Journal of Mathematics Research; Vol. 9, No. 4; August 2017
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

# On $FGDF$-modules

Alhousseynou BA[1], Sidy Demba Touré[1] & Oumar Diankha[1]

[1] Department of mathematiques and Computer Sciences, Faculty of Sciences and Technics, Cheikh Anta Diop University, Dakar, Senegal

Correspondence: Alhousseynou BA, Department of mathematiques and Computer Sciences, Faculty of Sciences and Technics, Cheikh Anta Diop University, Dakar, Senegal. E-mail: alhousseynou.ba@ucad.edu.sn

Received: April 24, 2017 Accepted: May 29, 2017 Online Published: July 25, 2017

doi:10.5539/jmr.v9n4p196 URL: https://doi.org/10.5539/jmr.v9n4p196

**Abstract**

Let $R$ be a unital ring and $M$ a unitary module not necessary over $R$. The $FGDF$-module is a generalization of $FGDF$-rings (Touré, Diop, Mohamed and Sangharé, 2014). In this work, we first give some properties of $FGDF$-modules. After that, we show that for a finitely generated module $M$, $M$ is a $FGDF$-module if and only if $M$ is of finite representation type module. Finally, we show that $M$ is a finitely generated $FGDF$-module if and only if every Dedekind finite module of $\sigma[M]$ is noetherian.

**Keywords:** finitely generated module, Dedekind finite module, $FGDF$-module

## 1. Introduction

We assume that $R$ is a unity ring and $M$ a unitary module not necessary over $R$. Let $M$ and $N$ be $R$-modules. $N$ is said to be generated by $M$ if there exist a set $\Lambda$ and an epimorphism $\varphi : M^{(\Lambda)} \longrightarrow N$. A submodule $K$ of $N$ is said subgenerated by $M$. The set of submodules of $N$ constitutes the category $\sigma[M]$. It is a full subcategory of $R$-Mod whose objects are submodules of a module generated by $M$ (Wisbauer, 1991).

A module $M$ is said to a prime module if for any submodule $N$ of $M$ $Ann(N) = Ann(M)$. A module $M$ is faithful if $Ann(M) = 0$. A module $M$ is said semisimple if it is direct sum of simple modules. A module $M$ is Hopfian if every surjective endomorphism of $M$ is an automorphism. $M$ is a Dedekind finite module if it is not isomorphic to any proper direct summand of itself. A module $M$ is said to be of finite representation type if it is of finite length and there are only a finite number of non isomorphic finitely generated indecomposable modules in $\sigma[M]$. A ring $R$ is said to be duo-ring if any one sided ideal is two sided.

The aim of this paper comes from to the following assertion. It is well know that in a commutative ring every finitely generated module is Dedekind finite but the converse is not always true. For instance the $\mathbb{Z}$-module $\mathbb{Q}$ is Dedekind finite but not finitely generated. In this paper, we study the modules $M$ for which every Dedekind finite module in $\sigma[M]$ is finitely generated. Those modules are called $FGDF$-modules.

## 2. Some Properties of $FGDF$ -modules

**Lemme 1:** (Ghorbani and Haghany, 2002) corollary 1.4

Let $R$ be a ring and $M$ a $R$-module. If $M$ is Hopfian then it is Dedekind finite.

**Proposition 1:**

Let $R$ be a ring and $M$ a $R$-module. If $M$ is a $FGDF$-module then, there exists a finite number of non-isomorphic simple modules in $\sigma[M]$.

**Proof**: Let $\{N_i\}_I$ be a complete system of non-isomorphic class of simple modules. Let $f : N_i \longrightarrow N_i$ an epimorphism with $f \neq 0$ and $i \in I$. As $N_i$ is simple then $\ker f = 0$, hence, $N_i$ is Hopfian for any $i \in I$. Let $N = \bigoplus_{i \in I} N_i \in \sigma[M]$. Since each $N_i$ is Hopfian and fully invariant then, $N$ is Hopfian. Therefore $N$ is Dedekind finite by lemma 1. Since $M$ is a $FGDF$-module then, $N$ is finitely generated. Hence $I$ is finite.

**Proposition 2:**

Let $R$ be a duo ring and $M$ a finitely generated and prime module over $End(M)$. If $M$ is a $FGDF$-module then, $M$ is a simple.

**Proof:**

As $M$ is finitely generated, we have an epimorphism $f : R \to M$. It is obvious to see that $R/Ann(M) \simeq M$ by the first theorem of isomorphism. It follows from 15.4 of (Wisbauer, 1991) that $\sigma[M] = R/Ann(M)$-Mod. Since $M$ is a $FGDF$-module, then $R/Ann(M)$ is a $FGDF$-ring. It results from (Touré, Diop and Sangharé, 2014) that $R/Ann(M)$ is artinian. Hence $M$ is artinian too. Therefore, there exists a simple submodule in $M$. Let $g : R \to N$ be an epimorphism with $N$ the simple submodule of $M$. Therefore $R/Ann(N) \simeq N$. Since $M$ is a prime module, then $R/Ann(M) = R/Ann(N)$ is simple. Thus $M$ is simple.

