarrow X_{BA}$ defined by $\sigma(n) = 1_B \delta\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) 1_A$ satisfy

(i) $\tau(am) = a\tau(m) + \delta_A(a)m$,

(ii) $\tau(mb) = \tau(m)b + m\delta_B(b)$,

(iii) $\sigma(na) = \sigma(n)a + n\delta_A(a)$,

(iv) $\sigma(bn) = b\sigma(n) + n\delta_B(b)n$.

Conversely, if δ_1 and δ_2 are derivation from A and B into X_{AA} and X_{BB}, respectively, and $\tau : M \longrightarrow X_{AB}$ and $\sigma : N \longrightarrow X_{BA}$ are any $\mathcal{R}$-linear maps satisfy in (i), (ii), (iii) and (iv), then the map $D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \delta_1(a) + \delta_2(b) + \tau(n) + \sigma(n)$ defines a derivation from S into X. If $X_{AB} = 0 = X_{BA}$, then we may assume that τ and σ are zero. Note that, in this case, $\delta_A(a)m = m\delta_B(b) = 0 = \delta_B(b)n = n\delta_n(a)$, for every $a \in A$, $b \in B$, $m \in M$ and $n \in N$.

Now, we have the following:

Theorem 5 If $X_{AB} = 0 = X_{BA}$, where X is a unitary S-module. Then

$$H^1(S, X) = H^1(A, X_{AA}) \oplus H^1(B, X_{BB})$$

Proof. Suppose that $X_{AB} = 0 = X_{BA}$ and consider the $\mathcal{R}$-linear map $\rho : Der(S, X) \longrightarrow H^1(A, X_{AA}) \oplus H^1(B, X_{BB})$ defined by

$$\delta \to (\delta_A + Inn(A, X_{AA}), \delta_B + Inn(B, X_{BB})).$$

If $\delta_1 \in Der(A, X_{AA})$ and $\delta_2 \in Der(B, X_{BB})$, then

$$D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \delta_1(a) + \delta_2(b)$$

is a derivation from S into X and

$$\begin{aligned} \rho(D) &= (\delta_A + N^1(A, X_{AA}), \delta_B + N^1(B, X_{BB})) \\ &= (\delta_1 + N^1(A, X_{AA}), \delta_2 + N^1(B, X_{BB})). \end{aligned}$$

The last equation is deduced from the fact that $\delta_A(a) = 1_A(\delta_1(a) + \delta_2(0))1_A = \delta_1(a)$ and $\delta_B(b) = 1_B(\delta_1(0) + \delta_2(b))$. Thus ρ is surjective.

If $\delta \in \ker\rho$, then $\delta_A \in Inn(A, X_{AA})$ and $\delta_B \in Inn(B, X_{BB})$. Then $\delta_A(a) = ax - xa$ for some $x \in X_{AA}$ and $\delta_B(b) = by - yb$ for some $y \in X_{BB}$. Then

$$\begin{aligned}
& D\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \delta_A(a) + \delta_B(b) = (ax - xa) + (by - yb) \\
&= \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} x - x \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} + x \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}\right) \\
&+ \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} y - \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix} y - y \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} x - x \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} + x \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} x\right) \\
&+ \left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} y - \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} y - y \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) \\
&= \Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix} x - x \begin{bmatrix} a & m \\ n & b \end{bmatrix} + \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix} x \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \\
&- \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix} x \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\Big) \\
&+ \Big(\begin{bmatrix} a & m \\ n & b \end{bmatrix} y - y \begin{bmatrix} a & m \\ n & b \end{bmatrix} - \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} y \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} y \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\Big) \\
&= \begin{bmatrix} a & m \\ n & b \end{bmatrix}(x + y) - (x + y)\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \delta_{x+y}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right).
\end{aligned}$$

Thus $D = \delta_{x+y}$. It is straightforward to show that

$$\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = 1_A\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_A - 1_B\delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right)1_A\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly,

$$\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = 1_B\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_B - \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}1_B\delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right)1_A.$$

Also,

$$\delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = 1_B\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_A\begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix} - \begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}1_B\delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right)1_A$$

and

$$\delta\left(\begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}\right) = 1_A\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\right)1_B\begin{bmatrix} a & 0 \\ n & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ n & b \end{bmatrix}1_A\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}\right)1_B.$$

These follow that

$$(\delta - D)\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = -\delta_{1_B\delta\left(\begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}\right)1_A}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right).$$

Therefore, we have $\delta - D \in Inn(S, X)$, and so $\delta \in Inn(S, X)$.