**Corollary 1:**

Let $R$ be a duo ring and $M$ a finitely generated, prime and faithful module over $End(M)$. If $M$ is a $FGDF$-module then, $R$ is a field.

**Proof:**

We have already shown in proposition 2 that $R/Ann(M)$ is isomorphic to a simple module $N$. As $M$ is a faithful then, $Ann(M) = Ann(N) = 0$. Hence $R$ is a field.

**Proposition 3:**

Let $M$ be a module over End(M), then the following conditions are verifyed:

(1) The homomorphism image of any $FGDF$-module is a $FGDF$-module;

(2) Let $M = \prod_{i\in I} M_i$ be a product of its submodules.

If $M$ is a $FGDF$-module then $M_i$ is a $FGDF$-module for each $i \in I$.

The converse is true if for any module $N$ of $\sigma[M]$ its submodules are fully invariant and $\sigma[M_i] \bigcap \sigma[M_j] = 0$ with $i \neq j$ in $I$ finite.

**Proof:**

(1) Let $f : M \to f(M) = L$ a homomorphism image of $M$. Therefore $L$ is generated by $M$. That means $L \in \sigma[M]$. Let's consider the subcategory $\sigma[L]$ and $K$ a Dedekind finite object of $\sigma[L]$. Since $K$ is also in $\sigma[M]$ and $M$ is a $FGDF$-module then $K$ is finitely generated. Hence $L$ is a $FGDF$-module.

(2) Assume $M = \prod_{i\in I} M_i$ a $FGDF$-module and $f : \prod_{i\in I} M_i \to M_i$ an epimorphism. It follows from (1) that, for any $i \in I$, $M_i$ is a $FGDF$-module.

Now consider, for each $i \in I$, $M_i$ is a $FGDF$-module. As $I$ is finite, then $\prod_{i\in I} M_i$ is isomorphic to $\bigoplus_{i\in I} M_i$. Suppose $N$ a Dedekind finite element of $\sigma[M]$. Since $\sigma[M_i] \bigcap \sigma[M_j] = 0$ with $i \neq j$ in $I$, it follows from proposition 2.2 of (Vanaja, 1996) that $N = \bigoplus_{i\in I} N_i$ and $N_i \in \sigma[M_i]$. As, for each $i \in I$, $N_i$ is Dedekind finite then, $N_i$ is finitely generated. Hence $N = \bigoplus_{i\in I} N_i$ is finitely generated since I is finite.

It is well know that a homomorphism image, a submodule or a factor of a Dedekind finite module is not in general a Dedekind finite module (Breaz, Cälugäreau and Schulz, 2011) but:

**Proposition 4:**

If $M$ is a $FGDF$-module, then the homomorphism image of every Dedekind finite module of $\sigma[M]$ is a Dedekind finite module.

**Proof:**

Let $N \in \sigma[M]$ be a Dedekind finite module, as $M$ is $FGDF$-module then $N$ is finitely generated. Assume that $f$ is a homomorphism image of $N$ such that $f : N \to f(N) = K$. It is well-know that the homomorphism image of a finitely generated module is finitely generated then, $K$ is finitely generated. Hence, $K$ is Dedekind finite module.

**Proposition 5:**

If $M$ is a $FGDF$-module then, every factor of a Dedekind finite module in $\sigma[M]$ is a Dedekind finite module. Moreover if $M$ is finitely generated then, every submodule of a Dedekind finite module in $\sigma[M]$ is Dedekind finite.

**Proof:** Let $N$ be a Dedekind finite object of $\sigma[M]$, hence $N$ is finitely generated. Thus for every submodule $L$ of $N$, $N/L$ is finitely generated, hence a Dedekind finite module.

Now let's show that any submodule $K$ of $N$ is a Dedekind finite. Since $N$ is finitely generated and is a module of over $R/Ann(M)$ which is artinian(proposition 2), therefore $N$ is noetherian by 15.21 of (Anderson and Fuller,1974). It is well know that any submodule of noetherian module is finitely generated. Therefore $K$ is finitely generated, hence of Dedekind finite.

**Corollary 2:**

Let $M = \bigoplus_{i\in I} M_i$ be a $FGDF$-module with $I$ finite.

$M$ is a Dedekind finite module if and only if $M_i$ is a Dedekind finite module for any $i \in I$.

**Proof:**

Assume $M$ is Dedekind finite module. It follows from proposition 5 that $M_i$ is a Dedekind finite module for any $i \in I$.

Now let's suppose that, for any $i \in I$, $M_i$ is a Dedekind finite module. As $M_i \in \sigma[M]$ then $M_i$ is finitely generated for any $i \in I$. Since $I$ is finite, therefore $M = \bigoplus_{i\in I} M_i$ is finitely generated. Hence, $M$ is a Dedekind finite module.