Conversely, let $\delta \in Inn(S, X)$. Then there exists $x \in X$ such that

$$\delta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} a & m \\ n & b \end{bmatrix}x - x\begin{bmatrix} a & m \\ n & b \end{bmatrix}.$$

So that

$$\begin{aligned}\delta_A(a) = 1_A\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_A &= 1_A\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}x - x\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_A \\ &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}1_A = 1_A - 1_Ax1_A\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \delta_{1_Ax1_A}(a).\end{aligned}$$

Similarly, $\delta_B(b) = \delta_{1_Bx\,1_B}(b)$. Hence δ_A and δ_B are inner and so $\delta \in \ker\rho$. Thus $Inn(S, X) = \ker\rho$. We conclude that

$$H^1(S, X) = \frac{Der(S, X)}{Inn(S, X)} = \frac{Der(S, X)}{\ker\,\rho} = H^1(A, X_{AA}) \oplus H^1(B, X_{BB}).$$

Corollary 6 $H^1(S, M) = 0 = H^1(S, N)$.

Proof. With $X = M$ $(X = N)$ we have

$$H^1(S, M) = H^1(A, 0) \oplus H^1(B, 0) \qquad \left(H^1(S, N) = H^1(A, 0) \oplus H^1(B, 0)\right)$$

and this is zero.

Corollary 7 $H^1(S, A) = 0$ where

$$S = \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle|\, a \in A\right\}.$$

Example 8 If $S = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_n \\ \mathbb{Z}_n & \mathbb{Z} \end{bmatrix}$ for $n > 1$, then $H^1(S, \mathbb{Z}_n) = 0$.

4. Stability of the First Hochschild Cohomology

Let A and $\mathcal{R}$ be Banach algebras such that A is a Banach $\mathcal{R}$-algebra with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b,\ (ab) \cdot \alpha = a(b \cdot \alpha) \tag{4}$$

for all $a, b \in A, \alpha \in \mathcal{R}$. Let X be a Banach A-bimodule and a Banach $\mathcal{R}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x,\ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \tag{5}$$

$$x \cdot (a \cdot \alpha) = (x \cdot a) \cdot \alpha,\ (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha) \tag{6}$$

$$a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,\ x \cdot (\alpha \cdot a) = (x \cdot \alpha) \cdot a \tag{7}$$

for all $a \in A, \alpha \in \mathcal{R}, x \in X$. Then we say that X is a Banach A-R-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathcal{R}, x \in X)$$

then X is called a *commutative* A-$\mathcal{R}$-module.

Let A and B are Banach $\mathcal{R}$-algebras with units 1_A and 1_B, respectively, M is a Banach $\mathcal{R}$-bimodule, left Banach A-module and right Banach B-module (A, B-module) and N is a Banach $\mathcal{R}$-bimodule, right Banach A-module and left B-module (B, A-module). Then $S = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mid a \in A, b \in B, m \in M, n \in N \right\}$ is a Banach $\mathcal{R}$-algebra equipped with the defined operations in section 1 and the following norm

$$\left\| \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right\| = \|a\|_A + \|b\|_B + \|m\|_M + \|n\|_N.$$

Let X be a unitary S-bimodule and X_{AA}, X_{BB}, X_{AB} and X_{BA} be similar to section 3. Assume that $X_{AB} = 0 = X_{BA}$. Let $\alpha \in \mathcal{R}$ and let $f_1, f_2, f_3 : S \longrightarrow X$ be mappings. Define

$$D_\alpha[f_1, f_2, f_3](s_1, s_2) = f_1(\alpha s_1 + \alpha s_2) - \alpha f_2(s_1) - \alpha f_3(s_2),$$

and

$$\delta[f_1, f_2, f_3](s_1, s_2) = s_1 f(s_2) - f(s_1 s_2) + f_3(s_1)s_2,$$

for all $s_1, s_2 \in S$. Similar to section 3, we obtain the mappings $f_A^i : A \longrightarrow X_{AA}$ and $f_B^i : B \longrightarrow X_{BB}$ for $i = 1, 2, 3$ that are defined as

$$f_A^i(a) = e_A f_i\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)e_A \text{ and } f_B^i(b) = e_B f_i\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)e_B,$$

for all $a \in A$ and $b \in B$.

Theorem 9 Let $\lambda, \gamma \in \mathbb{R}^+$ and $f_1, f_2, f_3 : S \longrightarrow X$ be mappings that satisfy

$$\|D_\alpha[f_1, f_2, f_3](s_1, s_2)\| \leq \lambda, \tag{8}$$

$$\|\delta[f_1, f_2, f_3](s_1, s_2)\| \leq \gamma. \tag{9}$$

If for any $s_i = 0,\ i = 1, 2$, we have $f_i(s_i) = 0$, then there exists a unique inner derivation D such that

$$\|f_1(s) - D(s)\| \leq 6\lambda, \tag{10}$$

$$\|f_2(s) - D(s)\| \leq 12\lambda, \tag{11}$$

$$\|f_3(s) - D(s)\| \leq 12\lambda, \tag{12}$$

for all $s \in S$.

Proof. Let $\alpha = 1_{\mathcal{R}}$ (unit of $\mathcal{R}$) and $s_2 = 0$, then

$$\|f_1(s_1) - f_2(s_1)\| \leq \lambda, \tag{13}$$

for all $s_1 \in S$. Similarly,

$$\|f_1(s_2) - f_3(s_2)\| \leq \lambda, \tag{14}$$

for all $s_2 \in S$. By repeating the above stated relations we obtain the desire.

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