## 3. Characterizations of $FGDF$ -modules

**Theorem 1:**

Let $R$ be a duo-ring and $M$ a finitely generated $End(R)$-module. Then, the following assertions are equivalent:

(1) $M$ is a $FGDF$-module;

(2) $M$ is of finite representation type.

**Proof:**

(1) $\Rightarrow$ (2) By the proposition 2, $M$ is artinian. It results from 15.21 of (Anderson and Fuller 1974) that $M$ is of finite length. It follows from proposition 1 that $M$ is of finite representation type.

(2) $\Rightarrow$ (1) We have already seen that $M \simeq R/Ann(M)$. Therefore $R/Ann(M)$ is a finite representation type. It results from theorem 3.3 (Fall and Sangharé, 2002), theorem 1.5 (Barry, Bazubwabo and Diop, 2010) and theorem 2.1 (Touré, Diop, Mohamed, Sangharé, 2014) that $M$ is a $FGDF$-module.

**Theorem 2:**

Let $M$ be a finitely generated module and $\{M_\lambda, \lambda \in \Lambda\}$ a finite set of modules. Then the following conditions are equivalent:

(1) $M$ is a $FGDF$-module;

(2) Every Dedekind finite module of $\sigma[M]$ is noetherian.

**Proof:**

(1) $\Rightarrow$ (2) Let $N$ be a Dedekind finite module of $\sigma[M]$, We have showed in proposition 5 that any submodule of $N$ is Dedekind finite. As $M$ is a $FGDF$-module, then any submodule of $N$ is finitely generated. Thus, $N$ is noetherian.

(2) $\Rightarrow$ (1) As $M$ is finitely generated, therefore $M$ is a Dedekind finite module, hence noetherian. Let $f : M^{(\Lambda)} \to N$ an epimorphism. It results from the first theorem of isomorphism that $M^{(\Lambda)}/\ker(f) \simeq N$. Therefore $N$ is noetherian. Hence any submodule of $N$ is finitely generated.

**Corollary 3:**

Let $M$ be a semisimple module. Then, the following assertions are equivalent:

(1) $M$ is a $FGDF$-module;

(2) Every Dedekind finite module of $\sigma[M]$ is finitely cogenerated.

**Proof:**

(1) $\Rightarrow$ (2) Let's show first that $M$ is finitely generated. As $M$ is semisimple, then $M = \bigoplus M_{i\in I}$ where $M_i$ is simple for any $i \in I$. Let $f : M_i \longrightarrow M_i$ an epimorphism with $f \neq 0$. Since $M_i$ is simple then $Ker f = 0$. Therefore, $M_i$ is Hopfian for any $i \in I$. As, for any $i \in I$, $M_i$ is fully invariant. Therefore $M$ is Hopfian, hence it is Dedekind finite. In particular $M \in \sigma[M]$ and as $M$ is a $FGDF$-module then $M$ is finitely generated.

Let $N$ be a Dedekind finite module of $\sigma[M]$ then, $N$ is finitely generated. It follows from proposition 2 that $N$ is a module over $R/Ann(M)$ which is an artinian ring. It results from 10.18 (Anderson and Fuller 1974) that $N$ is finitely cogenerated.

(2) $\Rightarrow$ (1) Let $N$ be Dedekind finite module, then $N$ is finitely cogenerated. As $M$ is semisimple, every module of $\sigma[M]$ is semisimple. Hence it follows from 10.6 (Anderson and Fuller 1974) that $N$ is finitely generated.

## References


Anderson, F. W., & Fuller, K. (1974). *Rings and categories of modules*, Springer-Verlag. https://doi.org/10.1007/978-1-4684-9913-1

Barry, A., Bazubwabo, E., & Diop, P. C. (2010). On Certain Class of Duo Rings Verifying Properties (I) and (S). *International Journal of Algebra, 4*(21), 1037-1044.

Breaz, S., Călugăreau, G., & Schulz, P. (2011). Modules with Dedekind finite endomorphism rings. *Mathematica, tome 53*(76), n1, 15-18

Fall A. L & Sangharé M.: *On I-duo rings*, Publication Mathematiques de Besancon, Theorie des nombres (2002) 1-6.

Ghorbani, A., & Haghany, A. (2002). On generalized hopfian module. *Journal of Algebra, 255*, 324-341. https://doi.org/10.1016/S0021-8693(02)00124-2

Touré, S. D., Diop, K., & Mohamed, S. M. O., Sangharé, M. (2014). A Characterization of Duo-Rings in which Every Dedekind Finite module is Finitely generated. *International Mathematical Forum, 9*(15), 733 - 736. https://doi.org/10.12988/imf.2014.4352

Vanaja, N. (1996). All finitely generated *M*-subgenerated modules are extending. *Comm. Algebra, 24*(2), 543-572. https://doi.org/10.1080/00927879608825585

Wisbauer, R. (1991). *Foundation of Module and Ring theory*. Gordon and Breach Science Publishers.


## Copyrights